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ARITHMETICO-GEOMETRICO-HARMONIC FUNCTIONAL MEAN IN CONVEX ANALYSIS

MUSTAPHA RAÏSSOULI AND HANANE BOUZIANE

RÉSUMÉ. En interprétant la conjuguée de Legendre-Fenchel comme un inverse dans un sens, nous étendons ici certaines notions d'opérateurs linéaires aux fonctions convexes. Étant donné un opérateur défini positif A et $0 < p < 1$, on donne l'analogue de A^p au cas fonctionnel et on déduit la définition de la moyenne géometrique de m fonctions convexes. Ensuite, on construit la moyenne arithmético-géométrico-harmonique au sens de l'analyse convexe laquelle, dans le cas quadratique, nous fournit celle des opérateurs positifs. On termine par une discussion concernant le lien de notre cas $p = 1/2$ avec la moyenne géometrique convexe définie dans [4].

ABSTRACT. Interpreting the notion of Legendre-Fenchel conjugate as an inverse in some sense, we introduce here some extensions of notions from linear operators to convex functions. Let A be a positive definite operator and $0 < p < 1$, we give an extension of A^p and so we define the geometric mean of m convex functions. Afterwards, we introduce the arithmetico-geometrico-harmonic functional mean from which we deduce that of three positive definite operators. At the end, we discuss the case $p = 1/2$ connected to a convex functional mean defined in [4].

1. Introduction. In the literature, an enormous amount of effort by some authors has been devoted to understand the notions of the scalar means because of their many interesting properties and applications [3, 7, 8, 20]. Let us describe some situations explaining the interest of these means. First, the arithmetico-geometric and geometrico-harmonic means obtain their importance from the fact that they can be used to compute elliptic integrals which are interesting in many scientific problems (see [12] and the reference cited therein). Secondly, the arithmetico-harmonic mean satisfies an "invariance property" which is very useful in application, in particular it is good tool in solving some functional equations [20]. The theory of operator means, extending that of the scalar ones, was stated in a large literature [1, 2, 4, 9, 12, 13, 14, 15, 18, 22]. These operator means arise in various contexts. As example, the geometric matrix mean appears as an illustration of an equivalent resistor for an electrical circuit with matrices arguments, [4, 26].

In the recent years, the extension of the notions of the means from operators to convex functions has undergone extensive developments [4, 10, 11, 25]. To explain the interest of these functional means, it is sufficient to recall that, as pointed in [11, 24],

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the "Legendre-Fenchel conjugate" f^* of a convex function f on a real Hilbert space H defined by

$$\forall x^* \in H, \quad f^*(x^*) = \sup\{\langle x^*, x\rangle - f(x) : x \in H\} \tag{1.1}$$

can be considered as that of "inverse" in some sense. That is to say, if f is a quadratical function associated to a positive definite operator A:

$$\forall x \in H, \quad f(x) := f_A(x) = \frac{1}{2}\langle Ax, x\rangle$$

then (1.1) yields

$$\forall x^* \in H, \quad f^*(x^*) = f_{A^{-1}}(x^*) = \frac{1}{2}\langle A^{-1}x^*, x^*\rangle$$

i.e. f^* is so quadratic associated to A^{-1}. Inspired by this latter idea, we can say that the theory of functional means contains that of scalar and operator ones.

We now sketch our fundamental goal more precisely. Let A be a positive definite operator and $0 < p < 1$. Recall [17] that the better p-iterate of A is given by

$$A^p = \frac{\sin(p\pi)}{\pi}\int_0^{+\infty} t^{p-1}A(tI+A)^{-1}dt, \tag{1.2}$$

or equivalently to the "convex" form

$$A^p = \frac{\sin(p\pi)}{\pi}\int_0^{+\infty} \frac{t^{p-1}}{1+t}A\left(\frac{t}{1+t}I + \frac{1}{1+t}A\right)^{-1}dt. \tag{1.3}$$

Since A is invertible, (1.3) is equivalent to

$$A^p = \frac{\sin(p\pi)}{\pi}\int_0^{+\infty} \frac{t^{p-1}}{1+t}\left(\frac{1}{1+t}I + \frac{t}{1+t}A^{-1}\right)^{-1}dt. \tag{1.4}$$

Our first purpose is to describe an extension of A^p to functional case. We suggest that a reasonable analogue of the right side of (1.4) to convex function is

$$\frac{\sin(p\pi)}{\pi}\int_0^{+\infty} \frac{t^{p-1}}{1+t}\left(\frac{1}{1+t}\sigma + \frac{t}{1+t}f^*\right)^* dt,$$

where $\sigma := \frac{1}{2}\|\cdot\|^2$, and we take as definition of $f^{(p)}$:

$$\forall x \in H, \quad f^{(p)}(x) = \frac{\sin(p\pi)}{\pi}\int_0^{+\infty} \frac{t^{p-1}}{1+t}\left(\frac{1}{1+t}\sigma + \frac{t}{1+t}f^*\right)^*(x)\,dt. \tag{1.5}$$

By analogy with the operator case, $f^{(p)}$ will be called the convex better p-iterate of f. When $p = 1/n$ ($n \geq 2$ integer), $A^{1/n}$ is the n-th positive operator root of A (i.e. the only positive operator B such that $B^n = A$) and analogously $f^{(1/n)}$ is called the convex n-th root of f.

A second aim of this paper is devoted in answer to the following question: what should be the analogue of $a^p b^{1-p}$ (a, b positive real numbers and $0 < p < 1$) when the variables a and b are convex functions? Note that, an analogue of $a^p b^{1-p}$ when a and b are positive definite operators is discussed in [22]: let $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_m) \in \mathbb{R}^m$ be a given probability vector (i.e. $0 \leq \sigma_i \leq 1$ and $\sum_{i=1}^m \sigma_i = 1$), the authors of [22] suggested that a reasonable analogue of the product $\prod_{i=1}^m a_i^{\sigma_i}$ ($a_i > 0$ real numbers) is

$$\prod_{i=1}^m A_i^{\sigma_i} = \exp\left(\sum_{i=1}^m \sigma_i \operatorname{Log} A_i\right), \tag{1.6}$$

where the variables A_i, $1 \leq i \leq m$, are positive definite operators. The extension of exponential from operator case to convex functional one is not done yet, that is where the difficulty lies to extend (1.6). To solve the above question following another approach, let us remark that $a^p b^{1-p}$ satisfies the identity

$$a^p b^{1-p} = \frac{\sin(p\pi)}{\pi} \int_0^{+\infty} \frac{t^{p-1}}{1+t} \left(\frac{t}{1+t} a^{-1} + \frac{1}{1+t} b^{-1} \right)^{-1} dt, \tag{1.7}$$

and we can suggest that a reasonable analogue of (1.7) to convex functions is

$$\forall x \in H, \quad (G_p(f,g))(x) = \frac{\sin(p\pi)}{\pi} \int_0^{+\infty} \frac{t^{p-1}}{1+t} \left(\frac{t}{1+t} f^* + \frac{1}{1+t} g^* \right)^* (x)\, dt. \tag{1.8}$$

When f and g are quadratical functions corresponding respectively to the positive definite operators A and B, we find, by a different method from that of [22], another extension of $a^p b^{1-p}$ to positive definite operators, that is

$$G_p(A,B) = \frac{\sin(p\pi)}{\pi} \int_0^{+\infty} t^{p-1} (tA^{-1} + B^{-1})^{-1} dt, \tag{1.9}$$

or equivalently

$$G_p(A,B) = \frac{\sin(p\pi)}{\pi} \int_0^{+\infty} t^{p-1} A(A+tB)^{-1} B\, dt, \tag{1.10}$$

whose the explicit form is given by

$$G_p(A,B) = B^{\frac{1}{2}} (B^{-\frac{1}{2}} A B^{-\frac{1}{2}})^p B^{\frac{1}{2}}. \tag{1.11}$$

This approach allowed us to describe an extension of $\sqrt[m]{a_1 a_2 \cdots a_m}$ (the geometric mean of m positive real numbers $a_1, a_2, \ldots, a_m$) to convex functions. Indeed, remarking that we can write

$$(a_1 a_2 \cdots a_m)^{1/m} = a_1^{1/m} \left((a_2 a_3 \cdots a_m)^{1/(m-1)} \right)^{1-1/m}$$

i.e. by the above notation

$$(a_1 a_2 \cdots a_m)^{1/m} = G_{1/m}(a_1, (a_2 a_3 \cdots a_m)^{1/(m-1)}),$$

we put for all integer $m \geq 2$ and f_i, $1 \leq i \leq m$, convex functions

$$\begin{cases} \mathscr{G}_1(f_1) = f_1, \\ \mathscr{G}_m(f_1, f_2, \ldots, f_m) = G_{1/m}(f_1, \mathscr{G}_{m-1}(f_2, f_3, \ldots, f_m)), \end{cases}$$

which defines the geometric mean of three or more convex functions.

This paper will be organized as follows. Section 2 contains some classical convex analysis results which will be needed throughout the paper. In section 3, after giving and studying the extension of A^p from operators to convex functions, we present a second order approximation of $(\sigma + \lambda f)^{(p)}$ when λ goes to 0^+ and we establish the differentiability of the map $s \mapsto (s\sigma + f)^{(p)}$. Section 4 is devoted to the extension of $a^p b^{1-p}$ from positive real numbers to convex functions, in particular an analogue of Young's inequality, $a^p b^{1-p} \leq pa + (1-p)b$, for functional case is obtained. In section 5, we introduce the arithmetic, geometric and harmonic mean of m convex functions ($m \geq 2$ integer) and we deduce a relationship between these three means extending that of positive real numbers. Section 6 of our paper is focused to define the arithmetico-geometrico-harmonic mean of three convex functions from which we deduce that of three positive definite operators. The final section displays the connection between our particular case $p = 1/2$ and a convex functional mean introduced in [4].

2. Background material and preliminary results. In this section, we recall some basic facts about convex analysis which are needed throughout the paper. We denote by H a real Hilbert space equipped with a scalar product $\langle \cdot, \cdot \rangle$ and its associated norm $\| \cdot \|$. Let $\mathcal{F}\left(H, \overline{\mathbb{R}}\right)$ be the space of maps defined from H into $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$. We can define on $\mathcal{F}\left(H, \overline{\mathbb{R}}\right)$ a partial ordering relation given by

$$\forall f, g \in \mathcal{F}\left(H, \overline{\mathbb{R}}\right), \quad f \leq g \iff \forall x \in H, \ f(x) \leq g(x),$$

where we extend the structure of $\mathbb{R}$ on $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ by setting

$$\forall x \in \overline{\mathbb{R}}, \quad -\infty \leq x \leq +\infty, \quad x + (+\infty) = +\infty.$$

Consider a function $f : H \to \overline{\mathbb{R}}$, whose epigraph

$$\mathrm{epi}(f) = \{ (x, \lambda) \in H \times \mathbb{R} : f(x) \leq \lambda \}$$

is convex. The cone of such functions not identically equal to $+\infty$ is usually denoted by $\mathrm{Conv}(H)$. Recall that the Legendre-Fenchel conjugate of $f \in \mathcal{F}\left(H, \overline{\mathbb{R}}\right)$ is the function $f^* \in \mathrm{Conv}(H)$ defined by [6, 27]:

$$\forall x^* \in H, \quad f^*(x^*) = \sup\{ \langle x^*, x \rangle - f(x) : x \in H \}.$$

It is well known that $f^{**} := (f^*)^* \leq f$ and $f \leq g \Rightarrow g^* \leq f^*$.

Remark 2.1. When H is a complex Hilbert space, the Legendre-Fenchel conjugate can be replaced by the extended one (see [25]):

$$\forall x^* \in H, \quad f^*(x^*) = \sup\{ \mathrm{Re}\left(\langle x^*, x \rangle\right) - f(x) : x \in H \}.$$

In what follows, we restrict ourselves to the case of real Hilbert space since analogous results for the complex ones can be stated in a similar manner.

Lemma 2.1 [16]. *For all* $f, g \in \mathcal{F}\left(H, \overline{\mathbb{R}}\right)$ *and* $0 < \alpha < 1$, *there holds*

$$(\alpha f + (1-\alpha) g)^* \leq \alpha f^* + (1-\alpha) g^*.$$

A convex-integral version of Lemma 2.1 is given in the following.

