A LAW OF LARGE NUMBERS FOR
RELAXED SCALAR CONSERVATION LAWS

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1. Introduction. Probabilistic methods have been used to obtain results about nonlinear evolution equations (see [BFG, COR, Fer, Rob]). Starting with an evolution equation one builds up a system composed of a large number of randomly interacting particles in such a way that formally the empirical laws converge to a weak solution of the equation. In some sense, this validates the equation. Usually, one has to modify or regularize the evolution equation to obtain such a validation. Here we start with the quasilinear scalar conservation law

\[ \partial_t u + \sum_{1 \leq a \leq d} A^a \partial_{x^a} u = 0 \]  

and we consider the Cauchy problem \( u(x, 0) = u_0(x) \), assuming that \( u_0 \) is a probability density. Generally, one considers weak solutions of special nature: irregular functions which are solutions in the sense of distributions. It would be nice to show that if one starts with a probability density \( u_0 \) then one has a unique solution \( u \) such that \( u_t \) is a probability density for all \( t > 0 \). Here we are not so precise. First we relax the equation. Written in the weak sense the equation becomes the following

\[ \langle \lambda_t, \varphi \rangle = \langle \lambda_0, \varphi \rangle + \int_0^t \langle \lambda_s, v(\theta) \cdot \nabla_x \varphi \rangle ds + \int_0^t \langle \tilde{\lambda}_s, \varphi \rangle ds, \]  

where \( v : \mathbb{R}_+ \to \mathbb{R}^d \) is a given function and \( \tilde{\lambda}_s \) is a nonlinear term of the form

\[ \langle \tilde{\lambda}_s, \varphi \rangle = \langle \lambda_s, \Lambda_1(\lambda_s) \varphi \rangle. \]

We then prove for some particular \( \Lambda_1 \) and \( v \), that starting with a probability measure \( \lambda_0 \), this relaxed equation has one and only one solution \( \lambda \) such that \( \lambda_t \) is a probability measure for all \( t \).

2. The Setup. We study the relaxed equation when \( \Lambda_1(\mu) \varphi \) is given by

\[ \Lambda_1(\mu) \varphi(x, \theta) = \int_0^1 \left[ \varphi(x, [G * \mu](x) \xi) - \varphi(x, \theta) \right] d\xi \]

where

\[ [G * \mu](x) = \int_{\mathbb{R}^d \times \mathbb{R}_+} G(x - y) \mu(dy, d\theta) \]

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and $G$ is a regular nonegative function which we take here equal to

$$G(x - y) = \frac{1}{(\sqrt{2\pi})^d} \exp\left\{ -\frac{||x - y||^2}{2} \right\}.$$  

We consider an interacting particle system associated with (1.2). The configuration of the system at time $t$ is denoted by $Z^n(t) = (Z^n_1(t), \ldots, Z^n_n(t))$ where $Z^n_j(t) = (x^n_j(t), \theta^n_j(t))$ is $\mathbb{R}^d \times \mathbb{R}_+$-valued. The process $\{Z^n(t), t \in [0, T]\}$ is Markov. We may assume that the trajectories are cadlag functions from $[0, T]$ to $(\mathbb{R}^d \times \mathbb{R}_+)^n$. The law $\mathbb{P}^n$ of this process is such that for any $f \in C^1_0([\mathbb{R}^d \times \mathbb{R}_+]^n)$, the process

$$f(Z^n(t)) - f(Z^n(0)) - \int_0^t (L^n + J^n)f(Z^n(s)) \, ds$$  

(2.1)

is a $\mathbb{P}^n$-martingale, where, for $z^n = ((x^n_1, \theta^n_1), \ldots, (x^n_n, \theta^n_n))$,

$$L^n f(z^n) = \sum_{j=1}^n v(\theta^n_j) \cdot \nabla_{x_j} f(z^n)$$

$$J^n f(z^n) = \sum_{j=1}^n \int_0^1 \left( f(z^n, \xi) - f(z^n) \right) \, d\xi$$

and $z^{n,j}(\xi)$ is the vector obtained from $z^n$ by replacing the component $\theta^n_j$ by $[G \ast \mu^n](x^n_j) \xi$ and leaving the other ones unchanged. The term $L^n$ corresponds to a free movement for the $x^n_j$'s while the term $J^n$ indicates that the $\theta^n_j$'s entail jumps at random times driven by an external field which is given here by the empirical measure of $z^n$:

$$\mu^n = \frac{1}{n} \sum_{j=1}^n \delta(x^n_j, \theta^n_j).$$

Let $\varphi \in C^1_0(\mathbb{R}^d \times \mathbb{R}_+)$ and consider the martingale $\{M^n_t(\varphi), t \in [0, T]\}$ obtained from (2.1) using the function

$$f((x^n_1, \theta^n_1), \ldots, (x^n_n, \theta^n_n)) = \frac{1}{n} \sum_{j=1}^n \varphi(x^n_j, \theta^n_j).$$

Simple computations give that

$$M^n_t(\varphi) = \langle \mu^n_t, \varphi \rangle - \langle \mu^n_0, \varphi \rangle - \int_0^t \langle \mu^n_s, \Gamma \varphi \rangle \, ds - \int_0^t \langle \mu^n_s, \Lambda_1(\mu^n_s) \varphi \rangle \, ds$$  

(2.2)

where $\Gamma \varphi(x, \theta) = v(\theta) \cdot \nabla_x \varphi(x, \theta)$ and $\mu^n_t$ is the empirical measure of $Z^n(t)$. Moreover, the quadratic variation $\langle M^n(\varphi) \rangle_t$ is given by

$$\langle M^n(\varphi) \rangle_t = \frac{1}{n} \int_0^t \langle \mu^n_s, \Lambda_2(\mu^n_s) \varphi \rangle \, ds$$  

(2.3)
where
\[
\Lambda_2(\mu) \varphi(x, \theta) = \int_0^1 (\varphi(x, [G * \mu](x)\xi) - \varphi(x, \theta))^2 \, d\xi.
\]

We shall use this martingale a few times in the sequel.

