Ann. Sci. Math. Québec **30** (2006), no 1, 109–116

SPECTRAL PROPERTIES OF NON-SELFADJOINT ELLIPTIC OPERATORS ASSOCIATED WITH BILINEAR FORMS

ALI SAMERIPOUR

RÉSUMÉ. L'auteur étudie les propriétés spectrales de l'opérateur elliptique non auto-adjoint

$$(Au)(x) = -\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial}{\partial x_j}\left\{\rho^{2\alpha}(x)a_{ij}(x)q(x)\frac{\partial u}{\partial x_i}(x)\right\}$$

défini sur l'espace de Hilbert $H_\ell = L^2(\Omega)^\ell$ associé à la forme bilinéaire

$$\mathscr{A}[u,v] = \int_\Omega \left\langle \rho^\alpha(x)a_{ij}(x)q(x)\frac{\partial u}{\partial x_i}(x), \rho^\alpha(x)\frac{\partial u}{\partial x_j}(x)\right\rangle_{\mathbb{C}^\ell} dx,$$

où $0 \le \alpha < \mu/n$, $\rho(x) = \operatorname{dist}\{x,S\}$, $q(x) \in C^2(\overline{\Omega}, \operatorname{End}\mathbb{C}^\ell)$, $a_{ij}(x) = a_{ji}(x) \in C^2(\overline{\Omega})$ pour tous $i,j \in \{1,\dots,n\}$ et il existe $M > 0$ tel que $|s|^2 \le M\sum_{i=1}^n\sum_{j=1}^n a_{ij}(x)s_i\overline{s_j}$ pour tous $x \in \overline{\Omega}$ et $s \in \mathbb{C}^n$. Il est supposé de plus que pour tout $x \in \overline{\Omega}$, la fonction matricielle $q(x)$ a des valeurs propres simples situées dans $\mathbb{C}\backslash Q$, où $Q = \{z \in \mathbb{C} : |\arg z| \le \varphi\}$, $\varphi \in (0,\pi)$.

ABSTRACT. The author investigates the spectral properties of the non-self adjoint elliptic operator

$$(Au)(x) = -\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial}{\partial x_j}\left\{\rho^{2\alpha}(x)a_{ij}(x)q(x)\frac{\partial u}{\partial x_i}(x)\right\}$$

defined on the Hilbert space $H_\ell = L^2(\Omega)^\ell$ associated with the bilinear form

$$\mathscr{A}[u,v] = \int_\Omega \left\langle \rho^\alpha(x)a_{ij}(x)q(x)\frac{\partial u}{\partial x_i}(x), \rho^\alpha(x)\frac{\partial u}{\partial x_j}(x)\right\rangle_{\mathbb{C}^\ell} dx,$$

where $0 \le \alpha < \mu/n$, $\rho(x) = \operatorname{dist}\{x,S\}$, $q(x) \in C^2(\overline{\Omega}, \operatorname{End}\mathbb{C}^\ell)$ $a_{ij}(x) = a_{ji}(x) \in C^2(\overline{\Omega})$ for all $i,j \in \{1,\dots,n\}$, and there exists $M > 0$ such that $|s|^2 \le M\sum_{i=1}^n\sum_{j=1}^n a_{ij}(x)s_i\overline{s_j}$ for all $x \in \overline{\Omega}$ and $s \in \mathbb{C}^n$. It is further assumed that for all $x \in \overline{\Omega}$, the matrix function $q(x)$ has simple eigenvalues lying in $\mathbb{C}\backslash Q$, where $Q = \{z \in \mathbb{C} : |\arg z| \le \varphi\}$, $\varphi \in (0,\pi)$.

1. Introduction. Let $\Omega_0 \subset \mathbb{R}^n$ be a bounded domain with a smooth boundary. Suppose that $S \subset \Omega_0$ is a closed manifold with finite dimension $\mu \in \{1,\dots,n-1\}$. Let $\Omega = \Omega_0\backslash S$ and consider the space $\mathscr{H}_\ell = W^2_{2,\alpha}(\Omega)^\ell = W^2_{2,\alpha}(\Omega)\times\cdots\times W^2_{2,\alpha}(\Omega)$ of vector functions

Reçu le 14 novembre 2004 et, sous forme définitive, le 23 mai 2005.