Lemma 2.2. *Let* $a, b \in \overline{\mathbb{R}}$ $(a < b)$, $F :]a, b[\times H \to \overline{\mathbb{R}}$ *and* $\psi :]a, b[\to [0, +\infty[$ *such that* $\int_a^b \psi(t) dt = 1$. *If we put*

$$\forall x \in H, \quad \Phi(x) = \int_a^b \psi(t) F(t, x) dt$$

then the conjugate functional $\Phi^* : H \to \overline{\mathbb{R}}$ *of* Φ *satisfies that*

$$\forall x^* \in H, \quad \Phi^*(x^*) \leq \int_a^b \psi(t) F^*(t, x^*) dt$$

where

$$F^*(t, x^*) = \sup_{x \in H} \{ \langle x^*, x \rangle - F(t, x) \}.$$

Proof. Let $x^* \in H$, we have successively

$$\begin{aligned}
\Phi^*(x^*) &= \sup_{x\in H}\left\{ \langle x^*, x\rangle - \int_a^b \psi(t)F(t,x)dt \right\} \\
&= \sup_{x\in H}\left\{ \int_a^b \psi(t)\big(\langle x^*, x\rangle - F(t,x)\big)\, dt \right\} \\
&\le \int_a^b \psi(t) \sup_{x\in H}\left\{ \langle x^*, x\rangle - F(t,x) \right\} dt,
\end{aligned}$$

and the lemma is proved. $\square$

For a function $f : H \to \widetilde{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$, $\mathrm{dom}(f)$ denotes its effective domain given by

$$\mathrm{dom}(f) = \{ x \in H : f(x) < +\infty \}.$$

The set of all lower semi-continuous convex functions defined from H into $\mathbb{R}\cup\{+\infty\}$ not identically equal to $+\infty$ is denoted by $\Gamma_\circ(H)$. It is well-known that if $f \in \mathrm{Conv}(H)$ then $f^* \in \Gamma_\circ(H)$, and, $f \in \Gamma_\circ(H)$ if and only if $f^{**} = f$.

We say that p is a subgradient of $f : H \to \widetilde{\mathbb{R}}$ at $x \in \mathrm{dom}(f)$ if for all $y \in H$ one has $f(y) \ge f(x) + \langle y - x, p\rangle$. The subdifferential of f at x is the (possibly empty) subset of H defined by

$$\partial f(x) = \left\{ p \in H : \forall y \in H,\ f(y) \ge f(x) + \langle y - x, p \rangle \right\}.$$

If we denote by $\mathrm{int}(\mathrm{dom}(f))$ the topological interior of $\mathrm{dom}(f)$, we recall that if $f \in \Gamma_\circ(H)$ and $\mathrm{int}(\mathrm{dom}(f))$ is nonempty then, for all $x \in \mathrm{int}(\mathrm{dom}(f))$, f is continuous at x and $\partial f(x) \neq \emptyset$.

Lemma 2.3. *Let $f_1, f_2, \dots, f_m \in \Gamma_\circ(H)$ and $(\alpha_1, \alpha_2, \dots, \alpha_m) \in \mathbb{R}^m$ such that $0 \le \alpha_i \le 1$ and $\sum_{i=1}^m \alpha_i = 1$. The following equivalence holds*

$$\mathrm{dom}\left(\sum_{i=1}^m \alpha_i f_i\right) = \mathrm{dom}\left(\sum_{i=1}^m \alpha_i f_i^*\right)^* \iff \forall (i,j),\ \mathrm{dom}(f_i) = \mathrm{dom}(f_j). \tag{2.1}$$

Proof. The case $m = 2$ and $(\alpha_1, \alpha_2) = (1/2, 1/2)$ is proved in [10]. The result works for the general case by the analogous arguments as in [10]. $\square$

Let $\sigma = \frac{1}{2}\|\cdot\|^2 \in \Gamma_\circ(H)$ be the only self-conjugate function. We recall [6] that for all real $\lambda > 0$ and $f \in \Gamma_\circ(H)$, the function $x \to (\lambda\sigma + f)^*(x)$ defined from H into $\mathbb{R}$ is Fréchet-differentiable.

Theorem 2.1 [5, 21, 23]. *For all $f \in \Gamma_\circ(H)$ and $\lambda > 0$, one has*

$$\lambda(\lambda\sigma + f^*)^* + (\sigma + \lambda f)^* = \sigma. \tag{2.2}$$

Taking $\lambda = a/b$, an elementary verification yields that Theorem 2.1 is equivalent to the general form:

$$\forall f \in \Gamma_\circ(H),\ \forall a, b > 0, \quad a(a\sigma + bf^*)^* + b(b\sigma + af)^* = \sigma, \tag{2.3}$$

from which we deduce the “convex” one

$$\frac{1}{1+t}\left(\frac{1}{1+t}\sigma+\frac{t}{1+t}f^*\right)^*+\frac{t}{1+t}\left(\frac{t}{1+t}\sigma+\frac{1}{1+t}f\right)^*=\sigma, \tag{2.4}$$

for all $f \in \Gamma_\circ(H)$ and $t > 0$.

We mention that the assumption “H is a Hilbert space” is not a necessary hypothesis throughout the paper but only to simplify the presentation for the reader. To explain this situation more precisely, let $(E, \|\cdot\|)$ be a (reflexive or not) Banach space and $(E^*, \|\cdot\|_*)$ its topological dual. If we denote by $\langle\cdot,\cdot\rangle_{E,E^*}$ the bracket (symmetric) duality between E and E^*, it is known that the Legendre-Fenchel conjugate $f \longrightarrow f^*$ is well defined (even in a locally convex vector space). With this, if $\sigma = \frac{1}{2}\|\cdot\|^2$ then $\sigma^* = \frac{1}{2}\|\cdot\|_*^2$ and hence, all the basic definitions above, together with Lemmas 2.1, 2.2, 2.3, are still true in the Banach space case. However, a version of Theorem 2.1 for the general case is not obvious, since the two functions $(\lambda\sigma^* + f^*)^*$ and $(\sigma + \lambda f)^*$ are respectively defined in the different spaces E and E^*. We are unable to extend this result for a no-Hilbert space. For more information, we refer the reader to [10] where a similar extension is written in a Banach space.

We end this section by stating some basic notions and results about quadratical functional case that are needed later. Let $\mathscr{L}(H)$ be the Banach space of continuous linear operators defined from H into H equipped with the norm

$$\forall A \in \mathscr{L}(H), \quad \|A\| = \sup_{x\neq 0}\frac{\|Ax\|}{\|x\|} = \sup\left\{\, \|Ax\| : \|x\| \le 1 \,\right\}.$$

Let (A_n) be a sequence of operators defined from H into H. We say that (A_n) converges to A in the operator norm (or uniformly) if $\|A_n - A\|$ tends to 0 when n tends to infinity. We say that (A_n) converges to A, in the strong operator topology (or strongly), if $(A_n x)_n$ converges (in H) to Ax for all $x \in H$.

Recall that A is positive (resp. positive definite) if A is symmetric and $\langle Ax, x\rangle \geqslant 0$ for each $x \in H$ (resp. there exists $\gamma > 0$ such that $\langle Ax, x\rangle \ge \gamma\|x\|^2$ for every $x \in H$). We write $A \leqslant B$ if and only if A and B are symmetric and $B - A$ is positive. If we take $f_A(x) = \frac{1}{2}\langle Ax, x\rangle$ for all $x \in H$, where A is a positive definite operator, then $f_A^* = f_{A^{-1}}$ and so

$$(\alpha\sigma + (1-\alpha)f_A)^* = f^*_{\alpha I+(1-\alpha)A} = f_{(\alpha I+(1-\alpha)A)^{-1}}\,, \tag{2.5}$$

for all positive operator A and $\alpha \in]0,1[$.

The subdifferential of f_A is given by $\partial f_A(x) = \{Ax\}$, for all $x \in H$.

3. Convex n-th functional root.

As already pointed, this section will be devoted to extend the better p-iterate operator to convex functions. We keep the same notations as above.

3.1. Definitions and first properties.

Let $f \in \mathrm{Conv}(H)$ be a given function. For a fixed $x \in H$, the map

$$t \mapsto \left(\frac{1}{1+t}\sigma+\frac{t}{1+t}f^*\right)^*(x)$$

is differentiable on $]0,1[$ (see [23]) and hence continuous on $]0,1[$. We can then introduce the following definition.

Definition 3.1. Let $f \in \operatorname{Conv}(H)$, $0 < p < 1$, and define

$$\forall x \in H, \quad f^{(p)}(x) = \frac{\sin(p\pi)}{\pi} \int_0^{+\infty} \frac{t^{p-1}}{1+t} \left(\frac{1}{1+t}\sigma + \frac{t}{1+t} f^* \right)^* (x)\, dt. \tag{3.1}$$

The functional $f^{(p)}$ is called the *convex better p-iterate* of f. When $p = 1/n$ with $n \geq 2$ an integer, $f^{(1/n)}$ is said to be the *convex n-th root* of f.

By a change of variable $t = (\tan s)^2$, it is easy to verify that

$$\forall x \in H, \quad f^{(p)}(x) = \frac{2}{\pi} \sin(p\pi) \int_0^{\frac{\pi}{2}} (\tan t)^{2p-1} \left((\cos^2 t)\sigma + (\sin^2 t) f^* \right)^* (x)\, dt. \tag{3.2}$$

Remark 3.1. From relation (3.1) we see immediately that $f^{(p)} \in \operatorname{Conv}(H)$. The condition $f \in \operatorname{Conv}(H)$ is not necessary to define $f^{(p)}$ but only for reason of simplicity. For the same reason, we omit the x in (3.1) and in similar other relations.

Proposition 3.1. *The following properties hold:*

1. *For all $f \in \operatorname{Conv}(H)$ and $\alpha \in \mathbb{R}$, $(f+\alpha)^{(p)} = f^{(p)} + p\alpha$.*
2. *For all $a > 0$, $(a\sigma)^{(p)} = a^p \sigma$, in particular $\sigma^{(p)} = \sigma$.*

Proof. 1. We can write

$$\begin{aligned} (f+\alpha)^{(p)} &= \frac{\sin(p\pi)}{\pi} \int_0^{+\infty} \frac{t^{p-1}}{1+t} \left(\frac{1}{1+t}\sigma + \frac{t}{1+t}(f+\alpha)^* \right)^* dt \\ &= \frac{\sin(p\pi)}{\pi} \int_0^{+\infty} \frac{t^{p-1}}{1+t} \left(\frac{1}{1+t}\sigma + \frac{t}{1+t} f^* - \frac{t}{1+t}\alpha \right)^* dt \\ &= \frac{\sin(p\pi)}{\pi} \int_0^{+\infty} \left[\frac{t^{p-1}}{1+t} \left(\frac{1}{1+t}\sigma + \frac{t}{1+t} f^* \right)^* + \frac{t^p}{(1+t)^2}\alpha \right] dt \\ &= f^{(p)} + \frac{\sin(p\pi)}{\pi}\alpha \int_0^{+\infty} \frac{t^p}{(1+t)^2} dt. \end{aligned}$$

A simple calculation, with relation (1.2), yields

$$\frac{\sin(p\pi)}{\pi} \int_0^{+\infty} \frac{t^p}{(1+t)^2} dt = p \frac{\sin(p\pi)}{\pi} \int_0^{+\infty} \frac{t^{p-1}}{1+t} dt = p, \tag{3.3}$$

and the desired result is obtained.

2. For $a > 0$, we have

$$\begin{aligned} (a\sigma)^{(p)} &= \frac{\sin(p\pi)}{\pi} \int_0^{+\infty} \frac{t^{p-1}}{1+t} \left(\frac{1}{1+t}\sigma + \frac{t}{1+t}(a\sigma)^* \right)^* dt \\ &= \frac{\sin(p\pi)}{\pi} \int_0^{+\infty} \frac{t^{p-1}}{1+t} \left(\frac{1}{1+t}\sigma + \frac{t}{a(1+t)}\sigma \right)^* dt \\ &= \frac{\sin(p\pi)}{\pi} \int_0^{+\infty} \frac{t^{p-1}}{1+t} \frac{a(1+t)}{a+t} \sigma dt = \left(\frac{\sin(p\pi)}{\pi} \int_0^{+\infty} \frac{a t^{p-1}}{a+t} dt \right) \sigma \end{aligned}$$

and by (1.2) we obtain

$$(a\sigma)^{(p)} = \frac{\sin(p\pi)}{\pi}\int_0^{+\infty}\frac{a^p(t/a)^{p-1}}{(1+(t/a))}\,\frac{1}{a}\,dt = a^p\sigma\,.$$

The proof is complete. □

Theorem 3.1. *With respect to the pointwise ordering on* $\mathcal{F}\left(H,\overline{\mathbb{R}}\right)$*, the map* $f\mapsto f^{(p)}$ *is:*

1. *Increasing on* $\operatorname{Conv}(H)$*: for all* $f,g\in\operatorname{Conv}(H)$ *such that* $f\geq g$*,* $f^{(p)}\geq g^{(p)}$*.*
2. *Concave on* $\Gamma_\circ(H)$*: for all* $f,g\in\Gamma_\circ(H)$ *and* $\alpha\in\,]0,1[\,$*, one has*

$$(\alpha f+(1-\alpha)g)^{(p)}\geq\alpha f^{(p)}+(1-\alpha)g^{(p)}\,.$$

Proof. 1. Let $f,g\in\operatorname{Conv}(H)$ such that $f\geq g$. Clearly,

$$\frac{1}{1+t}\sigma+\frac{t}{1+t}g^*\geq\frac{1}{1+t}\sigma+\frac{t}{1+t}f^*$$

and

$$\left(\frac{1}{1+t}\sigma+\frac{t}{1+t}f^*\right)^*\geq\left(\frac{1}{1+t}\sigma+\frac{t}{1+t}g^*\right)^*$$

from which the desired result follows.