3. Preliminaries. In this section we study the continuity of some mappings on a particular set of measures. The main result is Proposition 3.2 which we shall use later in the proof of Lemma 4.4.

Let \( P_1 \) be the set of \( M^1(\mathbb{R}^d \times \mathbb{R}^+) \) such that
\[
m_1(\mu) := \int_{\mathbb{R}^d \times \mathbb{R}^+} \left(\|x\| + \|\theta\|\right) \mu(dx, d\theta) < \infty.
\]

We give \( P_1 \) the Polish topology induced by the metric
\[
\rho(\mu, \nu) = \inf\{ \langle \eta, \|x - y\| + |\theta - \tau| \rangle \mid \eta \text{ is a coupling of } \mu \text{ and } \nu \}.
\]

Proposition 3.1 below characterizes this topology.

**Proposition 3.1.** If \( \mu_n \to \mu \) in \( P_1 \) then \( \langle \mu_n, \varphi \rangle \to \langle \mu, \varphi \rangle \) for any continuous function \( \varphi \) such that \( |\varphi(x, \theta)| \leq C (1 + \|x\| + \|\theta\|) \). In particular, \( \mu_n \to \mu \) weakly.

Let \( \nu \) be a probability measure on \( \mathbb{R}^d \). Put
\[
[G * \nu](x) = \int_{\mathbb{R}^d} G(x - y) \nu(dy)
\]
where
\[
G(x - y) = \frac{1}{(\sqrt{2\pi})^d} \exp\left\{-\frac{\|x - y\|^2}{2}\right\}.
\]

**Lemma 3.1.** If \( \nu_n \to \nu \) weakly then \( G * \nu_n \to G * \nu \) uniformly on compact subsets of \( \mathbb{R}^d \).

**Proof.** Let \( K \) be a compact subset of \( \mathbb{R}^d \) and \( \varepsilon > 0 \). There exists \( \delta > 0 \) such that
\[
\|z_1 - z_2\| < \delta \Rightarrow |G * \lambda(z_1) - G * \lambda(z_2)| < \frac{\varepsilon}{3}
\]
for any probability measure \( \lambda \). Indeed, the function \( G \) is uniformly continuous on \( \mathbb{R}^d \). Hence there exists \( \delta > 0 \) such that
\[
\|z_1 - z_2\| < \delta \Rightarrow \|G(z_1) - G(z_2)\| < \frac{\varepsilon}{3}.
\]
Consequently we have
\[
\|z_1 - z_2\| < \delta \Rightarrow \|G(z_1 - y) - G(z_2 - y)\| < \varepsilon, \quad \forall y \in \mathbb{R}^d
\]
\[
\Rightarrow |G * \lambda(z_1) - G * \lambda(z_2)| \leq \int_{\mathbb{R}^d} \|G(z_1 - y) - G(z_2 - y)\| \lambda(dy) < \varepsilon.
\]
Since $K$ is compact, one can cover $K$ with a finite number of balls of radius $\delta$. The weak convergence of $(\nu_n)$ to $\nu$ implies the pointwise convergence of $G * \nu_n$ to $G * \nu$. Thus there exists an integer $n_0$ such that
\[
n \geq n_0 \Rightarrow |G * \nu_n(z) - G * \nu(z)| < \frac{\varepsilon}{3}
\]
for each ball’s center $z$. Now let $x \in K$ and let $z_0$ be the center of the ball containing $x$. The absolute value $|G * \nu_n(x) - G * \nu(x)|$ is bounded by
\[
|G * \nu_n(x) - G * \nu_n(z_0)| + |G * \nu_n(z_0) - G * \nu(z_0)| + |G * \nu(z_0) - G * \nu(x)|.
\]
The first and third term are smaller than $\varepsilon/3$ because $\|x - z_0\| < \delta$. The second one is also smaller than $\varepsilon/3$ when $n \geq n_0$. \hfill $\square$

**Lemma 3.2.** Let $\varphi \in C_{b}^{1}(\mathbb{R}^d \times \mathbb{R}_+)$ and suppose $\mu_n \to \mu$ weakly. Then $\Lambda_1(\mu_n) \varphi \to \Lambda(\mu) \varphi$ uniformly on compact subsets of $\mathbb{R}^d \times \mathbb{R}_+$

**Proof.** The function $\varphi$ (being in $C_{b}^{1}(\mathbb{R}^d \times \mathbb{R}_+)$) is uniformly continuous. Therefore, we have
\[
\forall \varepsilon > 0, \exists \delta > 0, \quad \|(x, \theta) - (x', \theta')\| < \delta \Rightarrow |\varphi(x, \theta) - \varphi(x', \theta')| < \varepsilon.
\]
In particular, we obtain
\[
\forall \varepsilon > 0, \exists \delta > 0, \quad |\theta - \theta'| < \delta \Rightarrow |\varphi(x, \theta) - \varphi(x, \theta')| < \varepsilon, \quad \forall x \in \mathbb{R}^d. \tag{3.1}
\]

Let $K$ be a compact subset of $\mathbb{R}^d \times \mathbb{R}_+$. There exists a compact subset $K' \subset K'$ and $a > 0$ such that $K \subset K' \times [0, a]$. Write $\nu_n$ and $\nu$ for the marginals of $\mu_n$ and $\mu$ on $\mathbb{R}^d$. Since $G * \nu_n$ converge to $G * \nu$ uniformly on $K'$, there exists an integer $n_0$ such that
\[
n \geq n_0 \Rightarrow |G * \nu_n(x) - G * \nu(x)| < \delta, \quad \forall x \in K'.
\]

Hence, recalling (3.1),
\[
\forall n \geq n_0, \forall x \in K', \forall \xi \in [0, 1], \quad |\varphi(x, G * \nu_n(x)\xi) - \varphi(x, G * \nu(x)\xi)| < \varepsilon.
\]

Therefore, provided $n \geq n_0$ and $x \in K$,
\[
|\Lambda_1(\mu_n) \varphi(x) - \Lambda_1(\mu) \varphi(x)| \leq \int_0^1 |\varphi(x, [G * \nu_n](x)\xi) - \varphi(x, [G * \nu](x)\xi)| \ d\xi < \varepsilon. \hfill \square
\]

Now let $v : \mathbb{R}_+ \to \mathbb{R}^d$ be a Lipschitz function with constant $c > 1$: $\|v(\theta) - v(\tau)\| \leq c|\theta - \tau|$. For $\varphi \in C_{b}^{1}(\mathbb{R}^d \times \mathbb{R}_+)$, recall that $\Gamma \varphi(x, \theta) = v(\theta) \cdot \nabla_x \varphi(x, \theta)$.