$u(x) = (u_1(x),\dots,u_\ell(x))$ defined on Ω with finite norm

$$|u|_+ = \left(\sum_{i=1}^{n}\int \rho^{2\alpha}(x)\left|\frac{\partial u}{\partial x_i}(x)\right|^2_{\mathbb{C}^\ell} dx + \int_\Omega |u(x)|^2_{\mathbb{C}^\ell}\right)^{1/2}.$$

This paper extends results given in [5–6] and explores new conditions. For example, an asymptotic formula is given for the distribution of eigenvalues of the operator A when $\mu \in \{1,\dots,n-1\}$ and $\alpha < \mu/n$.

More specifically, we investigate the spectral properties of the non-self adjoint elliptic differential operator

$$(Au)(x) = -\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial}{\partial x_j}\left\{\rho^{2\alpha}(x)a_{ij}(x)q(x)\frac{\partial u}{\partial x_i}(x)\right\}$$

defined on $H_\ell = L^2(\Omega)^\ell$ which is associated to the bilinear form

$$\mathscr{A}[u,v] = \int_\Omega <\rho^\alpha(x)a_{ij}(x)q(x)\frac{\partial u}{\partial x_i}(x), \rho^\alpha(x)\frac{\partial u}{\partial x_j}(x)>_{\mathbb{C}^\ell} dx,$$

where $0 \le \alpha < \mu/n$, $\rho(x) = \operatorname{dist}\{x,S\}$, $q(x) \in C^2(\overline{\Omega}, \operatorname{End}\mathbb{C}^\ell)$, $a_{ij}(x) = a_{ji}(x) \in C^2(\overline{\Omega})$ for all $i,j \in \{1,\dots,n\}$, and there exists $M > 0$ such that all $x \in \overline{\Omega}$ and $s \in \mathbb{C}^n$

$$|s|^2 \le M\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}(x)s_i\overline{s_j}.$$

We further assume that for all $x \in \overline{\Omega}$, the matrix function $q(x)$ has simple eigenvalues $\mu_1(x),\dots,\mu_\ell(x)$ in the complex plane outside the sector $Q = \{z \in \mathbb{C} : |\arg z| \le \varphi\}$, $\varphi \in (0,\pi)$. In other words, $\mu_i(x) \in C^2(\overline{\Omega})$, $\mu_i(x) \in \mathbb{C}\backslash Q$ and $\mu_i(x) \ne \mu_j(x)$ for all $i \ne j$.

Proceeding as in [4], we extend the domain of the operator A to the closed set

$$D(A) = \left\{u \in \mathring{\mathscr{H}}_\ell \cap W^2_{2,\mathrm{loc}}(\Omega)^\ell : \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial}{\partial x_j}\left(\rho^{2\alpha}a_{ij}q\frac{\partial u}{\partial x_i}\right) \in H_\ell\right\}.$$

Here $W^2_{2,\mathrm{loc}}(\Omega)^\ell = W^2_{2,\mathrm{loc}}(\Omega)\times\cdots\times W^2_{2,\mathrm{loc}}(\Omega)$ is the ℓ-fold product of the space $W^2_{2,\mathrm{loc}}(\Omega)$ of functions $u(x)$ on Ω such that

$$\sum_{i=0}^{2}\int_J |u^{(i)}(x)|^2 dx < \infty$$

on some open set $J \subset \Omega$.

We denote by $\mathring{\mathscr{H}}_\ell$ the closure of $C_0^\infty(\Omega)^\ell$ with respect to the above norm, where $C_0^\infty(\Omega)$ is the space of infinitely differentiable functions with compact support in Ω.

When $\ell = 1$, we note $\mathscr{H} = \mathscr{H}_1$ and $\mathring{\mathscr{H}} = \mathring{\mathscr{H}}_1$. Here and in the sequel, we take $\arg z \in (-\pi,\pi]$ and $\|T\|$ denotes the norm of the bounded operator T on H_ℓ.