2. Let $f,g\in\Gamma_\circ(H)$ and $\alpha\in\,]0,1[\,$. By definition, we have

$$(\alpha f+(1-\alpha)g)^{(p)}=\frac{\sin(p\pi)}{\pi}\int_0^{+\infty}\frac{t^{p-1}}{1+t}\left(\frac{1}{1+t}\sigma+\frac{t}{1+t}(\alpha f+(1-\alpha)g)^*\right)^*dt\,.$$

According to relation (2.4), we can write

$$\frac{1}{1+t}\left(\frac{1}{1+t}\sigma+\frac{t}{1+t}(\alpha f+(1-\alpha)g)^*\right)^* \\ =\sigma-\frac{t}{1+t}\left(\frac{t}{1+t}\sigma+\frac{1}{1+t}(\alpha f+(1-\alpha)g)\right)^*.$$

But

$$\frac{t}{1+t}\sigma+\frac{1}{1+t}(\alpha f+(1-\alpha)g) \\ =\alpha\left(\frac{t}{1+t}\sigma+\frac{1}{1+t}f\right)+(1-\alpha)\left(\frac{t}{1+t}\sigma+\frac{1}{1+t}g\right)$$

so, by Lemma 2.1, we have

$$\left(\frac{t}{1+t}\sigma+\frac{1}{1+t}(\alpha f+(1-\alpha)g)\right)^* \\ \leq\alpha\left(\frac{t}{1+t}\sigma+\frac{1}{1+t}f\right)^*+(1-\alpha)\left(\frac{t}{1+t}\sigma+\frac{1}{1+t}g\right)^*$$

It follows that

$$\begin{aligned}
\sigma-\frac{t}{1+t}\left(\frac{t}{1+t}\sigma+\frac{1}{1+t}(\alpha f+(1-\alpha)g)\right)^{*}
&\geq \sigma-\alpha\frac{t}{1+t}\left(\frac{t}{1+t}\sigma+\frac{1}{1+t}f\right)^{*}-(1-\alpha)\frac{t}{1+t}\left(\frac{t}{1+t}\sigma+\frac{1}{1+t}g\right)^{*}\\
&\geq \alpha\left\{\sigma-\frac{t}{1+t}\left(\frac{t}{1+t}\sigma+\frac{1}{1+t}f\right)^{*}\right\}\\
&\qquad+(1-\alpha)\left\{\sigma-\frac{t}{1+t}\left(\frac{t}{1+t}\sigma+\frac{1}{1+t}g\right)^{*}\right\}.
\end{aligned}$$

Due to relation (2.4) again, we obtain

$$\begin{aligned}
&\frac{t^{p-1}}{1+t}\left(\frac{1}{1+t}\sigma+\frac{t}{1+t}\big(\alpha f+(1-\alpha)g\big)^{*}\right)^{*}\\
&\qquad\geq \alpha\left\{\frac{t^{p-1}}{1+t}\left(\frac{1}{1+t}\sigma+\frac{t}{1+t}f^{*}\right)^{*}\right\}+(1-\alpha)\left\{\frac{t^{p-1}}{1+t}\left(\frac{1}{1+t}\sigma+\frac{t}{1+t}g^{*}\right)^{*}\right\}
\end{aligned}$$

which yields

$$\big(\alpha f+(1-\alpha)g\big)^{(p)}\geq \alpha f^{(p)}+(1-\alpha)g^{(p)}.$$

The proof is complete. □

Open problem. *Is the map $f\to f^{(p)}$ concave on* Conv(H)*? The previous proof uses Theorem 2.1 which holds for $f\in\Gamma_\circ(H)$. We do not know if Theorem 2.1 is still true for $f\in$* Conv(H).

Proposition 3.2. *For all $f\in$* Conv(H)*, there holds*

$$((1-p)\sigma+pf^{*})^{*}\leq f^{(p)}\leq(1-p)\sigma+pf. \tag{3.4}$$

Proof. First, we show that $f^{(p)}\leq(1-p)\sigma+pf$. From the relation

$$f^{(p)}=\frac{\sin(p\pi)}{\pi}\int_0^{+\infty}\frac{t^{p-1}}{1+t}\left(\frac{1}{1+t}\sigma+\frac{t}{1+t}f^{*}\right)^{*}dt$$

we derive, with Lemma 2.1, that

$$f^{(p)}\leq\frac{\sin(p\pi)}{\pi}\int_0^{+\infty}\frac{t^{p-1}}{(1+t)^2}\sigma\,dt+\frac{\sin(p\pi)}{\pi}\int_0^{+\infty}\frac{t^{p}}{(1+t)^2}f\,dt.$$

Using (1.2), (3.3) and the equality

$$\frac{t^{p-1}}{(1+t)^2}=\frac{t^{p-1}}{1+t}-\frac{t^{p}}{(1+t)^2}$$

we deduce the desired result.

We now show that $((1-p)\sigma+pf^{*})^{*}\leq f^{(p)}$. By virtue of Lemma 2.2, we have

$$\begin{aligned}
(f^{(p)})^{*}&\leq\frac{\sin(p\pi)}{\pi}\int_0^{+\infty}\frac{t^{p-1}}{1+t}\left(\frac{1}{1+t}\sigma+\frac{t}{1+t}f^{*}\right)^{**}dt\\
&\leq\frac{\sin(p\pi)}{\pi}\left(\int_0^{+\infty}\frac{t^{p-1}}{(1+t)^2}dt\right)\sigma+\frac{\sin(p\pi)}{\pi}\left(\int_0^{+\infty}\frac{t^{p}}{(1+t)^2}dt\right)f^{*}.
\end{aligned}$$

Similarly to the previous inequality,

$$(f^{(p)})^* \leq (1-p)\sigma + pf^*$$

and so

$$((1-p)\sigma + pf^*)^* \leq (f^{(p)})^{**} \leq f^{(p)}.$$

This completes the proof. □

Corollary 3.1. *Let $f \in \mathrm{Conv}(H)$, then one has*

$$f^{(p)} = \sigma \Longleftrightarrow f = \sigma$$

and

$$f^{(p)} \geq \sigma \Longleftrightarrow f \geq \sigma \text{ (resp. } f^{(p)} \leq \sigma \Longleftrightarrow f \leq \sigma).$$

Proof. Follows immediately from Proposition 3.2. □

Remark 3.2. Relation (3.4) implies that $\mathrm{dom}(f) \subset \mathrm{dom}\left(f^{(p)}\right)$, so $\mathrm{dom}\left(f^{(p)}\right) = H$ if $\mathrm{dom}(f) = H$. It follows that the integrals of (3.1) and (3.2) converge, to a finite real number, for all $x \in \mathrm{dom}(f)$.

Now, we present an example explaining that the functional $f^{(p)}$ defined above is an extension of the operator case.

Example 3.1. Let A be a positive definite operator from H into H and define the function $f(x) = \frac{1}{2}\langle Ax, x\rangle$ for all $x \in H$. Using (2.5), a simple calculation gives

$$\forall x \in H, \quad f^{(p)}(x) = \frac{1}{2}\langle A^p x, x\rangle$$

where A^p is defined by (1.2). In particular, if $H = \mathbb{R}$ and $f(x) = \frac{1}{2}ax^2$ $(a > 0)$, we obtain $f^{(p)}(x) = \frac{1}{2}a^p x^2$.

Saying it another way, if we denote by f_A the quadratical function associated to a positive definite operator A we have $f_A^{(p)} = f_{A^p}$. Hence the above functional theory contains that of operators.

Given two positive definite operators A and B, Theorem 3.1 and Proposition 3.2 imply that :

i) If $A \geq B$ then $A^p \geq B^p$.

ii) For all $\alpha \in]0,1[$, $(\alpha A + (1-\alpha)B)^p \geq \alpha A^p + (1-\alpha)B^p$.

iii) $A\left((1-p)A + pI\right)^{-1} \leq A^p \leq (1-p)I + pA$.

3.2. Second order functional limited developments of $(\sigma + \lambda f)^{(p)}$. We begin this subsection by recalling the following two lemmas which will be used in the sequel.

Lemma 3.1 [23]. *Let $f \in \Gamma_\circ(H)$ such that* int(dom(f)) *is nonempty and $\lambda > 0$ sufficiently small. Then the following estimation*

$$(\sigma + \lambda f)^*(x) = \sigma(x) - \lambda f(x) + \lambda^2 \sigma(p_f(x)) + \lambda^2 \theta_\lambda(f)(x)$$

holds for all $x \in$ int(dom(f)) *where $\lim_{\lambda \to 0^+} \theta_\lambda(f) = 0$ in the pointwise convergence, and $p_f(x) = P_{\partial f(x)}(0)$ is the unique projection point from 0 to the nonempty closed convex set $\partial f(x)$.*

Lemma 3.2 [24]. *For a fixed* $f \in \Gamma_\circ(H)$ *such that* $\mathrm{int}(\mathrm{dom}(f)) \neq \emptyset$ *and* $x \in \mathrm{int}(\mathrm{dom}(f))$, *let us put*

$$\forall \lambda > 0,\ \forall t \in]0,1[\,, \quad \phi_\lambda(t)(x) = \frac{f(x) - (\lambda t \sigma + f^*)^*(x)}{\lambda t}.$$

Then $(\phi_\lambda)_\lambda$ *converges uniformly, with respect to t, to* $\sigma(p_f)$ *when* λ *tends to* 0^+:

$$\forall x \in \mathrm{int}(\mathrm{dom}(f)), \quad \lim_{\lambda \to 0^+} \sup_{t \in]0,1[} \left| \phi_\lambda(t)(x) - \sigma(p_f(x)) \right| = 0.$$

By using the above lemmas, we wish to establish the next theorems.

Theorem 3.2. *Let* $f \in \Gamma_\circ(H)$ *such that* $\mathrm{int}(\mathrm{dom}(f))$ *is nonempty and* $\lambda > 0$ *sufficiently small. The second order functional limited development*

$$(\sigma + \lambda f)^{(p)}(x) = \sigma(x) + p\lambda f(x) + \frac{p(p-1)}{2}\lambda^2 \sigma(p_f(x)) + \lambda^2 \theta_\lambda(f)(x)$$

holds for every $x \in \mathrm{int}(\mathrm{dom}(f))$, *where* $\lim_{\lambda \to 0^+} \theta_\lambda(f) = 0$ *for the pointwise convergence, and* $p_f(x) = P_{\partial f(x)}(0)$ *is the unique projection point from 0 to the nonempty closed convex set* $\partial f(x)$*.*

Proof. To simplify the writing, we omit the $x \in \mathrm{int}(\mathrm{dom}(f))$. Using the definition of $f^{(p)}$ given by (3.2) one has

$$(\sigma + \lambda f)^{(p)} = \frac{2}{\pi}\sin(p\pi) \int_0^{\frac{\pi}{2}} (\tan t)^{2p-1} \left((\cos^2 t)\sigma + (\sin^2 t)(\sigma + \lambda f)^* \right)^* dt.$$

Relation (2.3) gives

$$(\cos^2 t)\left((\cos^2 t)\sigma + (\sin^2 t) f^*\right)^* + (\sin^2 t)\left((\sin^2 t)\sigma + (\cos^2 t) f\right)^* = \sigma; \tag{3.5}$$

hence

$$\begin{aligned}\left((\cos^2 t)\sigma + (\sin^2 t)(\sigma + \lambda f)^*\right)^* &= \frac{\sigma}{\cos^2 t} - (\tan^2 t)\left((\sin^2 t)\sigma + (\cos^2 t)(\sigma + \lambda f)\right)^* \\ &= \frac{\sigma}{\cos^2 t} - (\tan^2 t)\left(\sigma + (\cos^2 t)\lambda f\right)^*\end{aligned}$$

and therefore

$$(\sigma + \lambda f)^{(p)} = \frac{2}{\pi}\sin(p\pi) \int_0^{\frac{\pi}{2}} (\tan t)^{2p-1} \left[\frac{1}{\cos^2 t}\sigma - (\tan^2 t)\left(\sigma + (\cos^2 t)\lambda f\right)^* \right] dt.$$