**Proposition 3.2.** Let $\varphi \in C_{b}^{1}(\mathbb{R}^d \times \mathbb{R}_+)$. Then $\mu \mapsto \langle \mu, \Lambda_1(\mu) \varphi \rangle$ and $\mu \mapsto \langle \mu, \Gamma \varphi \rangle$ are continuous mappings on $\mathcal{P}_1$.  

\[\text{4}\]
Proof. For the first mapping, it suffices to show that \( \langle \mu_n, \Lambda_1(\mu_n) \varphi \rangle \to \langle \mu, \Lambda_1(\mu) \varphi \rangle \) if \( \mu_n \to \mu \) weakly. Let us write
\[
\langle \mu_n, \Lambda_1(\mu_n) \varphi \rangle - \langle \mu, \Lambda_1(\mu) \varphi \rangle = \langle \mu_n, \Lambda_1(\mu_n) \varphi - \Lambda_1(\mu) \varphi \rangle + (\langle \mu_n, \Lambda_1(\mu) \varphi \rangle - \langle \mu, \Lambda_1(\mu) \varphi \rangle).
\]
The second term goes to zero because \( \Lambda_1(\mu) \varphi \) is bounded and continuous. For the first one, fix \( \varepsilon > 0 \) and choose \( K \) compact in \( \mathbb{R}^d \times \mathbb{R}_+ \) and such that \( \mu_n(K) > 1 - \varepsilon \) for all \( n \). This is possible because \( (\mu_n) \) is tight (being weakly convergent). Now write
\[
\langle \mu_n, \Lambda_1(\mu_n) \varphi - \Lambda_1(\mu) \varphi \rangle = \langle \mu_n, (\Lambda_1(\mu_n) \varphi - \Lambda_1(\mu) \varphi) 1_{K^c} \rangle + \langle \mu_n, (\Lambda_1(\mu_n) \varphi - \Lambda_1(\mu) \varphi) 1_K \rangle.
\]
On one hand, we have \( \| \Lambda_1(\mu_n) \varphi - \Lambda_1(\mu) \varphi \| \leq 2 \| \varphi \| \) so
\[
\left| \langle \mu_n, (\Lambda_1(\mu_n) \varphi - \Lambda_1(\mu) \varphi) 1_{K^c} \rangle \right| \leq 2 \| \varphi \| \mu_n(K^c) \leq 2 \| \varphi \| \varepsilon.
\]
On the other hand, \( \Lambda(\mu_n) \varphi \) converges uniformly to \( \Lambda(\mu) \varphi \) on \( K \). Hence, there exists \( n_0 \) such that \( |\Lambda_1(\mu_n) \varphi(x) - \Lambda_1(\mu) \varphi(x)| < \varepsilon \) for all \( x \in K \) and \( n \geq n_0 \). For those \( n \), we have
\[
\left| \langle \mu_n, \Lambda_1(\mu_n) \varphi - \Lambda_1(\mu) \varphi \rangle \right| \leq \varepsilon \mu_n(K) + 2 \| \varphi \| \varepsilon.
\]
Since \( \varepsilon \) is arbitrary, we obtain the result.

The continuity of the second mapping is immediate because, by the hypothesis made on \( v \), \( \Gamma \varphi \) is a continuous function such that \( |\Gamma \varphi(x, \theta)| \leq C_\varphi \left( \| v(0) \| + c \theta \right). \)

4. The Law of Large Numbers. We are interested by the process \( \{ \mu^n_t, t \in [0, T] \} \) where \( \mu^n_t \) is the empirical measure of \( Z^n(t) \). We look at \( \mu^n_t \) as an element of \( \mathcal{P}_1 \). From the definition of \( \rho \), one easily deduce the inequality
\[
\rho(\mu^n_t, \mu^n_t) \leq \frac{1}{n} \sum_{j=1}^n \| x^n_j(t) - x^n_j(s) \| + |\theta^n_j(t) - \theta^n_j(s)|
\]
using the coupling
\[
\eta = \frac{1}{n} \sum_{j=1}^n \delta_{Z^n_j(s)} \otimes \delta_{Z^n_j(t)}.
\]
Therefore, the trajectories \( t \mapsto \mu^n_t \) belong to \( D([0, T], \mathcal{P}_1) \). The law of \( \{ \mu^n_t, t \in [0, T] \} \) is denoted by \( P^n \). Our main result is the following.

Theorem 4.1. Assume the following conditions hold:
(a) \( v : \mathbb{R}_+ \to \mathbb{R}^d \) is Lipschitz with constant \( c > 1 : \| v(\theta) - v(\tau) \| \leq c |\theta - \tau| \).
(b) \( \sup_n \mathbb{E}^n \left[ \frac{1}{n} \sum_{j=1}^n \left( \| x^n_j(0) \|^2 + \theta^n_j(0)^2 \right) \right] < \infty \),
(c) \( \{ \mu^n_0 \} \) converges in distribution to \( \lambda_0 \in \mathcal{P}_1 \).

Then the sequence \( (P^n) \) converges weakly to \( \delta_\lambda \) where \( \lambda \) is the unique solution of (1.2).

The proof of Theorem 4.1 is mainly based on Lemma 4.1 which gives the propagation of the moment condition (b). As usual, the method is to prove first that the sequence \( (P^n) \) is relatively compact. This is Proposition 4.1. Then we prove in Proposition 4.2 that any limit point of the sequence \( (P^n) \) is concentrated on the solutions of (1.2). To end the proof it is then sufficient to show the uniqueness of the solutions of (1.2) (Proposition 4.3).