2. Spectral properties. Let the operator A be defined as in Section 1, and let $Q \subset \mathbb{C}$ be some closed sector with vertex at 0. Let also $S \subset Q\backslash\mathbb{R}_+$ be a closed sector with vertex at 0, where $\mathbb{R}_+$ denotes the positive real numbers. Furthermore, suppose that the eigenvalues $\mu_1(x),\dots,\mu_\ell(x)$ of the matrix function $q(x)$ belong to $C^2(\overline{\Omega})$, so that the oscillation of the values of their argument does not exceed $\pi/16$. We then have the following result.

Theorem 1. *If S is defined as above, then for any $\lambda \in S$ whose modulus is sufficiently large, the inverse operator $(A-\lambda I)^{-1}$ exists and is continuous in $L^2(\Omega)^\ell$. Furthermore, there exist sufficiently large numbers $M_S > 0$ and $C_S > 0$ depending on S such that for all $|\lambda| \geq C_S$,*

$$\|(A-\lambda I)^{-1}\| \leq M_S |\lambda|^{-1}. \tag{1}$$

Let $\lambda_1, \lambda_2, \ldots$ denote the eigenvalues of the operator A in Q, enumerated in non-decreasing order of their absolute values (taking into account their algebraic multiplicity). Let $N(t) = \operatorname{card}\{j : |\lambda_j| \leq t\}$ for all $t > 0$. According to Theorem 1, the operator A has finitely many eigenvalues in any closed sector Q. Therefore,

$$\lim_{j\to\infty} \arg\lambda_j = 0.$$

Theorem 2. *Under the above assumptions and the condition $\alpha < \mu/n$, one has*

$$N(t) \sim (2\pi)^{-n} v_n t^{n/2} \sum_{i=1}^{\ell} \int_\Omega \rho^{-n\alpha}(x) \mu_i{}^{-n/2}(x) / \sqrt{\det a(x)}\, dx,$$

as $t \to \infty$, where v_n denotes the volume of the unit ball in $\mathbb{R}^n$ and $a(x) = (a_{ij}(x))_{i,j=1}^n$.

Proof of Theorem 1. It is based on estimates of the resolvent of the operators $A_1, \ldots, A_\ell$ defined on $\mathscr{H} = \mathscr{H}_1$ by

$$\begin{aligned} (A_k y)(x) &= -\sum_{i=1}^n \sum_{j=1}^n \frac{\partial}{\partial x_j} \left\{ \rho^{2\alpha}(x) a_{ij}(x) \mu_k \frac{\partial y}{\partial x_i}(x) \right\}, \\ D(A_k) &= \left\{ y \in \mathring{\mathscr{H}} \cap W^2_{2,\mathrm{loc}}(\Omega) : \sum_{i=1}^n \sum_{j=1}^n \frac{\partial}{\partial x_j} \rho^{2\alpha} a_{ij} \mu_k \frac{\partial y}{\partial x_i} \in H \right\}. \end{aligned}$$

Using the partition of unity and the assumed condition on the eigenvalues, we construct non-negative functions $\varphi_{k_1}(x), \ldots, \varphi_{k_m}(x) \in C^\infty(\overline{\Omega})$ with the property that for all $x \in \overline{\Omega}$, $x_1, x_2 \in \operatorname{supp} \varphi_{k_j}$ and $j \in \{1, \ldots, m\}$,

$$\sum_{i=1}^m \varphi_{k_i}^2(x) \equiv 1 \quad \text{and} \quad |\arg\{\mu_k(x_1)\mu_k^{-1}(x_2)\}| < \frac{\pi}{16}.$$

We then construct functions $\mu_{k_r}(x) \in C^2(\overline{\Omega})$ such that $\mu_{k_r}(x) = \mu_k(x)$ for all $x \in \operatorname{supp} \varphi_{k_r}$ and $\mu_{k_r}(x) \notin S$ for all $x \in \overline{\Omega}$. For $k \in \{1, \ldots, \ell\}$ and $r \in \{1, \ldots, m\}$, we have

$$\left| \arg\{\mu_{k_r}(x_1)\mu_{k_r}^{-1}(x_2)\} \right| \leq \frac{\pi}{8} \tag{2}$$

for all $x_1, x_2 \in \operatorname{supp}(\varphi_{k_r})$. From these conditions, we can first prove Theorem 1 in the simple case $\mathscr{H} = \mathscr{H}_1$. The general case $\mathscr{H}_\ell$ then follows by diagonalisation.