By Lemma 3.1, we have

$$\left(\sigma + (\cos^2 t)\lambda f\right)^* = \sigma - (\cos^2 t)\lambda f + (\cos^4 t)\lambda^2 \sigma(p_f) + (\cos^4 t)\lambda^2 \theta_\lambda^t(f), \tag{3.6}$$

with $\theta_\lambda^t(f) \to 0$ when $\lambda \to 0^+$. It follows that

$$\begin{aligned}&\frac{1}{\cos^2 t}\sigma - (\tan^2 t)\left(\sigma + (\cos^2 t)\lambda f\right)^* \\ &\qquad = \frac{1}{\cos^2 t}\sigma - (\tan^2 t)(\sigma - (\cos^2 t)\lambda f + (\cos^4 t)\lambda^2 \sigma(p_f) + (\cos^4 t)\lambda^2 \theta_\lambda^t(f)) \\ &\qquad = (\sigma + (\sin^2 t)\lambda f - (\sin^2 t \cos^2 t)\lambda^2 \sigma(p_f) - (\sin^2 t \cos^2 t)\lambda^2 \theta_\lambda^t(f)\end{aligned}$$

which yields

$$
\begin{aligned}
(\sigma+\lambda f)^{(p)} &= \frac{2}{\pi}\sin(p\pi)\left(\int_0^{\frac{\pi}{2}}(\tan t)^{2p-1}dt\right)\sigma \\
&\quad+\frac{2}{\pi}\sin(p\pi)\left(\int_0^{\frac{\pi}{2}}(\tan t)^{2p-1}(\sin^2 t)dt\right)\lambda f \\
&\quad-\frac{2}{\pi}\sin(p\pi)\left(\int_0^{\frac{\pi}{2}}(\tan t)^{2p-1}(\sin^2 t\cos^2 t)dt\right)\lambda^2\sigma(p_f) \\
&\quad-\frac{2}{\pi}\sin(p\pi)\left(\int_0^{\frac{\pi}{2}}(\tan t)^{2p-1}(\sin^2 t\cos^2 t)\theta_\lambda^t(f)dt\right)\lambda^2
\end{aligned}
$$

Using the change of variable $s=(\tan t)^2$ and (1.2), (3.3), we can verify that

$$\frac{2}{\pi}\sin(p\pi)\int_0^{\frac{\pi}{2}}(\tan t)^{2p-1}dt=1 \tag{3.7}$$

$$\frac{2}{\pi}\sin(p\pi)\int_0^{\frac{\pi}{2}}(\tan t)^{2p-1}(\sin^2 t)dt=p \tag{3.8}$$

$$\frac{2}{\pi}\sin(p\pi)\int_0^{\frac{\pi}{2}}(\tan t)^{2p-1}(\sin^2 t\cos^2 t)dt=\frac{p(1-p)}{2}.$$

Then we deduce that

$$(\sigma+\lambda f)^{(p)}=\sigma+p\lambda f+\frac{p(p-1)}{2}\lambda^2\sigma(p_f)+\lambda^2\theta_\lambda(f)$$

where

$$\theta_\lambda(f)=-\frac{2}{\pi}\sin(p\pi)\int_0^{\frac{\pi}{2}}(\tan t)^{2p-1}(\sin^2 t\cos^2 t)\theta_\lambda^t(f)dt. \tag{3.9}$$

Formula (3.6) and Theorem 2.1 give

$$\theta_\lambda^t(f)=\frac{f-\left((\cos^2 t)\lambda\sigma+f^*\right)^*}{(\cos^2 t)\lambda}-\sigma(p_f)$$

and therefore, by Lemma 3.2, $\theta_\lambda^t(f)\to 0$, uniformly in $t\in\,]0,\pi/2[$, as $\lambda\to 0^+$. This fact, combined with (3.9), yields $\lim_{\lambda\to 0^+}\theta_\lambda(f)=0$ for the pointwise convergence. The proof is complete. □

Theorem 3.3. *With the same hypothesis as in Theorem 3.2, we have the functional limited development*

$$(\sigma+\lambda f)^{*(p)}(x)=\sigma(x)-p\lambda f(x)+\frac{p(p+1)}{2}\lambda^2\sigma(p_f(x))+\lambda^2\theta_\lambda(f)(x)$$

for all $x\in\operatorname{int}(\operatorname{dom}(f))$, *where* $\lim_{\lambda\to 0^+}\theta_\lambda(f)=0$ *in the pointwise convergence, and* $p_f(x)=P_{\partial f(x)}(0)$ *is the unique projection point from 0 to the nonempty closed convex* $\partial f(x)$.

Proof. According to (3.2), we have

$$\begin{aligned}(\sigma+\lambda f)^{*(p)} &= \frac{2}{\pi}\sin(p\pi)\int_0^{\frac{\pi}{2}}(\tan t)^{2p-1}\left((\cos^2 t)\sigma+(\sin^2 t)((\sigma+\lambda f)^*)^*\right)^* dt\\ &= \frac{2}{\pi}\sin(p\pi)\int_0^{\frac{\pi}{2}}(\tan t)^{2p-1}\left((\cos^2 t)\sigma+(\sin^2 t)(\sigma+\lambda f)\right)^* dt\\ &= \frac{2}{\pi}\sin(p\pi)\int_0^{\frac{\pi}{2}}(\tan t)^{2p-1}\left((\cos^2 t+\sin^2 t)\sigma+(\sin^2 t)\lambda f\right)^* dt\\ &= \frac{2}{\pi}\sin(p\pi)\int_0^{\frac{\pi}{2}}(\tan t)^{2p-1}\left(\sigma+(\sin^2 t)\lambda f\right)^* dt.\end{aligned}$$

Thanks to Lemma 3.1, we find

$$\left(\sigma+(\sin^2 t)\lambda f\right)^*=\sigma-(\sin^2 t)\lambda f+(\sin^4 t)\lambda^2\sigma(p_f)+(\sin^4 t)\lambda^2\theta_\lambda^t(f)$$

where $\theta_\lambda^t(f)\to 0$ when λ goes to 0^+. It follows that

$$\begin{aligned}(\sigma+\lambda f)^{*(p)}=&\frac{2}{\pi}\sin(p\pi)\left(\int_0^{\frac{\pi}{2}}(\tan t)^{2p-1}dt\right)\sigma\\ &-\frac{2}{\pi}\sin(p\pi)\left(\int_0^{\frac{\pi}{2}}(\tan t)^{2p-1}(\sin^2 t)dt\right)\lambda f\\ &+\frac{2}{\pi}\sin(p\pi)\left(\int_0^{\frac{\pi}{2}}(\tan t)^{2p-1}(\sin^4 t)dt\right)\lambda^2\sigma(p_f)\\ &+\frac{2}{\pi}\sin(p\pi)\left(\int_0^{\frac{\pi}{2}}(\tan t)^{2p-1}(\sin^4 t)\theta_\lambda^t(f)dt\right)\lambda^2.\end{aligned}$$

For the third integral, we have

$$\begin{aligned}&\frac{2}{\pi}\sin(p\pi)\int_0^{\frac{\pi}{2}}(\tan t)^{2p-1}(\sin^4 t)dt\\ &\qquad=\frac{2}{\pi}\sin(p\pi)\int_0^{\frac{\pi}{2}}(\tan t)^{2p-1}(\sin^2 t-\sin^2 t\cos^2 t)dt\\ &\qquad=p-\frac{p(1-p)}{2}=\frac{p(p+1)}{2}\end{aligned}$$

which, combined with (3.7) and (3.8), gives

$$(\sigma+\lambda f)^{*(p)}=\sigma-p\lambda f+\frac{p(p+1)}{2}\lambda^2\sigma(p_f)+\lambda^2\theta_\lambda(f)$$

where

$$\theta_\lambda(f)=\frac{2}{\pi}\sin(p\pi)\int_0^{\frac{\pi}{2}}(\tan t)^{2p-1}(\sin^4 t)\theta_\lambda^t(f)dt$$

and

$$\theta_\lambda^t(f)=\frac{f-\left((\sin^2 t)\lambda\sigma+f^*\right)^*}{(\sin^2 t)\lambda}-\sigma(p_f).$$

Now, applying Lemma 3.2, we get $\lim_{\lambda\to 0^+}\theta_\lambda(f)=0$ as in the previous proof. ☐

Example 3.2. 1. Let $H=\mathbb{R}$ and consider

$$f(x)=\frac{1}{2}ax^2 \quad (a>0)$$

Theorems 3.2 and 3.3, along with Example 3.1, give the classical results

$$\begin{aligned}(1+\lambda a)^p &= 1+p\lambda a+\frac{p(p-1)}{2}\lambda^2a^2+\lambda^2\theta_\lambda(a)\\ (1+\lambda a)^{-p} &= 1-p\lambda a+\frac{p(p+1)}{2}\lambda^2a^2+\lambda^2\theta_\lambda(a)\end{aligned}$$

with $\theta_\lambda(a)\to 0$ as $\lambda\to 0^+$.

2. More generally, let H be a Hilbert space and f be the function

$$f(x)=\frac{1}{2}\langle Ax,x\rangle$$

where A is a positive linear operator from H into H. In this case, after a simple computation, Theorems 3.2 and 3.3, with Example 3.1, yield

$$\begin{aligned}(I+\lambda A)^p &= I+p\lambda A+\frac{p(p-1)}{2}\lambda^2A^2+\lambda^2\theta_\lambda(A)\\ (I+\lambda A)^{-p} &= I-p\lambda A+\frac{p(p+1)}{2}\lambda^2A^2+\lambda^2\theta_\lambda(A)\end{aligned}$$

where $\theta_\lambda(A)\to 0$, in the strong operator convergence, when $\lambda\to 0^+$.

These examples justify again that the functional $f^{(p)}$ defined above is a reasonable analogue of A^p for convex functions.

3.3. Differentiability of $s\to(s\sigma+f)^{(p)}$. In this subsection, we study the differentiability of the maps $s\to(s\sigma+f)^{(p)}$ and $s\to(s\sigma+f)^{*(p)}$ computing explicitely their gradients.

Proposition 3.3. *Let $f\in\Gamma_\circ(H)$ be a given function such that* $\operatorname{int}(\operatorname{dom}(f))\neq\emptyset$. *For a fixed $x\in\operatorname{int}(\operatorname{dom}(f))$, the application $s\to(s\sigma+f)^{(p)}(x)$ from $\mathbb{R}_+^*:=\,]0,+\infty[$ into $\widetilde{\mathbb{R}}$ is differentiable and*

$$\begin{aligned}&\frac{d}{ds}(s\sigma+f)^{(p)}(x)\\ &\qquad=\frac{2}{\pi}\sin(p\pi)\int_0^{\frac{\pi}{2}}(\tan t)^{2p+1}(\sin^2 t)\sigma\left(\nabla\left((\sin^2 t)\sigma+(\cos^2 t)(s\sigma+f)\right)^*\right)(x)dt\end{aligned}$$

for all $s\in\mathbb{R}_+^*$.

Proof. For all $s\in\mathbb{R}_+^*$ and $\lambda\to 0^+$, we can write

$$((\lambda+s)\sigma+f)^{(p)}=\frac{2}{\pi}\sin(p\pi)\int_0^{\frac{\pi}{2}}(\tan t)^{2p+1}\left((\cos^2 t)\sigma+(\sin^2 t)((\lambda+s)\sigma+f)^*\right)^*dt$$

and according to (3.5), we have

$$
\begin{aligned}
\big((\cos^2 t)\sigma\ &+(\sin^2 t)((\lambda+s)\sigma+f)^*\big)^*\\
&=\frac{1}{\cos^2 t}\sigma-(\tan^2 t)\left((\sin^2 t)\sigma+(\cos^2 t)((\lambda+s)\sigma+f)\right)^*\\
&=\frac{1}{\cos^2 t}\sigma-(\tan^2 t)\left((\cos^2 t)\lambda\sigma+(\sin^2 t+(\cos^2 t)s)\sigma+(\cos^2 t)f\right)^*
\end{aligned}
$$

Define $\bar{f}:=(\sin^2 t+(\cos^2 t)s)\sigma+(\cos^2 t)f$. We observe that $\bar{f}\in\Gamma_\circ(H)$ and $\mathrm{dom}(\bar{f})=\mathrm{dom}(f)$. By Theorem 2.1 and Lemma 3.1, we obtain

$$
\begin{aligned}
\big((\cos^2 t)\sigma+(\sin^2 t)\ &((\lambda+s)\sigma+f)^*\big)^*\\
&=\frac{1}{\cos^2 t}\sigma-(\tan^2 t)\left(\bar{f}^*-(\cos^2 t)\lambda\sigma(p_{\bar{f}^*})-(\cos^2 t)\lambda\bar{\theta}^t_\lambda(f)\right)\\
&=\frac{1}{\cos^2 t}\sigma-(\tan^2 t)\left((\sin^2 t)\sigma+(\cos^2 t)(s\sigma+f)\right)^*\\
&\qquad+(\sin^2 t)\lambda\sigma(p_{\bar{f}^*})+(\sin^2 t)\lambda\bar{\theta}^t_\lambda(f)\\
&=\left((\cos^2 t)\sigma+(\sin^2 t)(s\sigma+f)^*\right)^*+(\sin^2 t)\lambda\sigma(p_{\bar{f}^*})+(\sin^2 t)\lambda\bar{\theta}^t_\lambda(f)
\end{aligned}
$$