**Lemma 4.1.** Under conditions (a) and (b) of Theorem 4.1, we have

\[
\sup_{0 \leq t \leq T} \sup_n \mathbb{E}^n \left[ \frac{1}{n} \sum_{j=1}^n \left( \| x^n_j(t) \|^2 + \theta^n_j(t)^2 \right) \right] < \infty.
\]

**Proof.** Consider the function \( \varphi(x, \theta) = \| x \|^2 + \theta^2 \). First we have

\[
\langle \mu^n_s, \Gamma \varphi \rangle = \frac{2}{n} \sum_{j=1}^n \langle \varepsilon^n_j(s) \rangle \cdot x^n_j(s)
\]

\[
\leq \frac{2}{n} \sum_{j=1}^n \| \varepsilon^n_j(s) \| \| x^n_j(s) \|
\]

\[
\leq \frac{1}{n} \sum_{j=1}^n \left( \| \varepsilon^n_j(s) \|^2 + \| x^n_j(s) \|^2 \right).
\]

Because of (a), \( \| \varepsilon(\theta) \| \leq \| \varepsilon(0) \| + c \theta \), so

\[
\langle \mu^n_s, \Gamma \varphi \rangle \leq \frac{1}{n} \sum_{j=1}^n \left( [\| \varepsilon(0) \| + c \theta^n_j(s)]^2 + \| x^n_j(s) \|^2 \right)
\]

\[
\leq 2 \| \varepsilon(0) \|^2 + \frac{1}{n} \sum_{j=1}^n [2c^2 \theta^n_j(s)^2 + \| x^n_j(s) \|^2]
\]

and therefore

\[
\langle \mu^n_s, \Gamma \varphi \rangle \leq 2 \| \varepsilon(0) \|^2 + 2c^2 \langle \mu^n_s, \varphi \rangle. \tag{4.1}
\]

Second, we have

\[
\Lambda(\mu^n_s) \varphi(x, \theta) = \frac{[G \ast \mu^n_s(x)]^2}{3} - \theta^2
\]

so that

\[
\langle \mu^n_s, \Lambda(\mu^n_s) \varphi \rangle \leq \frac{1}{n} \sum_{j=1}^n \left[ \frac{[G \ast \mu^n_s(x^n_j(s))]^2}{3} + \langle \mu^n_s, \theta^2 \rangle \right].
\]

But \( G \) is bounded and \( \mu^n_s \) is a probability measure. Thus \( G \ast \mu^n_s(z) \) is bounded by a constant \( c' \) which does not depend on \( n, s \) or \( z \). It implies that

\[
\langle \mu^n_s, \Lambda(\mu^n_s) \varphi \rangle \leq (c')^2 + \langle \mu^n_s, \varphi \rangle. \tag{4.2}
\]
We now use the martingale (2.2), (4.1) and (4.2) to obtain
\[
E^n \left[ \langle \mu^n_t, \varphi \rangle \right] \leq E^n \left[ \langle \mu^n_0, \varphi \rangle \right] + t \left[ (\epsilon')^2 + 2\|v(0)\|^2 \right] + (2\epsilon^2 + 1) \int_0^T E^n \left[ \langle \mu^n_s, \varphi \rangle \right] ds.
\]

Gronwall lemma implies that
\[
E^n \left[ \langle \mu^n_t, \varphi \rangle \right] \leq \left\{ E^n \left[ \langle \mu^n_0, \varphi \rangle \right] + t \left[ (\epsilon')^2 + 2\|v(0)\|^2 \right] \right\} e^{(2\epsilon^2 + 1)t}.
\]

Since
\[
E^n \left[ \frac{1}{n} \sum_{j=1}^n \|X_j^n(t)\|^2 + \theta_j^n(t)^2 \right] = E^n \left[ \langle \mu^n_t, \varphi \rangle \right],
\]
the result follows from hypothesis (b). \( \square \)

Remark. The function \( \varphi \) above is not bounded. Strictly speaking the previous calculations would have to be done using truncation.

Lemma 4.2. Under conditions (a) and (b) of Theorem 4.1, there exists a constant \( C_T \) such that
\[
\forall n \geq 1, \forall m \geq 1, \quad P \left\{ \sup_{0 \leq t \leq T} \left[ \frac{1}{n} \sum_{j=1}^n \|X_j^n(t)\|^2 + \theta_j^n(t)^2 \right] > C_T 2^n \right\} \leq \frac{1}{2^n}.
\]

Proof. Again we take \( \varphi(x, \theta) = \|x\|^2 + \theta^2 \) in (2.2) to obtain the inequality:
\[
\sup_{0 \leq t \leq T} \left[ \frac{1}{n} \sum_{j=1}^n \|X_j^n(t)\|^2 + \theta_j^n(t)^2 \right] \leq \langle \mu^n_0, \varphi \rangle + \sup_{0 \leq t \leq T} \left[ M^n_t(\varphi) \right]
\]
\[
+ \int_0^T \left\{ \left| \langle \mu^n_s, \Gamma \varphi \rangle \right| + \left| \langle \mu^n_s, \Lambda(\mu^n_s) \varphi \rangle \right| \right\} ds. \quad (4.3)
\]

Put
\[
D_T = \sup_{0 \leq t \leq T} \sup_n E^n \left[ \frac{1}{n} \sum_{j=1}^n \|X_j^n(t)\|^2 + \theta_j^n(t)^2 \right];
\]

(4.1) and (4.2) implies that
\[
E^n \left[ \int_0^T \left\{ \left| \langle \mu^n_s, \Gamma \varphi \rangle \right| + \left| \langle \mu^n_s, \Lambda(\mu^n_s) \varphi \rangle \right| \right\} ds \right] \leq [(\epsilon')^2 + 2\|v(0)\|^2 + (2\epsilon^2 + 1)D_T]T
\]
so
\[
P \left\{ \int_0^T \left\{ \left| \langle \mu^n_s, \Gamma \varphi \rangle \right| + \left| \langle \mu^n_s, \Lambda(\mu^n_s) \varphi \rangle \right| \right\} ds > \frac{a}{3} \right\}
\]
\[
\leq \frac{3}{a} [(\epsilon')^2 + 2\|v(0)\|^2 + (2\epsilon^2 + 1)D_T]T. \quad (4.4)
\]
Next we have
\[ |M^n_T(\varphi)| \leq \langle \mu^n_0, \varphi \rangle + \langle \mu^n_T, \varphi \rangle + \int_0^T \left\{ |\langle \mu^n_s, \Gamma \varphi \rangle| + |\langle \mu^n_s, \Lambda(\mu^n_s) \varphi \rangle| \right\} ds. \]

