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CONVERGENCE RATE OF THE BINOMIAL TREE SCHEME FOR CONTINUOUSLY PAYING OPTIONS

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RÉSUMÉ. Les Options à Paiements Continus (OPC) forment une classe de produits dérivés qui couvrent très naturellement les risques provenant de mouvements défavorables d'actifs transigés continûment. Nous étudions le taux de convergence des OPC évaluées par arbres binomiaux lorsque la prime d'exercice φ est $C^{(2)}$ par morceaux et assujettie à des conditions de bornitude. Nous montrons que lorsque φ est continue, le taux de convergence est n^{-1} alors qu'il est $n^{-\frac{1}{2}}$ si φ est discontinue.

ABSTRACT. Continuously Paying Options (CPOs) form a very natural class of derivatives for hedging risks coming from adverse movements of a continuously traded asset. We study the rate of convergence of CPOs evaluated under the binomial tree scheme when the payout function φ is piecewise $C^{(2)}$ subject to some boundedness conditions. We show that if φ is continuous, the rate of convergence is n^{-1} while it is $n^{-\frac{1}{2}}$ if φ is discontinuous.

1. Introduction

Security derivatives, sometimes also called portfolio insurances, play a critical role in modern finance. They are massively traded and allow those who purchase or sell them to better tailor their exposures to financial contingencies. A trader who, on the other hand, wants to buy protection against possible adverse value at time T of an asset whose price is S_t, $0 \leq t \leq T$, may elect to purchase a European option paying $\varphi(T, S_T)$ at time T. If, on the other hand, protection is needed at some still unknown time $0 < \tau \leq T$ in the future, the investor would naturally prefer an American option paying $\varphi(\tau, S_\tau)$ at the time τ of his choice. But sometimes, like it is the case for electricity, an underlying asset can be continuously purchased or traded over some period $0 \leq t \leq T$, and European or American options are clearly not adequate: an investor needs a *Continuously Paying Option* (CPO) to hedge his risk. For instance, in order to hedge the risk coming from continuously purchasing a commodity at a constant rate of Q units per year but at a random spot price S_t, $0 \leq t \leq T$, one can purchase a Call CPO with maturity T and strike k, that is, a CPO with payout $\varphi(t, x) = \max(x - k, 0)$. The CPO will guarantee that the commodity costs are limited to a maximum $(Qk)\, dt$ over any time interval of size dt prior to maturity. A CPO with maturity T and payout $\varphi : \mathbb{R}^2 \to \mathbb{R}$, written on a underlying S with value S_t at time $t \geq 0$, is a contract that

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provides its purchaser a continuous inflow of money at a rate of $\varphi(t, S_t)$ per year, that is, which pays $\int_a^b \varphi(t, S_t)dt$ over any time interval $[a, b]$ for $0 \leq a \leq b \leq T$.

Assume a constant risk free rate $\rho > 0$ and a complete and arbitrage free market. The value $v^{\mathcal{C}}_{\varphi,T}(0, x)$ of a CPO at time $t = 0$, knowing that $S_0 = x$, is clearly given by

$$v^{\mathcal{C}}_{\varphi,T}(0, x) := \int_0^T e^{-\rho t} E_{0,x} \left[\varphi(t, S_t)\right] dt, \tag{1}$$

where $E_{0,x} [\varphi(t, S_t)]$ denotes the conditional expectation of $\varphi(t, S_t)$, under the risk neutral probability measure, knowing that $S_0 = x$.

Let $S^{(n)}$ be the random walk obtained by the *Binomial Tree Scheme* (BTS) introduced by Cox *et al.* [5]. Due to its simplicity and flexibility, the BTS is undoubtedly the most well known way of pricing options —European, American, CPOs and others— when a closed form formula is not readily available. The purpose of this paper is to study the rate of convergence (the order of convergence) of the BTS to its Black–Scholes limit when evaluating CPOs.

The motivation in this paper is twofold. First the question of the rate of convergence of the BTS for CPOs is an interesting question in itself, as CPOs form an important and natural class of options and rates of convergence provide a natural way of measuring how fast convergence occurs. But furthermore, the value $u(t, x)$ at time t, knowing that $S_t = x$, of American type options with intrinsic value h and maturity T has been given in [2, 10, 13] for the American put, and in [16] under great generality for the closely related *randomized* American option, as the sum of a European option and a CPO. Since the rate of convergence of European options is known from [20], finding the rate of convergence of the BTS for CPOs is a most natural step towards calculating the long conjectured but still elusive rate of convergence of the BTS for American options.

Finding the rate of convergence of the value of security derivatives when the underlying process S is approximated by random walks $S^{(n)}$ is a quite natural problem which has attracted the attention of several researchers in recent years. The case of the BTS is of special interest because of its simplicity, flexibility and immense popularity. Under the BTS, the rate of convergence for European/American put was studied in [17, 18]. Lamberton [14] derived error estimates for the American put which show that the rate of convergence is at least of order $n^{-\frac{2}{3}}$. Heston and Zhou [8] studied the rate of convergence for American/European options and, in particular, its relation to the smoothness of the payoff function and ways to improve the convergence. Lamberton [15] studied the rate of convergence under various assumptions on the distribution of random walks approximating the underlying Brownian motion. In particular, he showed that for the American put under the BTS, the rate of convergence is at least of order $\left(\log (n)\, n^{-1}\right)^{\frac{4}{5}}$. In a remarkable paper, Walsh [20] obtained an explicit formula for the first terms of the error of the BTS for a general class of European options, showing that the order of convergence is n^{-1} for continuous payouts and $n^{-\frac{1}{2}}$ for the discontinuous ones. Carbone [1] studied the rate of convergence of approximations of quantities of the form $E\left[f\left(B_t, \sup_{s\leq t} B_s\right)\right]$ when the Brownian motion B_s is replaced by approximating random walks. Diener and Diener [6] proved that the rate of convergence is of order n^{-1} in the case of the European call, for a generalized BTS.