It follows that

$$
((\lambda+s)\sigma+f)^{(p)}=(s\sigma+f)^{(p)}+\lambda\frac{2}{\pi}\sin(p\pi)\int_0^{\frac{\pi}{2}}(\tan t)^{2p+1}(\sin^2 t)\sigma(p_{\bar{f}^*})dt+\lambda\theta_\lambda(f)
$$

with

$$
p_{\bar{f}^*}=\nabla\left((\sin^2 t+(\cos^2 t)s)\sigma+(\cos^2 t)f\right)^*
$$

$$
\theta_\lambda(f)=\frac{2}{\pi}\sin(p\pi)\int_0^{\frac{\pi}{2}}(\tan t)^{2p+1}(\sin^2 t)\bar{\theta}^t_\lambda(f)dt
$$

and where, by Lemma 3.1 and (2.3),

$$
\bar{\theta}^t_\lambda(f)=\frac{\bar{f}^*-\left((\cos^2 t)\lambda\sigma+\bar{f}\right)^*}{(\cos^2 t)\lambda}-\sigma(p_{\bar{f}^*}).
$$

Consequently, we have

$$
\frac{((\lambda+s)\sigma+f)^{(p)}-(s\sigma+f)^{(p)}}{\lambda}=\frac{2}{\pi}\sin(p\pi)\int_0^{\frac{\pi}{2}}(\tan t)^{2p+1}(\sin^2 t)\sigma(p_{\bar{f}^*})dt+\theta_\lambda(f)
$$

By Lemma 3.2 and similarly to the proof of Theorem 3.2, $\lim_{\lambda\to 0^+}\theta_\lambda(f)=0$ and this gives the desired result. □

Proposition 3.4. *Let* $f\in\Gamma_\circ(H)$ *be a given function such that* $\mathrm{int}(\mathrm{dom}(f))\neq\emptyset$. *For a fixed* $x\in\mathrm{int}(\mathrm{dom}(f))$, *the application* $s\to(s\sigma+f)^{*(p)}(x)$ *from* $\mathbb{R}^*_+$ *into* $\widetilde{\mathbb{R}}$ *is differentiable with gradient*

$$
\begin{aligned}
&\frac{d}{ds}(s\sigma+f)^{*(p)}(x)\\
&\quad=-\frac{2}{\pi}\sin(p\pi)\int_0^{\frac{\pi}{2}}(\tan t)^{2p+1}(\sin^2 t)\sigma\left(\nabla\left((\cos^2 t)\sigma+(\sin^2 t)(s\sigma+f)\right)^*\right)(x)dt
\end{aligned}
$$

for all $s \in \mathbb{R}_+^*$.

Proof. For all $s \in \mathbb{R}_+^*$ and $\lambda \to 0^+$, we can write

$$\begin{aligned}\big((\lambda+s)\sigma+f\big)^{*(p)} &= \frac{2}{\pi}\sin(p\pi)\int_0^{\frac{\pi}{2}}(\tan t)^{2p-1}\left((\cos^2 t)\sigma+(\sin^2 t)((\lambda+s)\sigma+f)\right)^* dt\\ &= \frac{2}{\pi}\sin(p\pi)\int_0^{\frac{\pi}{2}}(\tan t)^{2p-1}\left((\sin^2 t)\lambda\sigma+(\cos^2 t)\sigma+(\sin^2 t)(s\sigma+f)\right)^* dt\end{aligned}$$

Define $\tilde{f} := (\cos^2 t)\sigma + (\sin^2 t)(s\sigma+f)$. We observe that $\tilde{f} \in \Gamma_\circ(H)$ and $\mathrm{dom}(\tilde{f}) = \mathrm{dom}(f)$. By Theorem 2.1 and Lemma 3.1, we get

$$\begin{aligned}&\left((\sin^2 t)\lambda\sigma+(\cos^2 t)\sigma+(\sin^2 t)(s\sigma+f)\right)^*\\ &\qquad = \left((\cos^2 t)\sigma+(\sin^2 t)(s\sigma+f)\right)^* - \lambda\left((\sin^2 t)\sigma(p_{\tilde{f}^*}) - (\sin^2 t)\tilde{\theta}_\lambda^t(f)\right)\end{aligned}$$

which yields

$$\begin{aligned}&\big((\lambda+s)\sigma+f\big)^{*(p)}\\ &\qquad = (s\sigma+f)^{*(p)} - \lambda\frac{2}{\pi}\sin(p\pi)\int_0^{\frac{\pi}{2}}(\tan t)^{2p-1}(\sin^2 t)\sigma(p_{\tilde{f}^*})dt + \lambda\theta_\lambda(f)\end{aligned}$$

with

$$p_{\tilde{f}^*} = \nabla\left((\cos^2 t)\sigma+(\sin^2 t)(s\sigma+f)\right)^*$$

$$\theta_\lambda(f) = -\frac{2}{\pi}\sin(p\pi)\int_0^{\frac{\pi}{2}}(\tan t)^{2p+1}(\sin^2 t)\tilde{\theta}_\lambda^t(f)dt$$

and

$$\tilde{\theta}_\lambda^t(f) = \frac{\tilde{f}^* - \left((\sin^2 t)\lambda\sigma+\tilde{f}\right)^*}{(\sin^2 t)\lambda} - \sigma(p_{\tilde{f}^*}).$$

Lemma 3.2 implies that $\lim_{\lambda\to 0^+}\theta_\lambda(f) = 0$ which yields the desired result. □

Example 3.3. Let $A : H \to H$ be a positive operator and define

$$f(x) = \frac{1}{2}\langle Ax, x\rangle,\ x \in H.$$

Propositions 3.3 and 3.4 yield the classical results:

$$\forall s \in \mathbb{R}_+^*,\quad \frac{d}{ds}(sI+A)^p = p(sI+A)^{p-1}$$

$$\forall s \in \mathbb{R}_+^*,\quad \frac{d}{ds}(sI+A)^{-p} = -p(sI+A)^{-p-1}$$

and again the above functional theory contains that of operators.

4. Convex $(p, 1-p)$ **functional mean.** In this paragraph, we shall give an extension of $a^p b^{1-p}$ ($0 < p < 1$ and a, b two positive real numbers) to convex functions from which we deduce that of positive definite operators. The connection between the operator mean deduced here and one introduced in [22] will be discussed.

Definition 4.1. Let $f,g \in \mathrm{Conv}(H)$, $0<p<1$, and define

$$G_p(f,g)=\frac{\sin(p\pi)}{\pi}\int_0^{+\infty}\frac{t^{p-1}}{1+t}\left(\frac{t}{1+t}f^*+\frac{1}{1+t}g^*\right)^* dt. \tag{4.1}$$

$G_p(f,g)$ is called the $(p,1-p)$ *functional mean* of f and g.

Putting $s=(\tan t)^2$, it is easy to verify that

$$G_p(f,g)=\frac{2}{\pi}\sin(p\pi)\int_0^{\frac{\pi}{2}}(\tan t)^{2p-1}\left((\sin^2 t)f^*+(\cos^2 t)g^*\right)^* dt\,. \tag{4.2}$$

From the definition of $G_p(f,g)$, we deduce immediately that $G_p(f,g)\in \mathrm{Conv}(H)$ for all $f,g\in Conv(H)$. The following proposition summarizes the elementary properties of $G_p(f,g)$.

Proposition 4.1. *For all $f,g\in \mathrm{Conv}(H)$, the following statements hold:*

1. $G_p(f,g)=G_{1-p}(g,f)$, $G_p(f,\sigma)=f^{(p)}$, $G_p(f,0)=-pf^*(0)$.
2. *For all $a,b\in\mathbb{R}$,* $G_p((f+a),(g+b))=G_p(f,g)+pa+(1-p)b$.
3. *For all $a,b\in\mathbb{R}_+^*$,*
$$G_p(af,bg)=\sqrt{ab}\,G_p\left(\sqrt{\frac{a}{b}}f,\sqrt{\frac{b}{a}}g\right),$$
in particular $G_p(af,ag)=a\,G_p(f,g)$.
4. *For all $f_1,f_2,g_1,g_2\in \mathrm{Conv}(H)$,*
$$f_1\ge f_2 \text{ and } g_1\ge g_2 \Rightarrow G_p(f_1,g_1)\ge G_p(f_2,g_2)\,.$$

Proof. 1. These follow immediately from the definition of $G_p(f,g)$.

2. By definition, relation (4.1) gives

$$\begin{aligned}G_p(f+a,g+b)&=\frac{\sin(p\pi)}{\pi}\int_0^{+\infty}\frac{t^{p-1}}{t+1}\left(\frac{t}{1+t}(f+a)^*+\frac{1}{1+t}(g+b)^*\right)^* dt\\&=\frac{\sin(p\pi)}{\pi}\int_0^{+\infty}\frac{t^{p-1}}{t+1}\left(\frac{t}{1+t}(f^*-a)+\frac{1}{1+t}(g^*-b)\right)^* dt\\&=\frac{\sin(p\pi)}{\pi}\int_0^{+\infty}\frac{t^{p-1}}{t+1}\left[\left(\frac{t}{1+t}f^*+\frac{1}{1+t}g^*\right)^*+a\frac{t}{1+t}+b\frac{1}{1+t}\right] dt\\&=G_p(f,g)+\frac{\sin(p\pi)}{\pi}\int_0^{+\infty}\frac{at^p}{(1+t)^2}dt+\frac{\sin(p\pi)}{\pi}\int_0^{+\infty}\frac{bt^{p-1}}{(1+t)^2}dt\end{aligned}$$

and by virtue of (3.7), (3.8), we obtain $G_p(f+a,g+b)=G_p(f,g)+pa+(1-p)b$.

3. We have successively

$$\begin{aligned}G_p(af,bg) &= \frac{\sin(p\pi)}{\pi}\int_0^{+\infty}\frac{t^{p-1}}{t+1}\left(\frac{t}{1+t}(af)^* + \frac{1}{1+t}(bg)^*\right)^* dt\\ &= \frac{\sin(p\pi)}{\pi}\int_0^{+\infty}\frac{t^{p-1}}{t+1}\left(\sqrt{ab}\frac{t}{1+t}(\frac{\sqrt{a}}{\sqrt{b}}f)^*(\frac{\cdot}{\sqrt{ab}})\right.\\ &\qquad\qquad\left. + \sqrt{ab}\frac{1}{1+t}(\frac{\sqrt{b}}{\sqrt{a}}g)^*(\frac{\cdot}{\sqrt{ab}})\right)^* dt\\ &= \frac{\sin(p\pi)}{\pi}\int_0^{+\infty}\frac{t^{p-1}}{t+1}\sqrt{ab}\left(\frac{t}{1+t}(\frac{\sqrt{a}}{\sqrt{b}}f)^* + \frac{1}{1+t}(\frac{\sqrt{b}}{\sqrt{a}}g)^*\right)^* dt\\ &= \sqrt{ab}\, G_p\left(\sqrt{\frac{a}{b}}f, \sqrt{\frac{b}{a}}g\right).\end{aligned}$$

4. Analogous to the proof of Theorem 3.1, 1. □

Theorem 4.1. *For all $f,g \in \mathrm{Conv}(H)$, the following relationship holds:*

$$(pf^* + (1-p)g^*)^* \le G_p(f,g) \le pf + (1-p)g. \tag{4.3}$$

Proof. Similar to the proof of Proposition 3.2 by using Lemma 2.2. □

Example 4.1. Let $f(x) = \frac{1}{2}\langle Ax,x\rangle$ and $g(x) = \frac{1}{2}\langle Bx,x\rangle$ where A and B are two positive definite linear operators defined from H into H. It is easy to verify that

$$\forall x \in H, \quad G_p(f,g)(x) = \frac{1}{2}\big\langle\, (G_p(A,B))x, x\,\big\rangle$$

where

$$\begin{aligned}G_p(A,B) &= \frac{\sin(p\pi)}{\pi}\int_0^{+\infty} t^{p-1}\left(tA^{-1} + B^{-1}\right)^{-1} dt\\ &= \frac{\sin(p\pi)}{\pi}\int_0^{+\infty} t^{p-1}A\,(A+tB)^{-1}B\,dt.\end{aligned} \tag{4.4}$$

Proposition 4.2. *The explicit form of $G_p(A,B)$ is given by the formula*

$$G_p(A,B) = B^{\frac{1}{2}}(B^{-\frac{1}{2}}AB^{-\frac{1}{2}})^p B^{\frac{1}{2}}. \tag{4.5}$$

$G_p(A,B)$ is called the $(p,1-p)$ operator mean of A and B.