Therefore,
\[ E^n [ |M^n_T(\varphi)| ] \leq 2D_T + [(\epsilon')^2 + 2\|v(0)\|^2 + (2\epsilon^2 + 1)D_T]T \]
and Doob’s inequality gives
\[
P \left\{ \sup_{0 \leq t \leq T} |M^n_T(\varphi)| > \frac{a}{3} \right\} \leq \frac{3}{a} E^n [ |M^n_T(\varphi)| \leq \frac{3}{a} (2D_T + [(\epsilon')^2 + (\epsilon + 1)D_T]T). \tag{4.5} \]

Finally, \( P \{ \langle \mu^n_0, \varphi \rangle > a/3 \} \leq 3D_T/a \). But inequality (4.3) implies that the probability
\[ P \left\{ \sup_{0 \leq t \leq T} \left[ \frac{1}{n} \sum_{j=1}^n \| x^n_j(t) \|^2 + \theta^n_j(t)^2 \right] > a \right\} \]
is bounded above by the sum
\[ \frac{3}{a} D_T + P \left\{ \sup_{0 \leq t \leq T} |M^n_T(\varphi)| > \frac{a}{3} \right\} + P \left\{ \int_0^T \left\{ |\langle \mu^n_s, \Gamma \varphi \rangle| + |\langle \mu^n_s, \Lambda(\mu^n_s) \varphi \rangle| \right\} ds > \frac{a}{3} \right\}. \]

If we take
\[ C_T = 3 \left\{ 2D_T + 2[(\epsilon')^2 + 2\|v(0)\|^2 + (2\epsilon^2 + 1)D_T]T \right\} \]
the result follows from (4.4) and (4.5). \( \Box \)

To prove the relative compactness of \( P^n \), it will be necessary to show that several auxillary processes are tight. This is done in the following lemma. Recall that a sequence of real processes is called \( C \) tight if it is tight and any limit process has continuous paths.

**Lemma 4.3.** Suppose \( \varphi \in \mathcal{C}^1_0(\mathbb{R}^d \times \mathbb{R}_+) \) and that conditions (a) and (b) of Theorem 4.1 hold. Then the processes \( \{ \int_0^t \langle \mu^n_s, \Gamma \varphi \rangle ds, t \in [0, T] \}, \{ \int_0^t \langle \mu^n_s, \Lambda(\mu^n_s) \varphi \rangle ds, t \in [0, T] \} \) and \( \{ \langle M^n(\varphi) \rangle_t, t \in [0, T] \} \) are \( C \) tight.

**Proof.** Consider the first sequence of processes. According to [Jac, Prop. 2.11, p. 309], we have to prove that
a) \( \forall \varepsilon > 0, \exists a > 0, \exists \eta, \exists n_0 \geq 2 \) such that
\[ \sup_{n \geq n_0} P \left\{ \sup_{0 \leq t \leq T} \left| \int_0^t \langle \mu^n_s, \Gamma \varphi \rangle ds \right| > a \right\} \leq \varepsilon, \]

b) \( \forall \varepsilon > 0, \forall \eta > 0, \exists \delta > 0, \exists \eta, \exists n_0 \geq 2 \) such that
\[ \sup_{n \geq n_0} P \left\{ w \left( \int_0^t \langle \mu^n_s, \Gamma \varphi \rangle ds, \delta \right) > \eta \right\} \leq \varepsilon. \]
where \( w \) is the modulus of continuity:

\[
w(x, \delta) = \sup_{\epsilon, t, 0 \leq \|x\|, 1 \leq \delta} |x(s) - x(t)|.
\]

Since \( |\Gamma \varphi(x, \theta)| \leq C_\varphi \left( \|v(0)\| + c \theta \right) \), we have

\[
\sup_{0 \leq t \leq T} \left| \int_0^t \langle \mu^s_n, \Gamma \varphi \rangle \ ds \right| \leq C_\varphi \left( \|v(0)\|T + c \int_0^T \langle \mu^0_n, \theta \rangle \ ds \right).
\]

Therefore,

\[
P \left\{ \sup_{0 \leq t \leq T} \left| \int_0^t \langle \mu^s_n, \Gamma \varphi \rangle \ ds \right| > a \right\} \leq \frac{C_\varphi^2}{a^2} \left( 2\|v(0)\|^2T^2 + 2T c^2 \int_0^T \mathbb{E} \left[ \langle \mu^0_n, \theta^2 \rangle \right] \ ds \right).
\]

By Lemma 4.1, the expectation on the right-hand side is uniformly bounded in \( n \) and \( s \). Property a) then follows. Next remark that

\[
w(\int_0^t \langle \mu^s_n, \Gamma \varphi \rangle \ ds, \delta) \leq C_\varphi \left( \|v(0)\| + c \sup_{0 \leq t \leq T} \langle \mu^0_n, \theta \rangle \right) \delta.
\]

Choose \( \delta \) small enough to have \( C_\varphi \|v(0)\| \delta \leq \eta/2 \). Then,

\[
P \left\{ w(\int_0^t \langle \mu^s_n, \Gamma \varphi \rangle \ ds, \delta) > \eta \right\} \leq P \left\{ \sup_{0 \leq t \leq T} \langle \mu^0_n, \theta^2 \rangle > \frac{\eta^2}{4c^2C_\varphi^2} \delta^2 \right\}
\]

and property b) follows from Lemma 4.2.