Studying American perpetuities, Dupuis and Wang [7] showed that, for a class of processes, when the optimal stopping time is chosen amongst those taking values on the grid $\{kT/n : k = 0, 1, 2, \ldots, \infty\}$, the rate of convergence to the option's value is n^{-1}. Kifer [12] gives error estimates for binomial approximations of game options for a class of payoff functions. In [19], it is shown that, under the BTS, the rate of convergence for the American put is, uniformly over the time-space grid, at least $n^{-\frac{1}{3}}$. In [9], it is shown that, indeed, this rate of convergence is optimally $n^{-\frac{1}{2}}$. In [4], it is shown that, in the Skorohod topology, the stopping times of the BTS converge to stopping time of the limiting Black–Scholes model for the American put option. In [3] and [11], modifications of the BTS for smoothing the convergence of European put and call options are developed.

Now suppose that $\varphi(t,x)$ is smooth enough. Thanks to [20, Theorem 4.3], we know that for every t in the interval $iT/n \le t < (i+1)\,T/n$ and for $i = 0, \ldots, n-1$,

$$E_{0,x}\left[\varphi\left(t, S_t^{(n)}\right)\right] = E_{0,x}\left[\varphi(t, S_t)\right] + \mathcal{O}\left(i^{-1}\right) + \mathcal{O}\left(\sqrt{T/n}\right),$$

where the term $\mathcal{O}\big(\sqrt{T/n}\big)$ has been added to cope with the spatial error when t is not a multiple of T/n, [20, page 344], and where $E_{0,x}\big[\varphi(t, S_t^{(n)})\big]$ is the expectation of $\varphi\big(t, S_t^{(n)}\big)$ when $S_0^{(n)} = x$ and $S^{(n)}$ is the stochastic process obtained from the BTS applied to the discounted process. Even if the spatial error term $\mathcal{O}(\sqrt{T/n})$ is ignored —and it cannot be ignored— this still adds up to an error term of order $\mathcal{O}(\log(n)\, n^{-1})$.

In this paper we show that, for CPOs subject to some polynomial boundedness conditions on the piecewise $C^{(2)}$ payout function φ, the rate of convergence of the BTS is n^{-1} for those φ which are continuous, and it is $n^{-\frac{1}{2}}$ if φ is discontinuous.

1.1. Settings

Throughout this paper we assume that $\rho > 0$ is the (constant) risk free rate and that $S = (S, \mathcal{F}, E_{r,x})$ is a geometric Brownian motion with volatility σ and drift ρ under the risk free measure. Here,

$$E_{r,x}\left(f\left(S_t\right)\right) := E\left[\left. f\left(S_t\right)\right| S_r = x\right]$$

and $\mathcal{F}$ is the usual filtration. Thus for every $0 \le r \le t$, every $x \ge 0$, and every measurable function $f \ge 0$,

$$E_{r,x}\left(f\left(S_t\right)\right) = E\left[f\left(x \exp\left(\sigma W_{t-r} + \left(\rho - \frac{1}{2}\sigma^2\right)(t-r)\right)\right)\right]$$

where W is a Brownian motion.

Moreover, $S^{(n)}$ denotes the stochastic process obtained under the BTS applied to the discounted process. In other words, a process $\widetilde{S}^{(n)}$ is constructed using the BTS and $S^{(n)}$ is obtained through the relation $\widetilde{S}_t^{(n)} = e^{-\rho t} S_t^{(n)}$. Therefore, starting at time $t_0 = 0$ and position $\widetilde{S}_0^{(n)} = x$, process $\widetilde{S}^{(n)}$ jumps at each time $t_i^n := i\Delta_n$, $i = 1, 2, \ldots$, from its current state $\widetilde{S}_{t_i^n}^{(n)}$ to the state $\widetilde{S}_{t_{i+1}^n}^{(n)} = a\widetilde{S}_{t_i^n}^{(n)}$ with probability p_u and from its current state to $\widetilde{S}_{t_{i+1}^n}^{(n)} = a^{-1}\widetilde{S}_{t_i^n}^{(n)}$ with probability p_d where

$$\Delta_n := T/n, \quad a := \exp(\sigma\sqrt{\Delta_n}), \quad p_u := (1+a)^{-1} \quad \text{and} \quad p_d := a p_u.$$

Process $S^{(n)}$ is extended between times t_i^n, $0 \le i \le n$, $n = 1, 2, \ldots$, by the formula

$$S_t^{(n)} := \sum_{i=0}^{\infty} 1_{[t_i^n, t_{i+1}^n)}(t)\, S_{t_i^n}^{(n)}.$$

In a slight abuse of notation, we use

$$E_{r,x}\left(f\left(S_t^{(n)}\right)\right) = E\left[f\left(S_t^{(n)}\right)\Big|\, S_r^{(n)} = x\right].$$

For $\varphi : [0,T] \times \mathbb{R}_+ \to \mathbb{R}$, we denote by $v_{\varphi,T}^{\mathcal{E}}(t,x)$ the value at time t of the European option with payout φ and maturity T, knowing that $S_t = x$. Hence,

$$v_{\varphi,T}^{\mathcal{E}}(t,x) := E_{t,x}\left(e^{-\rho(T-t)}\varphi(T, S_T)\right).$$

Similarly, $v_{\varphi,T}^{\mathcal{C}}(0,x)$, given by (1), is the value at time $t = 0$ of the CPO with payout φ and maturity T when $S_0 = x$. Finally,

$$\widehat{v}_{\varphi,T}^{\mathcal{C}}(0,x) := \sum_{i=0}^{n-1} \frac{e^{-\rho t_i^n}}{n} E_{0,x}\left(\varphi(t_i^n, S_{t_i^n})\right)$$

is the Riemann left sum approximation of the integral in $v_{\varphi,T}^{\mathcal{C}}$. The expressions

$$v_{\varphi,T}^{\mathcal{E}(n)}(t,x), \quad v_{\varphi,T}^{\mathcal{C}(n)}(0,x) \quad \text{and} \quad \widehat{v}_{\varphi,T}^{\mathcal{C}(n)}(0,x)$$

are defined in the same manner with $S^{(n)}$ replacing S.