Proof. Relation (4.4) can be written as

$$G_p(A,B) = \frac{\sin(p\pi)}{\pi}\int_0^{+\infty} t^{p-1}B^{\frac{1}{2}}\left(I + tB^{\frac{1}{2}}A^{-1}B^{\frac{1}{2}}\right)^{-1}B^{\frac{1}{2}}dt,$$

and by (1.2) we obtain the desired result. □

Remark 4.1. $G_p(A,B)$ is an extension of $a^p b^{1-p}$ from positive real numbers to positive definite operators: (4.3) is an analogue of Young's inequality, $a^p b^{1-p} \leq pa + (1-p)b$, for convex functions. For positive definite operators, we obtain:

$$A\left((1-p)A + pB\right)^{-1} B \leq G_p(A,B) \leq pA + (1-p)B\,.$$

The operator mean $G_p(A,B)$ is generally different from $\exp(p\,\mathrm{Log}A + (1-p)\,\mathrm{Log}B)$, introduced in [22]. In fact, for $p = 1/2$, $G_{1/2}(A,B)$ coincides with $B^{\frac{1}{2}}(B^{-\frac{1}{2}}AB^{-\frac{1}{2}})^{\frac{1}{2}}B^{\frac{1}{2}}$ and it is already established

$$B^{\frac{1}{2}}(B^{-\frac{1}{2}}AB^{-\frac{1}{2}})^{\frac{1}{2}}B^{\frac{1}{2}} \neq \exp\left(\tfrac{1}{2}\,\mathrm{Log}A + \tfrac{1}{2}\,\mathrm{Log}B\right)$$

(see [22]). For $p \neq 1/2$, the same result works by the analogous arguments as in [22].

In addition, the explicit expression (4.5) is not practical to compute $G_p(A,B)$ even in the case $p = 1/n$ ($n \geq 2$ integer) and A, B are two positive definite matrices. For $n = 2$, an expansion of $G_{1/2}(A,B)$ from a convergent process of continued fraction development with matrices arguments can be found in [26]. For the general case, $G_p(A,B)$ can be characterized by the following result.

Theorem 4.2. *Let A and B be two positive definite operators and define the algorithm*

$$\begin{cases} Z_0 &= pA + (1-p)B, \\ Z_{k+1} &= (1-p)Z_k + pA(Z_k^{-1}B)^{\frac{1}{p}-1}, \quad k \geq 0. \end{cases}$$

Then the sequence $(Z_k)_k$ *converges, in the operator norm, to* $G_p(A,B)$.

Proof. Let $a > 0$ be a real number and consider the sequence

$$\begin{cases} x_0 &= pa + (1-p), \\ x_{k+1} &= (1-p)x_k + pa(x_k)^{1-\frac{1}{p}}, \quad k \geq 0. \end{cases}$$

Using the concavity of $x \mapsto \mathrm{Log}\,x$ ($x > 0$), it is not hard to prove that the real sequence $(x_k)_k$ converges to a^p. We deduce that the following algorithm

$$\begin{cases} X_0 &= pA + (1-p)I, \\ X_{k+1} &= (1-p)X_k + pA(X_k)^{1-\frac{1}{p}}, \quad k \geq 0, \end{cases}$$

converges, in the operator norm, to A^p. It follows that the iterative scheme

$$\begin{cases} Y_0 &= pB^{-\frac{1}{2}}AB^{-\frac{1}{2}} + (1-p)I, \\ Y_{k+1} &= (1-p)Y_k + pB^{-\frac{1}{2}}AB^{-\frac{1}{2}}(Y_k)^{1-\frac{1}{p}}, \quad k \geq 0 \end{cases}$$

converges uniformly to $(B^{-\frac{1}{2}}AB^{-\frac{1}{2}})^p$. Letting $Z_k = B^{\frac{1}{2}}Y_kB^{\frac{1}{2}}$ and using a simple manipulation, we deduce the desired result. □

Theorem 4.2 gives a practical algorithm for computation of $G_{1/n}(A,B)$ ($n \geq 2$ integer) when A and B are two given positive definite matrices.

5. Geometric mean of m convex functions. Let $f_1, f_2, \dots, f_m$ be m convex functions ($m \geq 2$ integer). The arithmetic and harmonic functional means of $f_1, f_2, \dots, f_m$, extending that of numbers and operators, can be respectively defined by

$$\mathscr{A}_m(f_1, f_2, \dots, f_m) = \sum_{i=1}^{m} \frac{1}{m} f_i, \tag{5.1}$$

$$\mathscr{H}_m(f_1, f_2, \dots, f_m) = \left(\sum_{i=1}^{m} \frac{1}{m} f_i^* \right)^*. \tag{5.2}$$

In the particular case $m = 2$, these means have been introduced and used in [4, 10, 25].

Our purpose in this section is to define the geometric mean of m convex functions in order to extend that of positive numbers and operators. The key idea for this extension comes from the fact that the previous means $\mathscr{A}_m$ and $\mathscr{H}_m$ can be written as follows

$$\mathscr{A}_m(f_1, f_2, \dots, f_m) = \frac{1}{m} f_1 + \frac{m-1}{m} \sum_{i=2}^{m} \frac{1}{m-1} f_i = \frac{1}{m} f_1 + (1 - \frac{1}{m}) \mathscr{A}_{m-1}(f_2, f_3, \dots, f_m),$$

$$\mathscr{H}_m(f_1, f_2, \dots, f_m) = \left(\frac{1}{m} f_1^* + (1 - \frac{1}{m}) (\mathscr{H}_{m-1}(f_2, f_3, \dots, f_m))^* \right)^*.$$

Moreover, for the scalar case, the geometric mean of m positive real numbers $a_1, \dots, a_m$ satisfies an analogous relation as $\mathscr{A}_m$ and $\mathscr{H}_m$, that is

$$\mathscr{G}_m(a_1, a_2, \dots, a_m) := \Big(\prod_{i=1}^{m} a_i \Big)^{1/m} = a_1^{1/m} \big(\mathscr{G}_{m-1}(a_2, a_3, \dots, a_m) \big)^{1-1/m}.$$

With this, given $f_i \in \mathrm{Conv}(H)$, $1 \leq i \leq m$, we put $G_1(f_1) = f_1$ and we define by induction

$$\mathscr{G}_m(f_1, f_2, \dots, f_m) = G_{1/m}\big(f_1, \mathscr{G}_{m-1}(f_2, f_3, \dots, f_m) \big), \tag{5.3}$$

where $G_{1/m}$ is defined by Definition 4.1 with $p = 1/m$.

Definition 5.1. For every $f_1, f_2, \dots, f_m \in \mathrm{Conv}(H)$, $\mathscr{G}_m(f_1, f_2, \dots, f_m)$ is called the *geometric functional mean* of $f_1, f_2, \dots, f_m$.

For $m = 2$, it is clear to see that $\mathscr{G}_2(f_1, f_2) = G_{1/2}(f_1, f_2)$, i.e. the geometric functional mean of f_1 and f_2, in the sense of Definition 5.1, coincides with their $(1/2, 1/2)$ functional mean in the sense of Definition 4.1. Further, by an induction on $m \in \mathbb{N}$ the properties of $\mathscr{G}_m$ can be deduced from that of $G_{1/m}$. In particular, we state the following proposition whose the proof is omitted.

Proposition 5.1. *For all* $f_1, f_2, \dots, f_m \in \mathrm{Conv}(H)$, *there holds*

$$\mathscr{H}_m(f_1, f_2, \dots, f_m) \leq \mathscr{G}_m(f_1, f_2, \dots, f_m) \leq \mathscr{A}_m(f_1, f_2, \dots, f_m), \tag{5.4}$$

where $\mathscr{A}_m$ *and* $\mathscr{H}_m$ *are respectively given by* (5.1) *and* (5.2).

Now, we study the case when f_i, $1 \leq i \leq m$, are quadratical functions. Let A_i, $1 \leq i \leq m$, be m positive definite operators defined from H into H and set

$$\forall x \in H, \quad f_i(x) = \frac{1}{2} \langle A_i x, x \rangle.$$

In this case, it is easy to establish that

$$\forall x\in H,\quad \big(\mathscr{G}_m(f_1,f_2,\ldots,f_m)\big)(x)=\frac{1}{2}\big\langle\big(\mathscr{G}_m(A_1,A_2,\ldots,A_m)\big)x,x\big\rangle,$$

with $\mathscr{G}_1(A)=A$ for all (positive) operator A and

$$\mathscr{G}_m(A_1,A_2,\ldots,A_m)=G_{1/m}\big(A_1,\mathscr{G}_{m-1}(A_2,A_3,\ldots,A_m)\big),\tag{5.5}$$

where $G_{1/m}$ is defined by (4.4) or (4.5), with $p=1/m$.

The analogue of (5.4) for operator is

$$\mathscr{H}_m(A_1,A_2,\ldots,A_m)\le\mathscr{G}_m(A_1,A_2,\ldots,A_m)\le\mathscr{A}_m(A_1,A_2,\ldots,A_m),\tag{5.6}$$

where the two extreme members

$$\mathscr{H}_m(A_1,A_2,\ldots,A_m)=\Big(\sum_{i=1}^{m}\frac{1}{m}A_i^{-1}\Big)^{-1}$$

and

$$\mathscr{A}_m(A_1,A_2,\ldots,A_m)=\sum_{i=1}^{m}\frac{1}{m}A_i$$

are respectively the harmonic and arithmetic means of $A_1,A_2,\ldots,A_m$. From (5.6), it is obvious that $\mathscr{G}_m(A_1,A_2,\ldots,A_m)$ is a positive definite operator.

Definition 5.2. Let $A_1,A_2,\ldots,A_m$, be m positive definite operators; $\mathscr{G}_m(A_1,A_2,\ldots,A_m)$ is called the *geometric operator mean* of $A_1,A_2,\ldots,A_m$.

As seen in the previous section, our geometric operator mean $\mathscr{G}_m(A_1,A_2,\ldots,A_m)$ is generally different from the operator

$$\exp\left(\frac{1}{m}\sum_{i=1}^{m}\operatorname{Log}A_i\right)$$

introduced in [22]. Indeed, for $m=2$,

$$\mathscr{G}_2(A_1,A_2):=G_{1/2}(A_1,A_2)=A_2^{\frac12}\left(A_2^{-\frac12}A_1A_2^{-\frac12}\right)^{\frac12}A_2^{\frac12}$$

which, as pointed out in [22], is different from $\exp\left(\frac12\operatorname{Log}A_1+\frac12\operatorname{Log}A_2\right)$.

In addition, by an induction on $m\ge 2$ in (5.5), the algorithm of Theorem 4.1 can be used to compute $\mathscr{G}_m(A_1,A_2,\ldots,A_m)$ when $A_1,A_2,\ldots,A_m$ are m given positive definite matrices.

6. Arithmetico-geometrico-harmonic functional mean. Let a, b, c be three positive real numbers and define the following sequences

$$\left\{\begin{array}{lcl}a_0&=&a\\ b_0&=&b\\ c_0&=&c\end{array}\right.\quad\text{et}\quad\left\{\begin{array}{lcl}a_{n+1}&=&\frac{1}{3}(a_n+b_n+c_n)\\ b_{n+1}&=&\sqrt[3]{a_nb_nc_n}\\ \frac{3}{c_{n+1}}&=&\frac{1}{a_n}+\frac{1}{b_n}+\frac{1}{c_n}\end{array}\right.\qquad(n\ge 0).$$

It is well known [19] that the real sequences (a_n), (b_n) and (c_n) converge to a same limit called the arithmetico-geometrico-harmonic mean of a,b and c.

In this section, we will extend this scalar algorithm to convex functions and we deduce that of positive definite operators. We preserve the same notations as in the above paragraph.