For the other processes, we must show similar statements. This is easier. Indeed, the random variables \( \sup_{0 \leq t \leq T} \left| \int_0^t \langle \mu^s_n, A(\mu^0_s) \varphi \rangle \ ds \right| \) and \( \sup_{0 \leq t \leq T} \left| \langle M^c(\varphi) \rangle \right| \) are bounded by \( 2T \|\varphi\|_\infty \) and \( 4T \|\varphi\|_\infty^2 \) respectively. The analogues of property a) then follow trivially by taking \( a \) large enough. Similarly, the modulus of continuity is bounded by \( 2\|\varphi\|_\infty \) and \( 4\|\varphi\|_\infty^2 \delta \) respectively. This time property b) is obtained by taking \( \delta \) small enough.

**Proposition 4.1.** Under conditions (a) and (b) of Theorem 4.1, the sequence \( (P^n) \) is relatively compact.

**Proof.** According to [Frn], it suffices to prove that:

1. There exists a sequence \( (K_m) \) of compact subsets of \( \mathcal{P}_1 \) such that

\[
\forall m, \forall n, \quad P^n \{ \exists t \in [0, T] \mid \mu^n_t \not\in K_m \} \leq 2^{-m}.
\]

2. For any \( \varphi \in C^1_b(\mathbb{R}^d \times \mathbb{R}^+) \), the real processes \( \{ \langle \mu^n_t, \varphi \rangle, t \in [0, T] \} \) are tight.

But for fixed \( \epsilon \) the set

\[
L(\epsilon) = \{ \mu \in \mathcal{P}_1 \mid \langle \mu, \|x\|^2 + \theta^2 \rangle \leq \epsilon \}
\]

is relatively compact. Hence, for any \( \mu \in L(\epsilon) \), there is a sequence \( (K_m) \) of compact sets of \( \mathcal{P}_1 \) such that

\[
P^n \{ \mu^n_t \not\in K_m \} \leq 2^{-m}.
\]

Since \( \{ \langle \mu^n_t, \varphi \rangle, t \in [0, T] \} \) is tight, we can choose \( (K_m) \) such that

\[
P^n \{ \mu^n_t \not\in K_m \} \leq 2^{-m}.
\]

For any \( \varphi \in C^1_b(\mathbb{R}^d \times \mathbb{R}^+) \), the real processes \( \{ \langle \mu^n_t, \varphi \rangle, t \in [0, T] \} \) are tight. But for fixed \( \epsilon \) the set

\[
L(\epsilon) = \{ \mu \in \mathcal{P}_1 \mid \langle \mu, \|x\|^2 + \theta^2 \rangle \leq \epsilon \}
\]

is relatively compact. Hence, for any \( \mu \in L(\epsilon) \), there is a sequence \( (K_m) \) of compact sets of \( \mathcal{P}_1 \) such that

\[
P^n \{ \mu^n_t \not\in K_m \} \leq 2^{-m}.
\]

Since \( \{ \langle \mu^n_t, \varphi \rangle, t \in [0, T] \} \) is tight, we can choose \( (K_m) \) such that

\[
P^n \{ \mu^n_t \not\in K_m \} \leq 2^{-m}.
\]
is compact in $\mathcal{P}_1$. Hence property 1) follows from Lemma 4.2 by taking $K_m = L(C_T 2^m)$ where $C_T$ is the constant appearing in that lemma. For 2), recall that

$$\langle \mu^n_t, \varphi \rangle = \langle \mu^n_0, \varphi \rangle + \int_0^t \langle \mu^n_s, \Gamma \varphi \rangle \, ds + \int_0^t \langle \mu^n_s, \Lambda_1(\mu^n_s) \varphi \rangle \, ds + M^n_t(\varphi).$$

We then use [JS, Cor. 3.33, p. 317] which says that the real processes $\{\langle \mu^n_t, \varphi \rangle, t \in [0, T] \}$ are tight provided

a) $\{\langle \mu^n_0, \varphi \rangle \}$ is tight,

b) $\{ \int_0^t \langle \mu^n_s, \Gamma \varphi \rangle \, ds, t \in [0, T] \}$ is $C$-tight,

c) $\{ \int_0^t \langle \mu^n_s, \Lambda(\mu^n_s) \varphi \rangle \, ds, t \in [0, T] \}$ is $C$-tight,

d) $\{ M^n_t(\varphi), t \in [0, T] \}$ is tight.

But a) is fulfilled because the random variables are bounded by $\|\varphi\|_{\infty}$; b) and c) were proved in Lemma 4.3. For d), we use [JS, Theo. 4.13, p. 322]: $\{ M^n_0(\varphi), t \in [0, T] \}$ is tight if $\{ M^n_t(\varphi), t \in [0, T] \}$ is $C$-tight. Again, this was proved in Lemma 4.3.

Lemma 4.4. Let $\varphi \in C^1_0(\mathbb{R}^d \times \mathbb{R}_+)$. The mapping $\Psi_\varphi : D([0, T], \mathcal{P}_1) \to D([0, T], \mathbb{R})$ defined by

$$\Psi_\varphi(w)(t) = \langle w(t), \varphi \rangle - \langle w(0), \varphi \rangle - \int_0^t \langle w(s), \Gamma \varphi \rangle \, ds - \int_0^t \langle w(s), \Lambda_1(w(s)) \varphi \rangle \, ds$$

is continuous.

Proof. Since 0 is a continuity point for any $w \in D([0, T], \mathcal{P}_1)$, the mapping $\Psi_\varphi^{(1)}(w)(t) = \langle w(0), \varphi \rangle$ is continuous. Next, we use the fact that the mapping

$$f(x)(t) = \int_0^t x(s) \, ds$$

from $D([0, T], \mathbb{R})$ into $D([0, T], \mathbb{R})$ is continuous. Therefore, it remains to show that

$$\Psi_\varphi^{(2)}(w)(t) = \langle w(t), \varphi \rangle, \quad \Psi_\varphi^{(3)}(w)(t) = \langle w(t), \Gamma \varphi \rangle \quad \text{and} \quad \Psi_\varphi^{(4)}(w)(t) = \langle w(t), \Lambda(w(t)) \varphi \rangle$$

are continuous mappings from $D([0, T], \mathcal{P}_1)$ into $D([0, T], \mathbb{R})$. This is fulfilled if $\mu \to \langle \mu, \Gamma \varphi \rangle$ and $\mu \to \langle \mu, \Lambda(\mu) \varphi \rangle$ are continuous mappings from $\mathcal{P}_1$ into $\mathbb{R}$. These properties were proved in Proposition 3.2.