Definition 1.1. We say that a function $f : [0,\infty) \to \mathbb{R}$ belongs to $\mathcal{K}_{N,L}^{(p)}$ if:

(i) There are $N+1$ functions $f_0, \ldots, f_N : \mathbb{R} \to \mathbb{R}$ and $N+1$ disjoint intervals $I_0 := [0, s_1]$, $I_1 := [s_1, s_2]$, $\ldots$, $I_N := [s_N, \infty)$ such that f_k is $C^{(2)}$ and $f(z) = f_k(z)$ for every z in the interior of I_k, $k = 0, \ldots, N$,

(ii) for every $z \ge 0$, $f(z) = \frac{1}{2}\left(f(z+) + f(z-)\right)$,

(iii) for every $z \ge 0$, $|f(z)| + |f'(z)| + |f''(z)| + |zf'(z)| + \left|z^2 f''(z)\right| \le L\left(1 + |z|^p\right)$,

(iv) $\sum_{i=1}^N |\Delta f(s_i)| \le L$, $\sum_{i=1}^N s_i\, |\Delta f'(s_i)| \le L$, $|\Delta f'(s_i)| \le L$ for $1 \le i \le N$

where $\Delta f(s) := f(s+) - f(s-)$ and $\Delta f'(s) := f'(s+) - f'(s-)$ for every real s.

Definition 1.2. We say that a function $\varphi : [0,T] \times \mathbb{R} \to \mathbb{R}$ belongs to $\mathcal{K}_{N,L}^{(p)}$ if the following conditions are satisfied.

(i) for every $0 \le t \le T$, the mapping $z \mapsto \varphi(t,z)$ belongs to $\mathcal{K}_{N,L}^{(p)}$,

(ii) $|\varphi(t,x) - \varphi(s,x)| \le L\,|t-s|$ for every $0 \le t, s \le T$ and every $x \ge 0$, that is, φ is uniformly in x Lipschitz in t with constant L.

Remark 1.3. Whenever needed, functions φ mapping $\mathbb{R}_+$ into $\mathbb{R}$ will be regarded as functions mapping $[0,T] \times \mathbb{R}_+$ into $\mathbb{R}$, with the trivial extension $\varphi(t,x) := \varphi(x)$, for every (t,x) in $[0,T] \times \mathbb{R}_+$.

Remark 1.4. Throughout this paper, p, N, L, T, σ and ρ are fixed. Expressions in terms of these parameters —and unless otherwise indicated only such expressions— will be referred to as constants.

1.2. Outline

In Section 2, a bound for the error between the value $v^{\mathcal{C}}_{\varphi,T}$ of a CPO and its approximation $\widehat{v}^{\mathcal{C}}_{\varphi,T}$ is found. Section 3 studies the error $v^{\mathcal{E}(n)}_{f,t} - v^{\mathcal{E}}_{f,t}$ as the maturity $t \to 0$. In Section 4, our main result is established: the rate of convergence of CPOs under the BTS. Section 5 contains auxiliary results.

2. Riemann sum approximations

Lemma 2.1. *For every $\beta > 0$, there exists a constant K_β such that*

$$\sup_{n\geq 1}\left(\sum_{i=1}^{n-1}\frac{1}{n^\beta i^{1-\beta}}\right) < K_\beta. \tag{2}$$

Proof. Let $\beta > 0$. We have

$$\beta^{-1} = \int_0^1 t^{\beta-1}dt = \lim_{n\to\infty}\sum_{i=1}^{n-1}\left(\frac{1}{n}\left(\frac{i}{n}\right)^{\beta-1}\right) = \lim_{n\to\infty}\sum_{i=1}^{n-1}\left(\frac{1}{n^\beta i^{1-\beta}}\right),$$

and therefore there exists a constant $K_\beta > 0$ such that (2) holds. □

Theorem 2.2. *There exists a constant $K \geq 0$ such that for every $\varphi \in \mathcal{K}^{(p)}_{N,L}$ and every $x \geq 0$ the following holds:*

(i) $\left|v^{\mathcal{C}}_{\varphi,T}(0,x) - \widehat{v}^{\mathcal{C}}_{\varphi,T}(0,x)\right| \leq \frac{K}{n}\left(1 + x^{p+1}\right)$,

(ii) $\left|v^{\mathcal{C}(n)}_{\varphi,T}(0,x) - \widehat{v}^{\mathcal{C}(n)}_{\varphi,T}(0,x)\right| \leq \frac{K}{n}$.

Proof. Let $\widehat{\varepsilon}^{\mathcal{C}}_{\varphi,T}(0,x) := v^{\mathcal{C}}_{\varphi,T}(0,x) - \widehat{v}^{\mathcal{C}}_{\varphi,T}(0,x)$ be the error between $v^{\mathcal{C}}_{\varphi,T}$ and its Riemann approximation $\widehat{v}^{\mathcal{C}}_{\varphi,T}$. Then

$$\begin{aligned}\widehat{\varepsilon}^{\mathcal{C}}_{\varphi,T}(0,x) &= \sum_{k=0}^{n-1}\int_{t^n_k}^{t^n_{k+1}} E_{0,x}\left(e^{-\rho t}\varphi\left(t,S_t\right) - e^{-\rho t^n_k}\varphi\left(t,S_t\right)\right)dt \\ &\quad + \sum_{k=0}^{n-1} e^{-\rho t^n_k}\int_{t^n_k}^{t^n_{k+1}} E_{0,x}\left(\varphi\left(t,S_t\right) - \varphi\left(t^n_k, S_{t^n_k}\right)\right)dt.\end{aligned}$$

Using the fact that $\varphi \in \mathcal{K}^{(p)}_{N,L}$ and Corollary 5.5, we get that, for constants K_1, K_2,

$$\begin{aligned}\left|\widehat{\varepsilon}^{\mathcal{C}}_{\varphi,T}(0,x)\right| &\leq \sum_{k=0}^{n-1}\int_{t^n_k}^{t^n_{k+1}} \rho L\left(t - t^n_k\right)E_{0,x}\left(1 + S^p_t\right)dt \\ &\quad + \sum_{k=1}^{n-1}\int_{t^n_k}^{t^n_{k+1}} K_1\left(1 + x^{p+1}\right)\frac{\left(t - t^n_k\right)}{\sqrt{t^n_k}}dt \\ &\quad + \left|\int_0^{t^n_1} E_{0,x}\varphi\left(t,S_t\right)dt\right| + \left|\frac{T}{n}\varphi\left(0,x\right)\right|.\end{aligned}$$