Define

$$(A_{k_r} y)(x) = -\sum_{i=1}^n \sum_{j=1}^n \frac{\partial}{\partial x_i} \left(\rho^{2\alpha}(x) a_{ij}(x) \mu_{k_r}(x) \frac{\partial y}{\partial x_i}(x) \right)$$

and

$$D(A_{k_r}) = \left\{ y \in \mathring{\mathscr{H}} \cap W^2_{2,\mathrm{loc}}(\Omega) : \sum_{i=1}^n \sum_{j=1}^n \frac{\partial}{\partial x_j} \left(\rho^{2\alpha} a_{ij} \mu_{k_r} \frac{\partial y}{\partial x_i} \right) \in H \right\}.$$

By (2), there exist complex numbers $Z_{k_r} \in \mathbb{C}$ with $|Z_{k_r}| = 1$ such that

$$c' \leq \operatorname{Re}\{Z_{k_r} \mu_{k_r}(x)\}, \quad c'|\lambda| \leq -\operatorname{Re}\{Z_{k_r} \lambda\}, \quad c' > 0. \tag{3}$$

for all $x \in \overline{\Omega}$, $\lambda \in S$. According to the uniformly elliptic condition, we have

$$c|s|^2 = c\sum_{i=1}^{n} |s_i|^2 \leq \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(x) s_i \overline{s_j}$$

for all $c > 0$, $s = (s_1, \ldots, s_n) \in \mathbb{C}^n$ and $x \in \Omega$.

Letting $s_i = \partial y / \partial x_i$ in the latter relation, one can see that

$$c\sum_{i=1}^{n} \left| \frac{\partial y}{\partial x_i}(x) \right|^2 \leq \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(x) \frac{\partial y}{\partial x_i}(x) \overline{\frac{\partial y}{\partial x_j}}(x).$$

From this, and in view of the first inequality in (3), we obtain

$$c_1 \sum_{i=1}^{n} \left| \frac{\partial y}{\partial x_i}(x) \right|^2 \leq \operatorname{Re}\{Z_{k_r} \mu_{k_r}(x)\} \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(x) \frac{\partial y}{\partial x_i}(x) \overline{\frac{\partial y}{\partial x_j}}(x).$$

Upon multiplying both sides of the latter relation by the positive term $\rho^{2\alpha}(x)$ and integrating on both sides, we find

$$\begin{aligned} c_1 \sum_{i=1}^{n} \int_\Omega \rho^{2\alpha}(x) \left| \frac{\partial y}{\partial x_i}(x) \right|^2 dx &\leq \operatorname{Re}\{Z_{k_r}\} \sum_{i=1}^{n}\sum_{j=1}^{n} \int_\Omega \rho^{2\alpha}(x) a_{ij}(x) \mu_{k_r}(x) \frac{\partial y}{\partial x_i}(x) \overline{\frac{\partial y}{\partial x_j}}(x) \\ &= \operatorname{Re}\{Z_{k_r}\} \left(-\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial y}{\partial x_j} \rho^{2\alpha} a_{ij} \mu_{k_r} \frac{\partial y}{\partial x_i}(x), y(x) \right) \\ &= \operatorname{Re}\{Z_{k_r}\} (A_{k_r} y, y). \end{aligned} \tag{4}$$

Here, $(\cdot,\cdot)$ denotes the inner product in H.

Note that the equality in (4) stems from the well-known theorem on m-sectorial operators. These operators are associated with the closed sectorial bilinear forms that are densely defined in $\mathscr{H}$. For further explanations, see Theorem 1 in Chapter 6 of [4].