Let $f,\ g,\ h \in \Gamma_\circ(H)$ such that

$$\mathrm{dom}(f) = \mathrm{dom}(g) = \mathrm{dom}(h) \tag{6.1}$$

and define the algorithm

$$\left\{\begin{array}{lcl} f_0 & = & f \\ g_0 & = & g \\ h_0 & = & h \end{array}\right. \quad \text{et} \quad \left\{\begin{array}{lcl} f_{n+1} & = & \mathscr{A}_3(f_n,g_n,h_n) \\ g_{n+1} & = & \mathscr{G}_3(f_n,g_n,h_n) \\ h_{n+1} & = & \mathscr{H}_3(f_n,g_n,h_n) \end{array}\right. \qquad (n \geq 0).$$

Theorem 6.1. *Let $f,\ g,\ h \in \Gamma_\circ(H)$ satisfying* (6.1). *The sequences (f_n), (g_n) and (h_n) converge pointwise to the same convex functional $\mathscr{AGH}(f,g,h)$, called the arithmetico-geometrico-harmonic mean of f, g and h, and verifying that*

$$\mathscr{H}_3(f,g,h) \leq \mathscr{AGH}(f,g,h) \leq \mathscr{A}_3(f,g,h)\,. \tag{6.2}$$

Proof. According to relation (5.4), we have immediately

$$\forall n \geq 0, \quad h_{n+1} \leq g_{n+1} \leq f_{n+1},$$

from which we deduce successively

$$f_{n+1} := \mathscr{A}_3(f_n,g_n,h_n) \leq \mathscr{A}_3(f_n,f_n,f_n) = f_n,$$

$$h_{n+1} := \mathscr{H}_3(f_n,g_n,h_n) \geq \mathscr{H}_3(h_n,h_n,h_n) = h_n^{**}.$$

By an induction on n, it is easy to see that $h_n \in \Gamma_\circ(H)$for all $n \in \mathbb{N}$, and so $h_n \leq h_{n+1}$ for every $n \geq 1$. Summarizing, we have shown

$$\mathscr{H}_3(f,g,h) = h_1 \leq \cdots \leq h_n \leq h_{n+1} \leq g_{n+1} \leq f_{n+1} \leq f_n \leq \cdots \leq f_1 = \mathscr{A}_3(f,g,h). \tag{6.3}$$

It follows that (f_n) (resp.(h_n)) is decreasing (resp. increasing) and lower bounded by h_1 (resp. upper bounded by f_1), then (f_n) and (h_n) both converge pointwise in $\widetilde{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$. Call their limits $\phi = \lim_n f_n$ and $\theta = \lim_n h_n$. The inequalities (6.3), combined with Lemma 2.3 and (6.1), yields

$$\begin{aligned} \mathrm{dom}\left(\mathscr{A}_3(f,g,h)\right) &= \mathrm{dom}(f_n) = \mathrm{dom}(g_n) = \mathrm{dom}(h_n) \\ &= \mathrm{dom}(\phi) = \mathrm{dom}(\theta) = \mathrm{dom}\left(\mathscr{H}_3(f,g,h)\right). \end{aligned} \tag{6.4}$$

Otherwise, relation $f_{n+1} := \mathscr{A}_3(f_n,g_n,h_n)$, which is equivalent to

$$\forall n \geq 0, \quad g_n = 3f_{n+1} - f_n - h_n, \tag{6.5}$$

implies that the sequence (g_n) converges pointwise in $\widetilde{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$. Putting $\psi = \lim_n g_n$ and using relations (6.4), (6.5), we obtain

$$\psi = 2\phi - \theta. \tag{6.6}$$

Thanks to (6.3) and (6.4), we deduce the following two formulas:

$$\theta \leq \psi \leq \phi \tag{6.7}$$

and

$$\mathrm{dom}(\phi) = \mathrm{dom}(\psi) = \mathrm{dom}(\theta). \tag{6.8}$$

Letting $n \to +\infty$ in the inequality $g_{n+1} \leq f_{n+1} := (f_n + g_n + h_n)/3$, we obtain $\psi \leq (\phi + \psi + \theta)/3$ and, by (6.8), we deduce that $2\psi \leq \phi + \theta$. Combining this last inequality with (6.6), we find $\phi \leq \theta$ which, by virtue of (6.7), yields that $\phi = \psi = \theta$. The proof is complete. □

Remark 6.1. Assumption (6.1) is not necessary for the pointwise convergence of f_n and h_n, but it is needed to have $\lim_n f_n = \lim_n h_n$. The next example explains this situation.

Example 6.1. Let $f = \Psi_E$, $g = \Psi_F$ and $h = \Psi_G$ where E, F, G are three closed subspaces of H and Ψ_E is the indicator function of E, i.e. $\Psi_E(x) = 0$ if $x \in E$ and $\Psi_E(x) = +\infty$ otherwise. Recall [27] that $f^* = \Psi_{E^\perp}$, $g^* = \Psi_{F^\perp}$ and $h^* = \Psi_{G^\perp}$ where $E^\perp$ is the orthogonal complement of E defined by: $E^\perp = \{x^* \in H : \langle x^*, x\rangle = 0 \text{ for all } x \in E\}$. By mathematical induction, we can easily verify that $f_n = \Psi_{E\cap F\cap G}$ and $h_n = \Psi_{(E^\perp \cap F^\perp \cap G^\perp)^\perp}$ for all $n \geq 1$. The sequences (f_n) and (h_n) converge pointwise but not to the same limit, since in general $E \cap F \cap G \neq (E^\perp \cap F^\perp G^\perp)^\perp$.

Take A, B, C three positive definite operators defined from H into H and define the following sequences

$$\begin{cases} A_0 &= A \\ B_0 &= B \\ C_0 &= C \end{cases} \quad \text{and} \quad \begin{cases} A_{n+1} &= \mathscr{A}_3(A_n, B_n, C_n) \\ B_{n+1} &= \mathscr{G}_3(A_n, B_n, C_n) \\ C_{n+1} &= \mathscr{H}_3(A_n, B_n, C_n) \end{cases} \qquad (n \geq 0).$$

The following corollary is a version of Theorem 6.1 for the quadratical case.

Corollary 6.1. *The sequences* (A_n)*,* (B_n) *and* (C_n) *converge, in the strong operator topology, to a same positive definite operator* $\mathscr{A}\mathscr{G}\mathscr{H}(A,B,C)$*, called the arithmetico-geometrico-harmonic mean of A, B and C, and satisfying that*

$$\mathscr{H}_3(A,B,C) \leq \mathscr{A}\mathscr{G}\mathscr{H}(A,B,C) \leq \mathscr{A}_3(A,B,C). \tag{6.9}$$

Proof. First, remark that for each $n \geq 0$, the operators A_n, B_n and C_n are positive definite. Let us put $f(x) = \frac{1}{2}\langle Ax, x\rangle$, $g(x) = \frac{1}{2}\langle Bx, x\rangle$ and $h(x) = \frac{1}{2}\langle Cx, x\rangle$ for every $x \in H$. By mathematical induction on $n \in \mathbb{N}$, it is not hard to see that for all $x \in H$, $f_n(x) = \frac{1}{2}\langle A_n x, x\rangle$, $g_n(x) = \frac{1}{2}\langle B_n x, x\rangle$ and $h_n(x) = \frac{1}{2}\langle C_n x, x\rangle$ where f_n, g_n and h_n (resp. A_n, B_n and C_n) were defined previously. According to Theorem 6.1, the real sequences $(\langle A_n x, x\rangle)_n$, $(\langle B_n x, x\rangle)_n$ and $(\langle C_n x, x\rangle)_n$ converge to a same limit which is trivially a positive quadratical function. By boundedness of sequences of positive operators, we deduce that the sequences (A_n), (B_n) and (C_n) converge strongly to the same positive definite operator. Relation (6.9) comes from (6.2) and the proof is complete. □

Remark 6.2. Let $f, g, h \in \Gamma_\circ(H)$ and put that

$$\Lambda(f,g,h) = (\mathscr{A}_3(f,g,h), \mathscr{G}_3(f,g,h), \mathscr{H}_3(f,g,h)).$$

If Λ^n denotes the n-th iterate of Λ, Theorem 6.1 can be summarized by the next version: there exists a convex functional $\mathscr{F} := \mathscr{A}\mathscr{G}\mathscr{H}(f,g,h)$ such that

$$\lim_{n\uparrow+\infty} \Lambda^n(f,g,h) = (\mathscr{F}, \mathscr{F}, \mathscr{F})$$

for the pointwise convergence. An analogous remark works for Corollary 6.1.

After proving the existence of the arithmetico-geometrico-harmonic functional mean, we will complete this section by a discussion about a notion called the "invariance property". In fact, in the case of standard (real) means more general result than that of [20] holds true. More precisely, we recall the following result.

Theorem 6.2. [20] *Let $I \subset \mathbb{R}$ be an interval and $k > 1$ a positive integer. Suppose that $M_1, M_2, \ldots, M_k : I^k \to I$ are means, that is to say,*

$$\min(x_1,\ldots,x_k) \leq M_i(x_1,\ldots,x_k) \leq \max(x_1,\ldots,x_k), \quad i = 1,\ldots,k$$

for all $x_1,\ldots,x_k \in I$. If the means $M_1,\ldots,M_k$ are continuous and at most one of them is not strict, then:

1. *there exists a continuous mean $K : I^k \to I$ such that the sequence of iterates of the mean-type mapping $(M_1, M_2, \ldots, M_k) : I^k \to I^k$ converges to a continuous mapping $(K, K, \ldots, K) : I^k \to I^k$,*
2. *K is invariant with respect to the mean-type mapping $(M_1, M_2, \ldots, M_k)$ which means that $K \circ (M_1, M_2, \ldots, M_k) = K$ (or K is the Gauss composition of the means $M_1, M_2, \ldots, M_k$),*
3. *a continuous mean which is invariant with respect to $(M_1, M_2, \ldots, M_k)$ is unique,*
4. *if $M_1, M_2, \ldots, M_k$ are strict then so is K.*

(A mean $M : I^k \to I$ is called strict if

$$\min(x_1,\ldots,x_k) \leq M(x_1,\ldots,x_k) \leq \max(x_1,\ldots,x_k)$$

for all $x_1,\ldots,x_k \in I$ such that $\min(x_1,\ldots,x_k) < \max(x_1,\ldots,x_k)$.)

Let $\mathscr{A}$, $\mathscr{G}$, $\mathscr{H}$ denote, respectively, the arithmetic, geometric and harmonic (real) mean. In the case $k = 2$, $M_1 = \mathscr{A}$, $M_2 = \mathscr{H}$ we hence get $K = \mathscr{G}$ and the invariance property reduces to the well known "harmony" identity

$$\mathscr{G}(\mathscr{A}(x,y), \mathscr{H}(x,y)) = \mathscr{G}(x,y), \quad \forall x,y > 0.$$

Applying the above theorem in the case $k = 3$, $M_1 = \mathscr{A}$, $M_2 = \mathscr{G}$, $M_3 = \mathscr{H}$ we obtain, as the invariant mean K, the standard arithmetico-geometrico-harmonic mean. Now a natural question arises wether the arithmetico-geometrico-harmonic functional mean $\mathscr{AGH}$ defined previously: what should be the analogue of Theorem 6.2 when the variables $x_1, x_2, \ldots, x_k$ are convex functions. In particular, the invariance property for the functional case is not obvious and appears to be interesting. For the moment, we are unable to give an answer for this latter point and we put the following conjecture.

Open problem. *Let $f, g, h \in \Gamma_\circ(H)$ satisfying* (6.1)*, is it true that*

$$\mathscr{AGH}(\mathscr{A}_3(f,g,h), \mathscr{G}_3(f,g,h), \mathscr{H}_3(f,g,h)) = \mathscr{AGH}(f,g,h)\,?$$

We conjecture that it is.

7. Discussion of the case $p = 1/2$**.** In [4], the authors introduced a convex geometric functional mean as follows: let $f, g \in \Gamma_\circ(H)$ such that $\mathrm{dom}(f) \cap \mathrm{dom}(g) \neq \emptyset$ and consider the algorithm:

$$\text{(AR)} \begin{cases} \alpha_0(f,g) &=& \frac{1}{2}f + \frac{1}{2}g, \\ \alpha_{n+1}(f,g) &=& \frac{1}{2}\alpha_n(f,g) + \frac{1}{2}(\alpha_n(f^*,g^*))^* \qquad (n \geq 0). \end{cases}$$

Theorem 7.1 [4]. *Let* $f,g\in\Gamma_\circ(H)$ *such that* $\operatorname{dom}(f)\cap\operatorname{dom}(g)\neq\emptyset$*. Then the sequence* $(\alpha_n(f,g))_n$ *converges pointwise to a convex function* $\tau(f,g)$ *called the convex geometric mean of* f *and* g*. In particular,* $\tau(f,\sigma):=f^{[1/2]}$ *is called the convex square root of* f*.*

In [4], the authors stated the definition of $\tau(f,g)$ together with its properties for f,g satisfying that

$$f,g\in\Gamma_\circ(H)\quad\text{and}\quad \operatorname{dom}\alpha_0(f,g)=\operatorname{dom}(\alpha_0(f^*,g^*))^*. \tag{7.1}$$

As pointed out in [25], the assumption (7.1) which, according to [10], is equivalent to $f,g\in\Gamma_\circ(H)$, $\operatorname{dom}(f)=\operatorname{dom}(g)$, is not necessary to define $\tau(f,g)$ as the pointwise limit of $(\alpha_n(f,g))_n$, but it is needed if we want the limits of the sequences $(\alpha_n(f,g))_n$ and $((\alpha_n(f^*,g^*))^*)_n$ to coincide and to have, as a consequence, the equality $\tau(f,g)=(\tau(f^*,g^*))^*$. See [25] for an example explaining this situation.

Using the algorithm (AR) and by the same arguments as in [4], Theorem 7.1 (and Proposition 7.1 below) is still true for all $f,g\in\operatorname{Conv}(H)$.

The elementary properties of $\tau(f,g)$ are summarized in the following.