Since

$$\sup_{0\le t\le T} E_{0,x}\varphi\left(t,S_t\right) \le \sup_{0\le t\le T} LE_{0,x}\left(1+S_t^p\right) \le K_2\left(1+x^p\right)$$

for some constant K_2, we get that, for some constant K_3,

$$\begin{aligned}\left|\widehat{\varepsilon}^{\mathcal{C}}_{\varphi,T}(0,x)\right| &\le K_2\left(1+x^p\right)\sum_{k=1}^{n-1}\frac{\rho L}{2}\left(\frac{T}{n}\right)^2 \\ &\quad +K_2\left(1+x^{p+1}\right)\left(\frac{1}{2}\sum_{k=1}^{n-1}\left(\frac{T}{n}\right)^2\sqrt{\frac{n}{Tk}}+\frac{2T}{n}\right) \\ &\le K_3\left(1+x^{p+1}\right)n^{-1}\left(1+\sum_{k=1}^{n-1}\frac{1}{\sqrt{n}\sqrt{k}}\right).\end{aligned}$$

Lemma 2.1 yields that $|\widehat{\varepsilon}^{\mathcal{C}}_{\varphi,T}(0,x)| \le K_4\left(1+x^{p+1}\right)n^{-1}$ for some K_4, proving (i).

As for (ii), it is derived in the same manner except that, instead of invoking Corollary 5.5, one uses the fact that $S_t^{(n)} = S_{t_i^n}^{(n)}$ for every $t \in [t_i^n t_{i+1}^n)$, $i \ge 0$, and the Lipschitz property of φ from which

$$\left|E_{0,x}\left(\varphi\left(t,S_t^{(n)}\right)-\varphi\left(t_i^n,S_{t_i^n}^{(n)}\right)\right)\right| \le L\left(t-t_i^n\right)$$

for $t_i^n \le t \le t_{i+1}^n$, $i=0,1,\ldots,n-1$. □

3. Convergence rate as the maturity goes to zero

For continuous $f \in \mathcal{K}^{(p)}_{N,L}$ and integer $n \ge 1$, we provide here an upper bound, up to a term of order $n^{-\frac{3}{2}}$, for the convergence rate of $v^{\mathcal{E}(n)}_{f,t} - v^{\mathcal{E}}_{f,t}$ as $t \to 0$.

Theorem 3.1. *Let $N, L, p \ge 0$. There exists a constant $K \ge 0$ such that for every $0 \le t \le T$, every $f \in \mathcal{K}^{(p)}_{N,L}$, every $x \ge 0$, and every integer $n \ge 1$, the error $\varepsilon_f(t,x) = v^{\mathcal{E}(n)}_{f,t}(0,x) - v^{\mathcal{E}}_{f,t}(0,x)$ satisfies*

$$|\varepsilon_f(t,x)| \le K\left(\sqrt[4]{t}n^{-1}+(\mathfrak{1}-\mathfrak{c})\,n^{-\frac{1}{2}}+n^{-\frac{3}{2}}\right)\left(1+x^{2p+2}\right) \tag{3}$$

where $\mathfrak{c}=1$ if f is continuous and $\mathfrak{c}=0$ otherwise.

Proof. Using the notation in Theorem 5.1 and Remark 5.2, we write

$$\varepsilon_f(t,x) = C_f(t,x)+D_f(t,x)+J_f(t,x)+H_f(t,x)+\mathcal{O}\left(n^{-\frac{3}{2}}\right),$$

where

$$\left|\mathcal{O}\left(n^{-\frac{3}{2}}\right)\right| \le C\left(1+x^p\right)n^{-\frac{3}{2}},$$

for some constant $C > 0$. Note that for every $t \in [0,T]$, $\sqrt{t} \le \sqrt[4]{T}\sqrt[4]{t}$, and for every $x \ge 0$ and $0 \le p \le q$,

$$1+x^p \le 2\left(1+x^q\right).$$

Hence to prove (3) we just need to show that there exists a constant $K > 0$ such that, for every f in $\mathcal{K}_{N,L}^{(p)}$,

$$|C_f(t,x)| \leq K\left(\mathfrak{c}\sqrt[4]{t}n^{-1} + (1-\mathfrak{c})\,n^{-\frac{1}{2}}\right)\left(1+x^{2p+1}\right), \tag{4}$$

$$|D_f(t,x)| \leq K\left(\mathfrak{c}\sqrt{t}n^{-1} + (1-\mathfrak{c})\,n^{-\frac{1}{2}}\right)\left(1+x^{2p+2}\right), \tag{5}$$

$$|J_f(t,x)| \leq K\left(\mathfrak{c}\sqrt{t}n^{-1} + (1-\mathfrak{c})\,n^{-\frac{1}{2}}\right)(1+x^{p}), \tag{6}$$

$$|H_f(t,x)| \leq K\left(\mathfrak{c}\sqrt{t}n^{-1} + (1-\mathfrak{c})\,n^{-\frac{1}{2}}\right)(1+x^{p}). \tag{7}$$

This is what is done in parts A-C below.

Part (A). Recall $C_f(t,x)$, Y_t and Z_t from (22), (17) and (18). Clearly, since Y_t and Z_t are linear combinations of powers of $w_t \sim N(0,1)$, one easily finds a constant $K_1 > 0$ such that, for every $0 \leq t \leq T$,

$$\sqrt{E_{0,x}(Y_t^2)} + \sqrt{E_{0,x}(Z_t^2)} \leq K_1.$$

Since $f \in \mathcal{K}_{N,L}^{(p)}$ we can also choose K_1 big enough such that

$$\begin{aligned}
\sqrt{E_{0,x}(f^2(S_t))} &\leq K_1(1+x^p),\\
e^{-\rho t}\frac{\sqrt{t}}{n}E_{0,x}\{Y_t f(S_t)\} &\leq \frac{\sqrt{t}}{n}\sqrt{E_{0,x}(Y_t^2)}\sqrt{E_{0,x}(f^2(S_t))}\\
&\leq \frac{\sqrt{t}}{n}(K_1)^2(1+x^p),\\
e^{-\rho t}\frac{\sqrt{t}}{n}E_{0,x}\{Y_t f(S_t)\} &\leq \frac{2\sqrt[4]{T}\sqrt[4]{t}}{n}(K_1)^2(1+x^p).
\end{aligned} \tag{8}$$

If $\mathfrak{c} = 0$, then (4) clearly follows. If $\mathfrak{c} = 1$, first note that $E_{0,x}\{Z_t\} = 0$, which can easily be verified since Z_t is a linear combination of powers of $w_t \sim N(0,1)$. Hence, according to Corollary 5.7, there exists $K_2 > 0$ such that

$$e^{-\rho t}E_{0,x}\{Z_t f(S_t)\} \leq E_{0,x}\{Z_t f(S_t)\} \leq K_2\sqrt[4]{t}\left(1+x^{2p+1}\right). \tag{9}$$

Putting (8) and (9) together obviously gives (4).