Proposition 4.2. Under conditions (a), (b) and (c) of Theorem 4.1, any limit point of $(P^n)$ is concentrated on the solutions of (1.2).

Proof. Consider the mapping $\Psi_\varphi$ of Lemma 4.4. It follows from (2.2) that $\Psi_\varphi(\mu^n)(t) = M^n_t(\varphi)$. Doob’s inequality and (2.3) give

$$E^n \left[ \sup_{0 \leq t \leq T} |M^n_t(\varphi)|^2 \right] = 4E^n \left[ M^n_T(\varphi)^2 \right] = \frac{4}{n} \int_0^T E^n \left[ \langle \mu^n_s, \Lambda(\mu^n_s) \varphi \rangle \right] \, ds \leq \frac{16T}{n} \|\varphi\|_{\infty}^2.$$
so that

\[
\lim_{n \to \infty} E^n \left[ \sup_{0 \leq t \leq T} |\Psi_n(\cdot)(t)|^2 \right] = \lim_{n \to \infty} E^n \left[ \sup_{0 \leq t \leq T} |\Psi(\mu^n)(t)|^2 \right] = \lim_{n \to \infty} E^n \left[ \sup_{0 \leq t \leq T} |M_t^n(\phi)|^2 \right] = 0. \tag{4.6}
\]

Let \( P \) be a limit point of \((P^n)\) and \((P^{n_k})\) a subsequence converging to \( P \). The mapping \( w \to \sup_{0 \leq t \leq T} |\Psi_n(w)(t)| \) is continuous on \( D([0,T], \mathcal{P}_1) \) because \( \Psi_n \) is. Moreover, by the computations above, \( \sup_{n_k} E^{n_k} \left[ \sup_{0 \leq t \leq T} |\Psi_n(\cdot)(t)|^2 \right] < \infty \). Therefore, the weak convergence of \((P^{n_k})\) yields

\[
\lim_{n \to \infty} E^{n_k} \left[ \sup_{0 \leq t \leq T} |\Psi_n(\cdot)(t)| \right] = E \left[ \sup_{0 \leq t \leq T} |\Psi(\cdot)(t)| \right]. \tag{4.7}
\]

The limits (4.6) and (4.7) imply that \( E \left[ \sup_{0 \leq t \leq T} |\Psi_n(\cdot)(t)| \right] = 0 \). Therefore, there exists a set \( \Omega_{\phi} \subset D([0,T], \mathcal{P}_1) \) such that \( P(\Omega_{\phi}) = 1 \) and for any \( w \in \Omega_{\phi} \) we have

\[
\langle w(t), \phi \rangle = \langle w(0), \phi \rangle + \int_0^t \langle w(s), \Gamma \phi \rangle ds + \int_0^t \langle w(s), \Lambda_1 \phi(s) \phi \rangle ds, \quad \forall t \in [0,T]. \tag{4.8}
\]

Let \( S \) be a countable subset of \( C_b^1(\mathbb{R}^d \times \mathbb{R}_+) \) which separates the probability measures on \( \mathbb{R}^d \times \mathbb{R}_+ \) and define

\[
\Omega_0 = \left( \bigcap_{\phi \in S} \Omega_{\phi} \right) \cap \{ w \in D([0,T], \mathcal{P}_1) \mid w(0) = \lambda_0 \}.
\]

Then by (c) of Theorem 4.1, \( P(\Omega_0) = 1 \), and every \( w \in \Omega_0 \) satisfies (4.8) for all \( \phi \in S \). Since \( S \) separates the probability measures, (4.8) is true for all \( \phi \in C_b(\mathbb{R}^d \times \mathbb{R}_+) \). Thus the measure \( P \) is concentrated on the solutions of (1.2). \( \square \)

To study the uniqueness of the solutions of (1.2), we use the bounded Lipschitz metric \( \rho_{BL} \) on \( \mathcal{M}^1(\mathbb{R}^d \times \mathbb{R}_+) \):

\[
\rho_{BL}(\nu_1, \nu_2) := \sup_{\phi \in C_b(\mathbb{R}^d \times \mathbb{R}_+) \mid \|\phi\|_{BL} \leq 1} |\langle \nu_1 - \nu_2, \phi \rangle|
\]

where

\[
\|\phi\|_{BL} := \max \left\{ \sup_{(x,\theta)} |\phi(x, \theta)|, \sup_{(x,\theta) \neq (y,\tau)} \frac{|\phi(x, \theta) - \phi(y, \tau)|}{\|x - y\| + |\theta - \tau|} \right\}.
\]

For \( \phi \in C_b(\mathbb{R}^d \times \mathbb{R}_+) \) we put

\[
J(\nu)\phi(x, \theta) = \int_0^1 \phi(x, [G + \nu]\xi) d\xi
\]

and we define a probability measure \( K(\nu) \) by \( \langle K(\nu), \phi \rangle = \langle \nu, J(\nu) \phi \rangle \).
Lemma 4.5. Let $\varphi \in C_b(\mathbb{R}^d \times \mathbb{R}_+)$ such that $\|\varphi\|_{\infty} \leq 1$. Then for any probability measure $\nu$, $\nu_1$ and $\nu_2$ we have

a) $J(\nu)\varphi \in C_b(\mathbb{R}^d \times \mathbb{R}_+)$ and $\|J(\nu)\varphi\|_{\infty} \leq \|G\|_{\infty} + 1$,

b) $\|J(\nu_1)\varphi - J(\nu_2)\varphi\|_{\infty} \leq \|G\|_{\infty} \rho_{\infty}(\nu_1, \nu_2),$

c) $\rho_{\infty}(K(\nu_1), K(\nu_2)) \leq (2\|G\|_{\infty} + 1)\rho_{\infty}(\nu_1, \nu_2).$