Proposition 7.1 [4]. *Let* $f,g\in\operatorname{Conv}(H)$ *satisfying that* $\operatorname{dom}(f)\cap\operatorname{dom}(g)\neq\emptyset$*, then the following statements hold:*

1. $\tau(f,g)=\tau(g,f)$ *and* $\tau(f,0)=-f^*(0)/2$*.*
2. $\tau(f+a,g+b)=\tau(f,g)+(a+b)/2$ *for all* $a,b\in\mathbb{R}$*.*
3. *If* $f\geq f_1$ *and* $g\geq f_2$ *(with* $f_1,f_2\in\operatorname{Conv}(H)$*) then* $\tau(f,g)\geq\tau(f_1,f_2)$*.*
4. $\mathscr{H}_2(f,g)\leq\tau(f,g)\leq\mathscr{A}_2(f,g)$*.*
5. *If* $f=\frac{1}{2}\langle A\cdot,\cdot\rangle$ *and* $g=\frac{1}{2}\langle B\cdot,\cdot\rangle$*, where A and B are two positive definite operators, then* $\tau(f,g)=\frac{1}{2}\langle\sigma(A,B)\cdot,\cdot\rangle$*, where* $\sigma(A,B)=A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\frac{1}{2}}A^{\frac{1}{2}}$ *is the geometric operator mean of A and B.*

From Proposition 7.1, we see that the properties of $\tau(f,g)$ are similar to those of $G_{1/2}(f,g)$ discussed in Section 4 (*cf.* Proposition 4.1, Theorem 4.1, Proposition 4.2). Moreover, an analogue "invariance property" from scalar means to functional ones was proved in [4], that is

$$\tau(\mathscr{A}_2(f,g),\mathscr{H}_2(f,g))=\tau(f,g),$$

for all $f,g\in\Gamma_\circ(H)$ such that $\operatorname{dom}(f)=\operatorname{dom}(g)$. By virtue of these analogies, we put the following.

Open problem. *Let* $f,g\in\operatorname{Conv}(H)$ *such that* $\operatorname{dom}(f)=\operatorname{dom}(g)=H$*. Is it true that*

$$\tau(f,g)=G_{1/2}(f,g)? \tag{7.2}$$

We conjecture that it is.

Of course, the equality (7,2) is true when f and g are two quadratical functions. If the assumption $\operatorname{dom}(f)=\operatorname{dom}(g)=H$ is not satisfied then (7.2) does not hold anymore. We refer the reader to [11] for more information.

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Résumé substantiel en français. La théorie des « moyennes » scalaires a connu, au cours des dernières dizaines d'années, de multiples developpements et d'intéressantes applications [3, 7, 8, 20]. Entre temps, une extension de cette théorie au cas des opérateurs a connu une large étude présentée avec profondeur et élégance [1, 2, 4, 9, 12, 13, 14, 15, 18, 22]. Au cours des récentes années, la théorie des opérateurs « moyennes » a connu à son tour une extension au cas des fonctions convexes [4, 10, 11, 25]. L'idée originale de cette extension vient du fait que la notion de « Legendre-Fenchel conjugaison » d'une fonction convexe peut être, dans un certain sens, interprétée comme un inverse en analyse convexe. Le travail présenté dans ce papier s'inscrit dans ce cadre et developpe un ensemble de notions et de résultats suite à ceux déjà présentés dans la littérature. Plus précisément, étant donné un opérateur défini positif A et $0 < p < 1$. Rappelons que, [17], A^p est donné par la formule (1.2) dont une forme "convexe" équivalente est (1.3). Puisque A est inversible alors A^p est encore donné par (1.4). Le premier but de ce travail est de donner une extension de A^p pour les fonctions convexes. Nous suggérons qu'un analogue raisonnable de l'expression (1.4), pour le cas fonctionnel, est donné par (1.5). Quand $p = 1/n$, où $n \geq 2$ est un nombre entier, $A^{1/n}$ est la racine n^{e} de l'opérateur positif A (c.-à-d. le seul opérateur positif B tel que $B^n = A$) et par analogie, $f^{(1/n)}$ sera appelée la racine n^{e} convexe de f. Parmi les propriétés de $f^{(p)}$ que nous avons étudiées, nous citons les suivantes.

- Pour tous $f, g \in \mathrm{Conv}(H)$, si $f \geq g$ alors $f^{(p)} \geq g^{(p)}$.
- Pour tous $f, g \in \Gamma_\circ(H)$ et $\alpha \in]0,1[$, on a : $(\alpha f + (1-\alpha)g)^{(p)} \geq \alpha f^{(p)} + (1-\alpha)g^{(p)}$.
- Pour tout $f \in \mathrm{Conv}(H)$, on a : $((1-p)\sigma + pf^*)^* \leq f^{(p)} \leq (1-p)\sigma + pf$.

Outre les assertions précédentes, nous avons aussi prouvé le résultat suivant.

Théorème 3.2. *Soit $f \in \Gamma_\circ(H)$ telle que $\mathrm{int}(\mathrm{dom}(f))$ soit non vide et $\lambda > 0$ assez petit alors le développement limité fonctionnel à l'ordre 2 suivant*

$$(\sigma + \lambda f)^{(p)}(x) = \sigma(x) + p\lambda f(x) + \frac{p(p-1)}{2}\lambda^2 \sigma(p_f(x)) + \lambda^2 \theta_\lambda(f)(x),$$

a lieu pour tout $x \in \mathrm{int}(\mathrm{dom}(f))$, où $\lim_{\lambda \to 0^+} \theta_\lambda(f) = 0$ au sens de la convergence simple, et $p_f(x) = P_{\partial f(x)}(0)$ est l'unique point projection de 0 sur le convexe fermé non vide $\partial f(x)$.

Le second objet de ce travail est de répondre à la question suivante : quel est l'analogue de $a^p b^{1-p}$ (a, b deux nombres réels positifs et $0 < p < 1$) quand les variables a et b sont des fonctions convexes ? Notons qu'un analogue de $a^p b^{1-p}$ quand a et b sont des opérateurs définis positifs, a été largement discuté dans [22] : plus généralement, soit $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_m) \in \mathbb{R}^m$ un vecteur probabilité donné (c.-à-d. $\sigma_i \geq 0$, $1 \leq i \leq m$ et $\sum_{i=1}^m \sigma_i = 1$), les auteurs de [22] ont suggéré qu'un analogue raisonnable de $\prod_{i=1}^m a_i^{\sigma_i}$ ($a_i > 0$ nombres réels) est donné par (1.6), où les variables A_i, $1 \leq i \leq m$, sont des opérateurs définis positifs. Si nous tentons d'étendre la formule (1.6) pour les fonctions convexes, la difficulté principale réside dans le fait que l'extension de l'exponentielle d'un opérateur au cas fonctionnel n'est pas encore achevée. Pour cela, notre travail decrit une approche, sous un angle fonctionnel, de l'extension de $a^p b^{1-p}$ du cas réel au cas fonctionnel. Remarquant que $a^p b^{1-p}$ étend a^p (avec $b = 1$) et vérifiant facilement l'identité (1.7), nous pouvons suggérer qu'un analogue raisonnable de l'expression (1.7) au cas des fonctions convexes est donné par (1.8).

Une fois que cette définition est donnée, nous étudions quelques propriétés de $G_p(f,g)$ pour $f,g \in \mathrm{Conv}(H)$. Notamment, on établit les assertions suivantes.

- $G_p(f,g) = G_{1-p}(g,f)$, $G_p(f,\sigma) = f^{(p)}$, $G_p(f,0) = -pf^*(0)$.
- Pour tout $a,b \in \mathbb{R}$, $G_p((f+a),(g+b)) = G_p(f,g) + pa + (1-p)b$.
- Pour tout $a,b \in \mathbb{R}_+^*$,
$$G_p(af,bg) = \sqrt{ab}\, G_p\left(\sqrt{\frac{a}{b}}f, \sqrt{\frac{b}{a}}g\right),$$
en particulier $G_p(af,ag) = a\, G_p(f,g)$.
- Pour tout $f_1,f_2,g_1,g_2 \in Conv(H)$,
$$f_1 \geq f_2 \text{ and } g_1 \geq g_2 \Rightarrow G_p(f_1,g_1) \geq G_p(f_2,g_2).$$
- $(pf^* + (1-p)g^*)^* \leq G_p(f,g) \leq pf + (1-p)g$.

Dans le cas où f et g sont deux fonctions quadratiques correspondant respectivement aux opérateurs définis positifs A et B, nous obtenons, par une méthode différente de celle de [22], une autre extension de $a^p b^{1-p}$ aux opérateurs définis positifs, à savoir la relation (1.9) laquelle est équivalente à (1.10) ou encore sous forme explicite à (1.11). Afin de rendre son calcul pratique et efficace pour $p = 1/n$ ($n \geq 2$ entier) et A,B matrices définies positives données, nous avons montré que $G_p(A,B)$ est caractérisé par le résultat ci-dessous.

Théorème 4.2. *Soit A et B deux opérateurs définis positifs, on définit l'algorithme suivant :*

$$\begin{cases} Z_0 &= pA + (1-p)B, \\ Z_{k+1} &= (1-p)Z_k + pA(Z_k^{-1}B)^{\frac{1}{p}-1}, \quad k \geq 0. \end{cases}$$

Alors la suite $(Z_k)_k$ converge, en norme, vers $G_p(A,B)$.

Notre approche permet aussi de décrire une extension de $\sqrt[m]{a_1a_2\cdots a_m}$ (moyenne géometrique de m réels positifs $a_1,a_2,\cdots,a_m$) aux fonctions convexes. En effet, remarquant que

$$(a_1a_2\cdots a_m)^{1/m} = a_1^{1/m}\left((a_2a_3\cdots a_m)^{1/(m-1)}\right)^{1-1/m}$$

c'est-à-dire, selon la notation précédente,

$$(a_1a_2\cdots a_m)^{1/m} = G_{1/m}(a_1,(a_2a_3\cdots a_m)^{1/(m-1)}),$$

nous définissons alors, pour tout nombre entier $m \geq 2$ et f_i, $1 \leq i \leq m$, fonctions convexes, la suite suivante :

$$\begin{cases} \mathscr{G}_1(f_1) = f_1, \\ \mathscr{G}_m(f_1,f_2,\ldots,f_m) = G_{1/m}(f_1,\mathscr{G}_{m-1}(f_2,f_3,\ldots,f_m)), \end{cases}$$

laquelle donne, par récurrence, la moyenne géometrique de trois ou plusieurs fonctions convexes.

Cette définition nous a conduit à construire la *moyenne arithmético-géometrico-harmonique* $\mathscr{AGH}(f,g,h)$ de trois fonctions convexes comme précisé par le théorème suivant (les opérateurs $\mathscr{A}_3$ et $\mathscr{H}_3$ ci-dessous sont définis par (5.1) et (5.2) pour $m = 3$).

Théorème 6.1. *Soient* f, g, $h \in \Gamma_\circ(H)$ *satisfaisant (6.1). Alors les suites* (f_n), (g_n) *et* (h_n) *définies par*

$$\begin{cases} f_0 &= f \\ g_0 &= g \\ h_0 &= h \end{cases} \quad et \quad \begin{cases} f_{n+1} &= \mathscr{A}_3(f_n,g_n,h_n) \\ g_{n+1} &= \mathscr{G}_3(f_n,g_n,h_n) \\ h_{n+1} &= \mathscr{H}_3(f_n,g_n,h_n) \end{cases} \qquad (n \geq 0).$$

convergent ponctuellement vers la même fonctionnelle convexe $\mathscr{AGH}(f,g,h)$*, appelée moyenne arithmetico-geometrico-harmonique de* f*,* g *et* h*, et satisfaisant*

$$\mathscr{H}_3(f,g,h) \leq \mathscr{AGH}(f,g,h) \leq \mathscr{A}_3(f,g,h)\,.$$

Dans le cas où les trois fonctions f,g,h sont quadratiques, nous obtenons le résultat suivant.

Corollaire 6.1. *Soient A, B, C trois opérateurs définis positifs de H dans H et considérons l'algorithme suivant :*

$$\begin{cases} A_0 &= A \\ B_0 &= B \\ C_0 &= C \end{cases} \quad and \quad \begin{cases} A_{n+1} &= \mathscr{A}_3(A_n,B_n,C_n) \\ B_{n+1} &= \mathscr{G}_3(A_n,B_n,C_n) \\ C_{n+1} &= \mathscr{H}_3(A_n,B_n,C_n) \end{cases} \qquad (n \geq 0).$$

Alors (A_n), (B_n) *et* (C_n) *convergent fortement vers le même opérateur (défini positif)* $\mathscr{AGH}(A,B,C)$*, appelé opérateur moyenne arithmético-géometrico-harmonique de A, B et C, et satisfaisant*

$$\mathscr{H}_3(A,B,C) \leq \mathscr{AGH}(A,B,C) \leq \mathscr{A}_3(A,B,C)$$

Enfin, l'article se termine par une discussion du cas $p = 1/2$ mis en lien avec la moyenne géometrique convexe étudiée dans [4].

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M. Raïssouli and H. Bouziane
AFA Group, AFACS UFR
Moulay Ismaïl University
Faculty of Sciences
Department of Mathematics and Computer Science
PO Box 4010, Zitoune
Meknès, Morocco
raissouli_10@hotmail.com