Part (B). Inequality (5) follows from the fact that, for some $K_3, K_4, K_5 > 0$,

$$D_f(t,x) \leq \frac{t}{n}K_3E_{0,x}\left\{S_t^2\left(1+S_t^p\right)\right\} \leq \frac{t}{n}K_4\left(x^2+x^{p+2}\right) \leq \frac{\sqrt{t}}{n}K_5\left(1+x^{2p+2}\right).$$

Part (C). Note that when f is continuous, $J_f(t,x) = 0$, for every $t, x > 0$. Hence (6) and (7) follow trivially from Lemma 5.3. □

4. The rate of convergence of CPOs under the BTS

In this section we prove our main result.

Theorem 4.1. (i) *There exists a constant* $K \geq 0$ *such that for every* $\varphi \in \mathcal{K}_{N,L}^{(p)}$ *and every* $x \geq 0$,

$$\left| v_{\varphi,T}^{\mathcal{C}(n)}(0,x) - v_{\varphi,T}^{\mathcal{C}}(0,x) \right| \leq K \left(1 + x^{2p+2}\right) \left(\frac{\mathfrak{c}}{n} + \frac{(1-\mathfrak{c})}{\sqrt{n}} \right), \tag{10}$$

where $\mathfrak{c} = 1$ *when* φ *is continuous and* $\mathfrak{c} = 0$ *otherwise.*

(ii) *The rate of convergence in (10) is sharp.*

Proof. (i) Note that $\widehat{v}_{\varphi,T}^{\mathcal{C}}$ and $\widehat{v}_{\varphi,T}^{\mathcal{C}(n)}$ are just the average value of the European options whose maturity ranges throught the grid times t_i^n, $i = 0, \dots, n-1$. Indeed

$$\widehat{v}_{\varphi,T}^{\mathcal{C}}(0,x) = \sum_{i=0}^{n-1} \frac{1}{n} v_{\varphi,t_i^n}^{\mathcal{E}}(0,x) \text{ and } \widehat{v}_{\varphi,T}^{\mathcal{C}(n)}(0,x) = \sum_{i=0}^{n-1} \frac{1}{n} v_{\varphi,t_i^n}^{\mathcal{E}(n)}(0,x).$$

Hence, since $v_{\varphi,t_0^n}^{\mathcal{E}}(0,x) = \varphi(0,x)$ and $v_{\varphi,t_0^n}^{\mathcal{E}(n)}(0,x) = \varphi(0,x)$,

$$\left| \widehat{v}_{\varphi,T}^{\mathcal{C}(n)}(0,x) - \widehat{v}_{\varphi,T}^{\mathcal{C}}(0,x) \right| \leq \sum_{i=1}^{n-1} \frac{1}{n} \left| v_{\varphi,t_i^n}^{\mathcal{E}(n)}(0,x) - v_{\varphi,t_i^n}^{\mathcal{E}}(0,x) \right|. \tag{11}$$

Now, from Theorem 3.1, there exists $K_1 > 0$ such that, for $i = 1, \dots, n-1$,

$$\left| v_{\varphi,t_i^n}^{\mathcal{E}(n)}(0,x) - v_{\varphi,t_i^n}^{\mathcal{E}}(0,x) \right| \leq K_1 \left(\mathfrak{c} \sqrt[4]{t_i^n} i^{-1} + \mathfrak{d} i^{-\frac{1}{2}} + i^{-\frac{3}{2}} \right) \left(1 + x^{2p+2}\right)$$

where $\mathfrak{d} = 1 - \mathfrak{c}$. Hence, replacing this expression in (11) gives

$$\begin{aligned} \left| \widehat{v}_{\varphi,T}^{\mathcal{C}(n)}(0,x) - \widehat{v}_{\varphi,T}^{\mathcal{C}}(0,x) \right| \leq\ & K_1 \left(1 + x^{2p+2}\right) \frac{\mathfrak{c}}{n} \sum_{i=1}^{n-1} \sqrt[4]{t_i^n} i^{-1} \\ & + K_1 \left(1 + x^{2p+2}\right) \frac{1-\mathfrak{c}}{\sqrt{n}} \sum_{i=1}^{n-1} \frac{i^{-\frac{1}{2}}}{\sqrt{n}} \\ & + K_1 \left(1 + x^{2p+2}\right) \frac{1}{n} \sum_{i=1}^{n-1} i^{-\frac{3}{2}}. \end{aligned} \tag{12}$$

From Lemma 2.1, there exists $K_2 > 0$ such that for every $n > 0$,

$$\sum_{i=1}^{n-1} \sqrt[4]{t_i^n} i^{-1} = \sum_{i=1}^{n-1} \frac{\left(\frac{iT}{n}\right)^{\frac{1}{4}}}{i} = \sum_{i=1}^{n-1} \frac{T^{\frac{1}{4}}}{n^{\frac{1}{4}} i^{1-\frac{1}{4}}} \leq K_2,$$

$$\sum_{i=1}^{n-1} \frac{i^{-\frac{1}{2}}}{\sqrt{n}} = \sum_{i=1}^{n-1} \frac{1}{n^{\frac{1}{2}} i^{1-\frac{1}{2}}} \leq K_2.$$

Using this in (12) gives that, for some constant $K_3 > 0$,

$$\left| \widehat{v}_{\varphi,T}^{\mathcal{C}(n)}(0,x) - \widehat{v}_{\varphi,T}^{\mathcal{C}}(0,x) \right| \leq K_3 \left(1 + x^{2p+2}\right) \left(\frac{\mathfrak{c}}{n} + \frac{1-\mathfrak{c}}{\sqrt{n}} \right). \tag{13}$$

Finally, from Theorem 2.2, there exists constants K_4, K_5 such that

$$\left|\widehat{v}_{\varphi,T}^{\mathcal{C}(n)}(0,x) - v_{\varphi,T}^{\mathcal{C}(n)}(0,x)\right| \leq \frac{K_4}{n}\left(1 + x^{p+1}\right), \tag{14}$$

$$\left|\widehat{v}_{\varphi,T}^{\mathcal{C}}(0,x) - v_{\varphi,T}^{\mathcal{C}}(0,x)\right| \leq \frac{K_5}{n}\left(1 + x^{p+1}\right). \tag{15}$$

Since $\left(1 + x^{p+1}\right) \leq 2\left(1 + x^{2p+2}\right)$, (13), (14) and (15) yield (10).