Proof. First observe that $\|G\|_{\infty} < \infty$ and $G * \nu(x) - G * \nu(y) \leq \|G\|_{\infty} \|x - y\|$. Clearly, $\|J(\nu)\varphi\|_{\infty} \leq \|\varphi\|_{\infty} \leq 1$. If $\|\varphi\|_{\infty} \leq 1$ then $|\varphi(x, \theta) - \varphi(y, \tau)| \leq \|x - y\| + |\theta - \tau|$ and consequently we have

$$\|\varphi(x, [G * \nu](x) \xi) - \varphi(y, [G * \nu](y) \xi)\| \leq \|x - y\| + \|G * \nu(x) - G * \nu(y)\| \xi$$

which implies

$$\|J(\nu)\varphi(x, \theta) - J(\nu)\varphi(y, \tau)\| \leq \int_0^1 |\varphi(x, [G * \nu](x) \xi) - \varphi(y, [G * \nu](y) \xi)| \, d\xi$$

$$\leq \left(\|G\|_{\infty} + 1\right) \left(\|x - y\| + |\theta - \tau|\right).$$

Conclusion a) follows. For b), we have

$$\|G * \nu_1(x) - G * \nu_2(x)\| = \left| \int_{\mathbb{R}^d \times \mathbb{R}_+} G(x - y)\nu_1(dy, d\theta) - \int_{\mathbb{R}^d \times \mathbb{R}_+} G(x - y)\nu_2(dy, d\theta) \right|$$

$$= \|G\|_{\infty} \left| \int_{\mathbb{R}^d \times \mathbb{R}_+} \frac{G(x - y)}{\|G\|_{\infty}}\nu_1(dy, d\theta) - \int_{\mathbb{R}^d \times \mathbb{R}_+} \frac{G(x - y)}{\|G\|_{\infty}}\nu_2(dy, d\theta) \right|$$

$$\leq \|G\|_{\infty} \sup_{\psi \in C_b(\mathbb{R}^d \times \mathbb{R}_+, \|\psi\|_{\infty} \leq 1} \left| \langle \nu_1 - \nu_2, \psi \rangle \right| = \|G\|_{\infty} \rho_{\infty}(\nu_1, \nu_2).$$

Therefore

$$\|\varphi(x, [G * \nu_1](x) \xi) - \varphi(x, [G * \nu_2](x) \xi)\| \leq \|G\|_{\infty} \rho_{\infty}(\nu_1, \nu_2) \xi$$

which implies

$$\|J(\nu_1)\varphi - J(\nu_2)\varphi\|_{\infty} \leq \int_0^1 |\varphi(x, [G * \nu_1](x) \xi) - \varphi(x, [G * \nu_2](x) \xi)| \, d\xi$$

$$\leq \|G\|_{\infty} \rho_{\infty}(\nu_1, \nu_2).$$

Now c) follows from a) and b). Indeed, suppose $\|\varphi\|_{\infty} \leq 1$, then

$$\left| \langle K(\nu_1) - K(\nu_2), \varphi \rangle \right| = \left| \langle \nu_1, J(\nu_1)\varphi \rangle - \langle \nu_2, J(\nu_2)\varphi \rangle \right|$$

$$\leq \left| \langle \nu_1, J(\nu_1)\varphi \rangle - \langle \nu_1, J(\nu_2)\varphi \rangle \right| + \left| \langle \nu_1, J(\nu_2)\varphi \rangle - \langle \nu_2, J(\nu_2)\varphi \rangle \right|$$

$$\leq \langle \nu_1, J(\nu_1)\varphi - J(\nu_2)\varphi \rangle + \langle \nu_1 - \nu_2, J(\nu_2)\varphi \rangle$$

$$\leq \|J(\nu_1)\varphi - J(\nu_2)\varphi\|_{\infty} + (\|G\|_{\infty} + 1)\rho_{\infty}(\nu_1, \nu_2)$$

$$\leq \left(2\|G\|_{\infty} + 1\right)\rho_{\infty}(\nu_1, \nu_2). \square$$
Proposition 4.3. If \( v \) satisfies hypothesis (a) of Theorem 4.1, the equation (1.2) has at most one solution for a given \( \lambda \neq 0 \).

Proof. Fix \( \varphi \in C_c(\mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R}_+) \) such that \( \| \varphi \|_{\text{ul}} \leq 1 \). Put \( T(t) \varphi(x, \theta) = \varphi(x + t v(\theta), \theta) \). An argument similar to that of [TW] shows that any solution of (1.2) must satisfy

\[
\forall t \in [0, T], \quad \langle \lambda_t, \varphi \rangle = e^{-t} \langle \lambda_0, T(t) \varphi \rangle + \int_0^t e^{-(t-s)} \langle K(\lambda_s), T(t-s) \varphi \rangle \, ds.
\]

Let \( \lambda^{(1)} \) and \( \lambda^{(2)} \) be two solutions of (1.2) with the same initial data \( \lambda_0 \). Then we have

\[
\forall t \in [0, T], \quad |\langle \lambda_t^{(1)} - \lambda_t^{(2)}, \varphi \rangle| \leq \int_0^t e^{-(t-s)} |\langle K(\lambda_s^{(1)}), T(t-s) \varphi \rangle| \, ds.
\]

Using hypothesis (a), we can easily show that \( \| T(t) \varphi \|_{\text{ul}} \leq t e + 1 \) so

\[
\forall t \in [0, T], \quad |\langle \lambda_t^{(1)} - \lambda_t^{(2)}, \varphi \rangle| \leq \int_0^t e^{-(t-s)} [(t-s) e + 1] \rho_{\text{ul}}(K(\lambda_s^{(1)}), K(\lambda_s^{(2)})) \, ds.
\]

Therefore, by Lemma 4.6,

\[
\forall t \in [0, T], \quad \rho_{\text{ul}}(\lambda_t^{(1)}, \lambda_t^{(2)}) \leq C (2 \| G \|_{\text{ul}} + 1) \int_0^t \rho_{\text{ul}}(\lambda_s^{(1)}, \lambda_s^{(2)}) \, ds
\]

and the result follows from Gronwall lemma.

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