Now combine the second inequality in (3) and

$$\int_\Omega |y(x)|^2 dx = (y, y) = \|y\|^2 > 0.$$

We get

$$c'|\lambda| \int_\Omega |y(x)|^2 dx \leq -\operatorname{Re}\{Z_{k_r} \lambda\} (y, y).$$

From this and the previous inequality, we find

$$\begin{aligned} c_1 \sum_{i=1}^{n} \int_\Omega \rho^{2\alpha}(x) \left| \frac{\partial y}{\partial x_i}(x) \right|^2 dx + c'|\lambda| \int_\Omega |y(x)|^2 dx &\leq \operatorname{Re} Z_{k_r} \{(A_{k_r} y, y) - \lambda (y, y)\} \\ &= \operatorname{Re} Z_{k_r} ((A_{k_r} - \lambda I) y, y) \\ &\leq \|Z_{k_r}\| \, \|y\| \, \|(A_{k_r} - \lambda I) y\| \\ &= \|y\| \, \|(A_{k_r} - \lambda I) y\|. \end{aligned} \tag{5}$$

Relation (5) implies that either

$$c'|\lambda| \, \|y(x)\|^2 = c'|\lambda| \int_\Omega |y(x)|^2 dx \leq \|y\| \, \|(A_{k_r} - \lambda I) y\|$$

or

$$|\lambda| \, \|y(x)\| \leq M_S \|(A_{k_r} - \lambda I) y\|. \tag{6}$$

This inequality ensures that the operator $A_{k_r} - \lambda I$ is one-to-one, which implies that $\ker(A_{k_r} - \lambda I) = 0$. Therefore the inverse operator $(A_{k_r} - \lambda I)^{-1}$ exists and is continuous.

Let $y = (A_{k_r} - \lambda I)^{-1} f$ with $f \in H$ in (6). It follows that

$$|\lambda| \, \|(A_{k_r} - \lambda I)^{-1} f\| \leq M_S \|(A_{k_r} - \lambda I)(A_{k_r} - \lambda I)^{-1} f\|,$$

because

$$(A_{k_r} - \lambda I)(A_{k_r} - \lambda I)^{-1} f = I(f) = f$$

and hence

$$|\lambda| \, \|(A_{k_r} - \lambda I)^{-1}(f)\| \leq M_S |f|.$$

However, $\lambda \neq 0$ and hence

$$\|(A_{k_r} - \lambda I)^{-1}(f)\| \leq M_S |\lambda|^{-1} |f|.$$

In other words,

$$\|(A_{k_r} - \lambda I)^{-1}\| \leq M_S |\lambda|^{-1}. \tag{7}$$

Let us now derive another estimate. Proceeding as in the argument leading to (5), we observe that

$$c_1 \sum_{i=1}^{n} \left\| \rho^{\alpha}(x) \frac{\partial y}{\partial x_i}(x) \right\|^2 = c_1 \sum_{i=1}^{n} \int_{\Omega} \rho^{2\alpha}(x) \left| \frac{\partial y}{\partial x_i}(x) \right|^2 dx \leq \| y \| \, \| (A_{k_r} - \lambda I) y \|.$$

Consequently,

$$c_1 \left\| \rho^{\alpha}(x) \frac{\partial y}{\partial x_i}(x) \right\|^2 \leq c_1 \sum_{i=1}^{n} \left\| \rho^{\alpha}(x) \frac{\partial y}{\partial x_i}(x) \right\|^2 \leq \| y \| \, \| (A_{k_r} - \lambda I) y \|.$$

Set $y = (A_{k_r} - \lambda I)^{-1} f$ with $f \in H$ in the latter relation. Proceeding as in the derivation of (7), we find

$$c_1 \left\| \rho^{\alpha} \frac{\partial}{\partial x_i} (A_{k_r} - \lambda I)^{-1} f \right\|^2 \leq \| (A_{k_r} - \lambda I)^{-1} f \| \, \| (A_{k_r} - \lambda I)(A_{k_r} - \lambda I)^{-1} f \|.$$

Considering that $(A_{k_r} - \lambda I)(A_{k_r} - \lambda I)^{-1} f = I(f) = f$, it follows that

$$c_1 \left\| \rho^{\alpha} \frac{\partial}{\partial x_i} (A_{k_r} - \lambda I)^{-1} f \right\|^2 \leq \| (A_{k_r} - \lambda I)^{-1} \| \, \| f \|^2.$$