(ii) The optimality of the rate of convergence can be easily verified with

$$\varphi(t,x) = \max\left(t - \frac{T}{2}, 0\right)x^2.$$

This is intuitively clear because the error can be decomposed into an average of European option errors which are known, thanks to [20, Theorem 4.3], to be optimally of order n^{-1}. □

5. Appendix

5.1. A version of Walsh's Theorem

Following Walsh [20], we use the notation $h := \sigma\sqrt{\frac{t}{n}}$ and let $\mathbb{N}_e^h := 2h\mathbb{Z}$ be the set of all even multiples of h, and $\mathbb{N}_o^h := h + \mathbb{N}_e^h$ the set of all odd multiples of h. We let $frac(z)$ denote the fractional part of z and set $x := S_0$. For every s, we set $\widetilde{s} := e^{-\rho t}s$ and, in particular, $\widetilde{S}_t := e^{-\rho t}S_t$. Additionally, let $w_t := \frac{W_t}{\sqrt{t}}$ and

$$X_t := \log\left(e^{-\rho t}S_t/x\right) = \sigma W_t - \frac{1}{2}\sigma^2 t, \tag{16}$$

$$Y_t := \frac{1}{6}\sigma w_t + \frac{1}{8}\sigma^2\sqrt{t} + \frac{1}{6}\sigma w_t^3 - \frac{1}{8}\sigma^2\sqrt{t}w_t^2 + \frac{1}{24}\sigma^3 t w_t, \tag{17}$$

$$Z_t := \frac{5}{12} - \frac{1}{6}w_t^2 - \frac{1}{12}w_t^4, \tag{18}$$

$$\theta(s) := frac\left(\log(s)(2h)^{-1}\right), \tag{19}$$

$$\vartheta(s) := 2\theta(s) - 1, \tag{20}$$

$$\Theta(s) := \frac{1}{3} + 2\theta(s)(1 - \theta(s)). \tag{21}$$

Finally, we consider the following expressions:

$$C_f(t,x) := e^{-\rho t}\frac{\sqrt{t}}{n}E_{0,x}\left\{Y_t f(S_t)\right\} + e^{-\rho t}\frac{1}{n}E_{0,x}\left\{Z_t f(S_t)\right\}, \tag{22}$$

$$D_f(t,x) := e^{-\rho t}\frac{t}{n}\frac{2}{3}\sigma^2 E_{0,x}\left\{S_t^2 f''(S_t)\right\}, \tag{23}$$

$$\widehat{p}(z,t) := \left(2\pi\sigma^2 t\right)^{-\frac{1}{2}}\exp\left(-\frac{(z + \frac{1}{2}\sigma^2 t)^2}{2\sigma^2 t}\right), \tag{24}$$

$$H_f(t,x) := e^{-\rho t}\frac{1}{n}\sigma^2 t\sum_i \left(s_i\Delta f'(s_i)\right)\Theta\left(\widetilde{s}_i/x\right)\widehat{p}(\log(\widetilde{s}_i/x), t), \tag{25}$$

$$J_f(t,x) := e^{-\rho t}\frac{1}{n}\left[-\sigma^2 t\sum_i \Delta f(s_i)\Theta\left(\widetilde{s}_i/x\right)\widehat{p}(\log(\widetilde{s}_i/x),t)\right.$$
$$-\frac{1}{3}\sum_{i:log(\widetilde{s}_i/x)\in\mathbb{N}_e^h}\log(\widetilde{s}_i/x)\Delta f(s_i)\widehat{p}(\log(\widetilde{s}_i/x),t)$$
$$\left.+\frac{1}{6}\sum_{i:log(\widetilde{s}_i/x)\in\mathbb{N}_o^h}\log(\widetilde{s}_i/x)\Delta f(s_i)\widehat{p}(\log(\widetilde{s}_i/x),t)\right]$$
$$+e^{-\rho t}\frac{\sqrt{t}}{\sqrt{n}}\sum_{i:log(\widetilde{s}_i/x)\notin\mathbb{N}_o^h}\vartheta\left(\widetilde{s}_i/x\right)\Delta f(s_i)\widehat{p}(\log(\widetilde{s}_i/x),t). \tag{26}$$

Using the above notation we can state our version of [20, Theorem 4.3].

Theorem 5.1. *Let $N, L, p \geq 0$. For every $f : \mathbb{R} \longrightarrow \mathbb{R}$ in $\mathcal{K}_{N,L}^{(p)}$,*

$$\varepsilon_f(t,x) = C_f(t,x) + D_f(t,x) + J_f(t,x) + H_f(t,x) + \mathcal{O}\left(n^{-\frac{3}{2}}\right) \tag{27}$$

where $\varepsilon_f(t,x) = v_{f,t}^{\mathcal{E}(n)}(0,x) - v_{f,t}^{\mathcal{E}}(0,x)$.