In view of (7), we then have

$$c_1 \left\| \rho^{\alpha} \frac{\partial}{\partial x_i} (A_{k_r} - \lambda I)^{-1} f \right\|^2 \leq M_S |\lambda|^{-1} \|f\|^2.$$

In the end, we have

$$\left\| \rho^{\alpha} \frac{\partial}{\partial x_i} (A_{k_r} - \lambda I)^{-1} \right\| \leq M'_S |\lambda|^{-1/2}. \tag{8}$$

From this, and according to Hardy's inequality, we conclude that

$$\| \rho^{2\alpha-1} (A_{k_r} - \lambda I)^{-1} \| \leq M_3 \sum_{i=1}^{n} \left\| \rho^{\alpha} \frac{\partial}{\partial x_i} (A_{k_r} - \lambda I)^{-1} \right\| \leq M' |\lambda|^{-1/2}. \tag{9}$$

For $k \in \{1, \ldots, \ell\}$, let us introduce the operator

$$G_k(\lambda) = \sum_{r=1}^{m} \varphi_{k_r} (A_{k_r} - \lambda I)^{-1} \varphi_{k_r} \tag{10}$$

in $\mathscr{H}_\ell$. Here φ_{k_r} is the operator of multiplication by the function $\varphi_{k_r}(x)$. Simple computations yield

$$\begin{aligned}(A_k-\lambda I)G_k(\lambda) &= I+\rho^{2\alpha-1}(x)\sum_{r=1}^{m}B_{k_r}(x)(A_{k_r}-\lambda I)^{-1}\varphi_{k_r}\\ &\quad+\rho^{2\alpha}(x)\sum_{i=1}^{n}\sum_{r=1}^{m}\gamma_{k_{ir}}(x)\frac{\partial}{\partial x_i}(A_{k_r}-\lambda I)^{-1}\varphi_{k_r}, \qquad (11)\end{aligned}$$

where β_{k_r} and $\gamma_{k_{ir}}(x)\in L_\infty(\Omega)$ are bounded functions while $\operatorname{supp}\beta_{k_r}$, $\operatorname{supp}\gamma_{k_{ir}}(x)\subset\operatorname{supp}\varphi_{k_r}$. From Equations (7) – (9), we see that for all $\lambda\in S$ with $|\lambda|\geq 1$,

$$\|(A_k-\lambda I)G_k(\lambda)-I\|\leq M|\lambda|^{-1/2}. \qquad (12)$$

Thus for $k\in\{1,\dots,\ell\}$, the representation

$$(A_k-\lambda I)^{-1}=G_k(\lambda)(I+F_k(\lambda)),\quad \|F_k(\lambda)\|\leq M|\lambda|^{-1/2} \qquad (13)$$

is valid for all $\lambda\in S$ with $|\lambda|\geq C_0$. We can thus get an estimate of the resolvent of the operators A_k. In fact, for $k\in\{1,\dots,\ell\}$, we have

$$\|(A_k-\lambda I)^{-1}\|\leq M|\lambda|^{-1/2}$$

for all $\lambda\in S$ such that $|\lambda|\geq 1$.

Now we diagonalise the matrix function $q(x)$, viz.

$$q(x)=U(x)\Lambda(x)U^{-1}(x),$$

where $U(x),U^{-1}(x)\in C^2(\overline{\Omega},\operatorname{End}\mathbb{C}^\ell)$ and $\Lambda(x)=\operatorname{diag}\{\mu_1(x),\dots,\mu_\ell(x)\}$.

Let $\Gamma(\lambda)=UB(\lambda)U^{-1}$, so that the operator $B(\lambda)$ in the direct sum

$$H_\ell=H\oplus\cdots\oplus H(\ell\text{-fold})$$

has the representation

$$B(\lambda)=\operatorname{diag}\{(A_1-\lambda I)^{-1},\dots,(A_\ell-\lambda I)^{-1}\},$$

where $\lambda\in S$ with $|\lambda|\geq C_S$ and $(Uu)(x)=U(x)u(x)$ for all $u\in H_\ell$. It is easily verified that

$$(A-\lambda I)\Gamma(\lambda)=I+\rho^{2\alpha-1}(x)q_0(x)B(\lambda)U^{-1}+\rho^{2\alpha}(x)\sum_{i=1}^{n}q_i(x)\frac{\partial}{\partial x_i}B(\lambda)U^{-1},$$