Proof. This is a mere rearrangement of the terms in [20, Theorem 4.3]. Indeed, upon defining

$$V_t := \left(\frac{5}{12}+\sigma^2\frac{t}{6}+\frac{\sigma^4t^2}{192}\right)-\frac{1}{6}\left(\frac{(X_t)^2}{\sigma^2 t}\right)-\frac{1}{12}\left(\frac{(X_t)^4}{\sigma^4t^2}\right), \tag{28}$$

then, apart from the use of a compact notation here, the error $\varepsilon_f(t,x)$ in [20, Theorem 4.3] is precisely expressed, up to a $\mathcal{O}(n^{-\frac{3}{2}})$ term, as

$$\varepsilon_f(t,x) = \frac{e^{-\rho t}}{n}E_{0,x}\left\{V_t f\left(S_t\right)\right\} + D_f(t,x) + J_f(t,x) + H_f(t,x). \tag{29}$$

By substituting $X_t = \sigma\sqrt{t}\left(\frac{W_t}{\sqrt{t}}\right) - \frac{1}{2}\sigma^2 t$ into (28), expanding the resulting expressions for $(X_t)^2$ and $(X_t)^4$, and rearranging the terms, it is trivial (especially with the help of a computer algebra system!) to verify that (27) is a mere rearrangement of the terms in [20, Theorem 4.3]. □

Remark 5.2. Intuitively, as the maturity $t \searrow 0$, the error $\varepsilon_f(t,x) \searrow 0$. And unsurprisingly, the proof of [20, Theorem 4.3] reveals that the term $\mathcal{O}(n^{-\frac{3}{2}})$ is uniform in t and $f \in \mathcal{K}_{N,L}^{(p)}$ and that indeed, there exists a constant $K > 0$ such that for every $0 \leq t \leq T$, every $f \in \mathcal{K}_{N,L}^{(p)}$ and every $x \geq 0$,

$$\left|\mathcal{O}\left(n^{-\frac{3}{2}}\right)\right| \leq K\left(1+x^p\right)n^{-\frac{3}{2}}.$$

Lemma 5.3. *Let $N, L, p \geq 0$ and let H_f and J_f be defined by (25) and (26). There exists a constant $K \geq 0$ such that for every $0 \leq t \leq T$, every $f \in \mathcal{K}_{N,L}^{(p)}$, every $x \geq 0$, and every integer $n \geq 1$, $|J_f(t,x)| \leq Kn^{-\frac{1}{2}}$ and $|H_f(t,x)| \leq K\sqrt{t}n^{-1}$.*

Proof. From (24) one easily gets

$$\sqrt{t}\,\widehat{p}(\log(\widetilde{s}_i/x),t) \leq \frac{1}{\sqrt{2\pi\sigma^2}}.$$

From (19) and (21) it is obvious that $|(2\theta(\widetilde{s}_i/x)-1)| \leq 2$ and $|\Theta(\widetilde{s}_i/x)| \leq 3$. Note that, with $u = zt^{-\frac{1}{2}}$, one can write

$$|z\widehat{p}(z,t)| = |u|\left(2\pi\sigma^2\right)^{-\frac{1}{2}}\exp\left(-\left(u+\frac{1}{2}\sigma^2\sqrt{t}\right)^2\left(2\pi\sigma^2\right)^{-1}\right) \leq K_1,$$

for some constant K_1. Hence $|\log(\widetilde{s}_i/x)\widehat{p}(\log(\widetilde{s}_i/x),t)| \leq K_1$, for every $t,x>0$ and for $i=0,\ldots,N$. By definition of $\mathcal{K}_{N,L}^{(p)}$, $\sum_i|\Delta f(s_i)| \leq L$ and $\sum_i|s_i\Delta f'(s_i)| \leq L$. Using these bounds in (26) and (25) one obtains

$$\begin{aligned}
|J_f(t,x)| &\leq e^{-\rho t}\frac{1}{n}\left[\sigma^2\sqrt{t}\sum_i\frac{3\,|\Delta f(s_i)|}{\sqrt{2\pi\sigma^2}}+\frac{1}{3}\sum_i K_1\,|\Delta f(s_i)|\right]\\
&\quad+e^{-\rho t}\frac{1}{\sqrt{n}}\sum_i 2\frac{|\Delta f(s_i)|}{\sqrt{2\pi\sigma^2}}\\
&\leq K_2\frac{1}{\sqrt{n}},\\
|H_f(t,x)| &\leq e^{-\rho t}\frac{1}{n}\sigma^2\sqrt{t}\sum_i\frac{|s_i\Delta f'(s_i)|}{\sqrt{2\pi\sigma^2}}3 \leq \frac{\sqrt{t}}{n}K_3,
\end{aligned}$$

for some constants K_2 and K_3. □

5.2. Bound estimates for the derivatives of European options

Proposition 5.4. *Let $N,L,p \geq 0$. There exists a constant K independent of L such that for every $0<t\leq T$, every $x\geq 0$ and every $f\in\mathcal{K}_{N,L}^{(p)}$,*

$$\left|\frac{d}{dt}E_{0,x}\left(f\left(S_t\right)\right)\right| \leq \frac{KL\left(1+x^{p+1}\right)}{\sqrt{t}}. \tag{30}$$

Proof. Let $\zeta(t,z) := \exp\left(\sigma\sqrt{t}z+ct\right)$ where $c := \rho-\frac{1}{2}\sigma^2$. Then $E_{0,x}\left(f\left(S_t\right)\right)$ can be written as

$$E_{0,x}\left(f\left(S_t\right)\right) = \int_{-\infty}^{\infty} f\left(x\zeta\left(t,z\right)\right)\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}\,dz$$

and if f is continuous, one verifies that

$$\begin{cases}
\frac{d}{dt}E_{0,x}\left(f\left(S_t\right)\right) & = \int_{-\infty}^{\infty}\frac{d}{dt}f\left(x\zeta\left(t,z\right)\right)\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}\,dz,\\
\frac{d}{dt}f\left(x\zeta\left(t,z\right)\right) & = f'\left(x\zeta\left(t,z\right)\right)\frac{1}{2}x\left(\sigma z+2\sqrt{t}c\right)\frac{\zeta(t,z)}{\sqrt{t}},\\
\left|\frac{d}{dt}f\left(x\zeta\left(t,z\right)\right)\right| & \leq L\left(1+x^p\zeta^p\left(t,z\right)\right)\frac{1}{2}x\left(\sigma\left|z\right|+2\sqrt{t}c\right)\frac{\zeta(t,z)}{\sqrt{t}}.
\end{cases}$$

A simple calculation yields (30) for a constant K independent of L.