where $q_i(x)\in C(\overline{\Omega},\operatorname{End}\mathbb{C}^\ell)$ for all $i\in\{0,\dots,n\}$. Using Equations (5) – (8), we have

$$(A-\lambda I)^{-1}=\Gamma(\lambda)(I+F(\lambda)),\quad \|F(\lambda)\|\leq M_S|\lambda|^{-1/2} \qquad (14)$$

for all $\lambda\in S$ with $|\lambda|\geq C_S$, as in (10). Thus $(A-\lambda I)^{-1}$ exists and is continuous. To prove (1), by the above definition of $\Gamma(\lambda)$, we can now use

$$\|\Gamma(\lambda)\|\leq M'|\lambda|^{-1},$$

and from (14) we conclude that

$$\|(A-\lambda I)^{-1}\|\leq M_S|\lambda|^{-1}.$$

This completes the proof of Theorem 1. □

Proof of Theorem 2. It suffices to use the results of Vulis and Solomyak [9]. □

Résumé substantiel en français. Soient $\Omega_0 \subset \mathbb{R}^n$ un domaine borné de frontière lisse et $S \subset \Omega_0$ une variété fermée de dimension finie $\mu \in \{1,\ldots,n-1\}$. Posons $\Omega = \Omega_0 \backslash S$ et considérons l'espace $\mathscr{H}_\ell = W^2_{2,\alpha}(\Omega)^\ell = W^2_{2,\alpha}(\Omega) \times \cdots \times W^2_{2,\alpha}(\Omega)$ des fonctions vectorielles $u(x) = (u_1(x),\ldots,u_\ell(x))$ définies sur Ω et de norme finie

$$|u|_+ = \left(\sum_{i=1}^n \int \rho^{2\alpha}(x) \left|\frac{\partial u}{\partial x_i}(x)\right|^2_{\mathbb{C}^\ell} dx + \int_\Omega |u(x)|^2_{\mathbb{C}^\ell}\right)^{1/2}.$$

Cet article étend des résultats obtenus en [5, 6]. De façon plus spécifique, on s'y intéresse aux propriétés spectrales de l'opérateur elliptique non auto-adjoint

$$(Au)(x) = -\sum_{i=1}^n \sum_{j=1}^n \frac{\partial}{\partial x_j}\left\{\rho^{2\alpha}(x) a_{ij}(x) q(x) \frac{\partial u}{\partial x_i}(x)\right\}$$

défini sur l'espace de Hilbert $H_\ell = L^2(\Omega)^\ell$ associé à la forme bilinéaire

$$\mathscr{A}[u,v] = \int_\Omega \left\langle \rho^\alpha(x) a_{ij}(x) q(x) \frac{\partial u}{\partial x_i}(x), \rho^\alpha(x) \frac{\partial u}{\partial x_j}(x)\right\rangle_{\mathbb{C}^\ell} dx,$$

où $0 \le \alpha < \mu/n$, $\rho(x) = \mathrm{dist}\{x,S\}$, $q(x) \in C^2(\overline{\Omega}, \mathrm{End}\,\mathbb{C}^\ell)$, $a_{ij}(x) = a_{ji}(x) \in C^2(\overline{\Omega})$ pour tous $i,j \in \{1,\ldots,n\}$ et il existe $M > 0$ tel que pour tous $x \in \overline{\Omega}$ et $s \in \mathbb{C}^n$,

$$|s|^2 \le M \sum_{i=1}^n \sum_{j=1}^n a_{ij}(x) s_i \overline{s_j}.$$

On suppose en outre que pour tout $x \in \overline{\Omega}$, la fonction matricielle $q(x)$ a des valeurs propres simples situées dans $\mathbb{C}\backslash Q$, où $Q = \{z \in \mathbb{C} : |\arg z| \le \varphi\}$, $\varphi \in (0,\pi)$. En d'autres mots, $\mu_i(x) \in C^2(\overline{\Omega})$, $\mu_i(x) \in \mathbb{C}\backslash Q$ et $\mu_i(x) \ne \mu_j(x)$ pour tous $i \ne j$. Ceci fait en sorte que l'oscillation des valeurs de leur argument ne dépasse jamais $\pi/16$.

Théorème 1. *Si S est tel que défini ci-haut et que $\lambda \in S$ est de module suffisamment grand, alors l'opérateur inverse $(A-\lambda I)^{-1}$ existe et il est continu dans $L^2(\Omega)^\ell$. De plus, il existe des nombres $M_S > 0$ et $C_S > 0$ dépendant de S tels que pour tout $|\lambda| \ge C_S$,*

$$\|(A-\lambda I)^{-1}\| \le M_S |\lambda|^{-1}.$$

Notons $\lambda_1, \lambda_2, \ldots$ les valeurs propres de l'opérateur A dans Q, énumérées en ordre croissant de valeur absolue (en tenant compte de leur multiplicité algébrique). Soit $N(t) = \mathrm{card}\{j : |\lambda_j| \le t\}$ pour tout $t > 0$. D'après le théorème 1, l'opérateur A possède un nombre fini de valeurs propres dans tout secteur fermé Q. Par conséquent,

$$\lim_{j\to\infty} \arg\lambda_j = 0.$$

Théorème 2. *Dans les mêmes conditions que ci-dessus et en supposant $\alpha < \mu/n$, on a*

$$N(t) \sim (2\pi)^{-n} v_n t^{n/2} \sum_{i=1}^\ell \int_\Omega \rho^{-n\alpha}(x) \mu_i{}^{-n/2}(x) / \sqrt{\det a(x)}\, dx,$$

quand $t \to \infty$, où v_n représente le volume de la boule unité dans $\mathbb{R}^n$ et $a(x) = (a_{ij}(x))^n_{i,j=1}$.

REFERENCES

1. K. Kh. Boĭmatov, *Asymptotics of the spectrum of nonselfadjoint systems of second-order differential operators*, Mat. Zametki **51** (1992), 8–16 [in Russian] ; [English translation in Math. Notes **51** (1992), 330–337].
2. K. Kh. Boĭmatov, *The spectral asymptotics of nonselfadjoint degenerate elliptic systems of differential operators*, Dokl. Akad. Nauk **330** (1993), 533–538 [in Russian] ; [English translation in Russian Acad. Sci. Dokl. Math. **47** (1993), 545–553].
3. K. Kh. Boĭmatov, *The spectral asymptotic behavior of differential and pseudodifferential operators. II*, Trudy Sem. Petrovsk. No. 10 (1984), 78–106 [in Russian] ; [English translation in J. Soviet Math. **35** (1986), 2744–2769].
4. T. Kato, *Perturbation Theory for Linear Operators*, Springer, New York, 1966.
5. A. Samiripur and K. Seddiki, *Distribution of the eigenvalues of nonselfadjoint elliptic systems that degenerate on the domain boundary*, Mat. Zametki **61** (1997), 463–467 [in Russian] ; [English translation in Math. Notes **61** (1997), 379–384].
6. A. Sameripour and K. Seddighi, *On the spectral properties of generalized non-self-adjoint elliptic systems of differential operators degenerated on the boundary of domain*, Bull. Iranian Math. Soc. **24** (1998), 15–32.
7. A. A. Shkalikov, *Tauberian type theorems on the distribution of zeros of holomorphic functions*, Mat. Sb. (N.S.) **123(165)** (1984), 317–347 [in Russian] ; [English translation in Math. USSR-sb. **51** (1985), 315–344].
8. I. L. Vulis and M. Z. Solomjak, *Spectral asymptotic analysis of the degenerate Steklov problem*, Vestnik Leningrad. Univ. No. 19 Mat. Meh. Astronom Vyp. 4 (1973), 148–150 [in Russian].
9. I. L. Vulis and M. Z. Solomjak, *Spectral asymptotic behavior of degenerate elliptic operators*, Dokl. Akad. Nauk SSSR **207** (1972), 262–265 [in Russian].

A. SAMERIPOUR
DEPARTMENT OF MATHEMATICS
LORESTAN UNIVERSITY
KHORRAMABAD, IRAN
asameripour@yahoo.com