Now if f is discontinuous, it can be written as the sum of a continuous function f_c in $\mathcal{K}_{N,L}^{(p)}$ and a piecewise constant function f_d where

$$f_d(x) = -\sum_{i=1}^{N} \Delta f(s_i) 1_{(-\infty,s_i]}(x) \quad \text{and} \quad f_c(x) = f(x) - f_d(x).$$

For piecewise constant functions, one easily verifies that, for some constant K_2 independent of L and $s_1, \ldots, s_N$,

$$\left| \frac{d}{dt} E_{0,x}(f_d(S_t)) \right| \leq \sum_{i=1}^{N} |\Delta f(s_i)| \frac{K_2}{\sqrt{t}} \leq L \frac{K_2}{\sqrt{t}} \leq \frac{K_2 L \left(1 + x^{p+1}\right)}{\sqrt{t}}$$

for every $x > 0$. □

Corollary 5.5. *Let* $N, L, p \geq 0$. (i) *There exists a constant* K *independent of* L *such that for every* $0 < t \leq t + h \leq T$*, every* $x \geq 0$ *and every* $\varphi \in \mathcal{K}_{N,L}^{(p)}$,

$$|E_{0,x}(\varphi(t+h, S_{t+h}) - \varphi(t, S_t))| \leq KL\left(1 + x^{p+1}\right) \frac{h}{\sqrt{t}}.$$

(ii) *There exists a constant* K *independent of* L *such that for every* $0 < h \leq T$*, every* $x \geq 0$ *and every continuous* $\varphi \in \mathcal{K}_{N,L}^{(p)}$,

$$|E_{0,x}(\varphi(h, S_h) - \varphi(0, x))| \leq KL\left(1 + x^{p+1}\right) \sqrt{h}.$$

Proof. (i) First note that

$$|E_{0,x}(\varphi(t+h, S_{t+h}) - \varphi(t, S_{t+h}))| \leq Lh$$

for every $0 < t \leq t + h \leq T$. Let $f(x) = \varphi(t, x)$ for every $x \geq 0$. Thanks to Proposition 5.4,

$$\left| \frac{d}{dt} E_{0,x}(f(S_t)) \right| \leq KL\left(1 + x^{p+1}\right) t^{-\frac{1}{2}}$$

for some constant K, independent of t, h, x and φ. Thus there exists $t^* \in (t, t+h)$ such that

$$|E_{0,x}(f(S_{t+h}) - f(S_t))| \leq \frac{KL\left(1 + x^{p+1}\right)}{\sqrt{t^*}} h \leq \frac{KL\left(1 + x^{p+1}\right)}{\sqrt{t}} h,$$

which yields the result.

(ii) Fix $0 < h \leq T$ and note that

$$|E_{0,x}(\varphi(h, S_h) - \varphi(0, S_h))| \leq Lh.$$

Since the mapping $t \to E_{0,x}(\varphi(0, S_t))$ is continuous in $[0, T]$ and differentiable in $(0, T]$,

$$E_{0,x}(\varphi(0, S_h)) - E_{0,x}(\varphi(0, S_0)) = \int_0^h \frac{d}{dt} E_{0,x}(f(S_t))\, dt.$$

Thanks to Proposition 5.4, there exists a constant K such that

$$\int_0^h \left| \frac{d}{dt} E_{0,x}(f(S_t)) \right| dt \leq \int_0^h \frac{KL\left(1 + x^{p+1}\right)}{\sqrt{t}} dt,$$

from which the result follows. □

Lemma 5.6. *For every $x > 0$ and h in $\mathcal{K}_{N,L}^{(p)}$, the function $g(z) := (h(z) - h(x))^2$ belongs to $\mathcal{K}_{N,L_x}^{(2p)}$ where $L_x := 54L^2(1+x^p)^2$.*

Proof. Thanks to the Lipschitz property of φ, one easily obtains that, for every real number z,

$$g(z) \leq 18L^2(1+x^p)^2\left(1+z^{2p}\right),$$

$$g'(z) \leq 18L^2(1+x^p)^2\left(1+z^{2p}\right)$$

and

$$g''(z) \leq 18L^2(1+x^p)^2\left(1+z^{2p}\right),$$

from which the result follows. □

Corollary 5.7. *Let $N, L, p \geq 0$ and let Z_t be a stochastic process such that, for every $t, x \geq 0$,*

(i) $E_{0,x}(Z_t) = 0$

(ii) $\sup_{0\leq t\leq T} E_{0,x}\left(Z_t^2\right) < \infty$.

Then there exists a constant $K > 0$ such that for every continuous h in $\mathcal{K}_{N,L}^{(p)}$, and every $0 < x$, $0 < t \leq T$, we have $|E_{0,x}(Z_t h(S_t))| \leq LK\sqrt[4]{t}\left(1+x^{2p+1}\right)$.

Proof. Fix $x > 0$ and h in $\mathcal{K}_{N,L}^{(p)}$. Because $E_{0,x}(Z_t) = 0$, we have that

$$E_{0,x}(Z_t h(S_t)) = E_{0,x}(Z_t(h(S_t) - h(x))). \tag{31}$$

Define the function $z \mapsto g(z) := (h(z) - h(x))^2$. From Schwarz's inequality,

$$|E_{0,x}(Z_t(h(z) - h(x)))| \leq \sqrt{E_{0,x}\left(Z_t^2\right)}\sqrt{E_{0,x}(g(S_t))}. \tag{32}$$

But according to Lemma 5.6, $g \in \mathcal{K}_{N,L_x}^{(2p)}$ where $L_x := 54L^2(1+x^p)^2$. Clearly there exists a constant K_1 independent of L such that $L_x \leq K_1L^2\left(1+x^{2p+1}\right)$. Note that $g(x) = 0$. Hence, thanks to Corollary 5.5, there exists a constant K_2 such that

$$\begin{aligned}\sqrt{E_{0,x}(g(S_t))} &= \sqrt{|E_{0,x}(g(S_t)) - g(x)|}\\ &\leq \sqrt{L_xK_2\sqrt{t}\,(1+x^{2p+1})}\\ &= \sqrt{K_1L^2K_2\sqrt{t}\,(1+x^{2p+1})^2}\\ &\leq L\sqrt{K_1K_2}\sqrt[4]{t}\left(1+x^{2p+1}\right).\end{aligned}$$

Thus, with (31) and (32), this shows that there exists a constant K independent of L such that $|E_{0,x}(Z_t h(S_t))| \leq LK\sqrt[4]{t}\left(1+x^{2p+1}\right)$ as wanted. □

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