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THE MÖBIUS CATEGORY OF A COMBINATORIAL INVERSE MONOID WITH ZERO

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RÉSUMÉ. Diverses propriétés (y compris les évaluations des fonctions de Möbius) et les liens entre les monoïdes avec inverses combinatoires 0-localement finis et les catégories de division de quasi-Möbius sont aisément déduits. Le monoïde polycyclique et le monoïde de McAlister sont étudiés par l'intermédiaire des catégories de division quasi-Möbius. La description du monoïde de McAlister impliquant le produit fibré diffère des descriptions de Lawson et Munn.

ABSTRACT. Various properties (including evaluations of Möbius functions) and connections between 0-locally finite combinatorial inverse monoids and quasi-Möbius-division categories are readily deduced. The polycyclic monoid and the McAlister monoid are investigated via quasi-Möbius-division categories. Our description of McAlister's monoid involving the pushout product differs from Lawson's and Munn's descriptions.

1. Introduction

According to Leech [11], an *abstract division category* is a pair (D, I), where D is a small category having finite pushouts, all of whose morphisms are epimorphisms, and I is an object of D (called the *quasi initial object*) such that for each object X in D, $Hom(I, X) \neq \varnothing$. A Loganathan and Leech category associated with an inverse monoid ([2, Chapter VII, Section 8]) is, in particular, a division category. So, for each inverse monoid S, there is a standard division category $(C(S), 1)$ defined by:

- $Ob(C(S)) = E(S)$, where $E(S)$ is the set of all idempotents of the inverse monoid S;

- $Hom(e, f) = \{(s, e) \in S \times E(S) \mid s^{-1}s \leq e \text{ and } ss^{-1}= f\}$, where $\leq$ is the natural partial order on an inverse semigroup;

- The composition of two morphisms is given by

$$g \xrightarrow{(t,g)} e \xrightarrow{(s,e)} f = g \xrightarrow{(st,e)} f;\ (s, e)(t, g) = (st, e).$$

All morphisms of $C(S)$ are epimorphisms, the quasi initial object is the identity element 1 of S and the square $[(s, e), (t, e), (t^{-1}ts^{-1}, ss^{-1}), (s^{-1}st^{-1}, tt^{-1})]$ is a pushout.

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Sometimes, instead of considering a standard division category of an inverse monoid S, it is preferable to consider another division category of S. In their application of groupoids of fractions to inverse semigroups, James and Lawson [3] use, for the connection between inverse monoids and division categories, the reduced standard division category of an inverse monoid. If S is an inverse monoid and F is an idempotent transversal of the $\mathcal{D}$-classes of S such that $1 \in F$, then the full subcategory $C_F(S)$ of the standard division category $C(S)$ defined by $Ob(C_F(S)) = F$ is a division category (with 1 as the quasi initial object) called the *reduced standard division category of S relative to the idempotent transversal F*. If F and F' are two idempotent transversals defined as above then $C_F(S)$ and $C_{F'}(S)$ are isomorphic. Furthermore, there exists an equivalence of division categories $(C(S), 1) \approx (C_F(S), 1)$; see [3].

The concept of Möbius category was introduced by Leroux [14]. We briefly summarize the terminology from the theory of Möbius categories needed to understand this paper. For a *decomposition-finite category C* (*i.e.*, a small category in which any morphism α has only finitely many non-trivial factorizations $\alpha = \beta\gamma$), the *incidence algebra* $A(C)$ over the field of complex numbers $\mathbb{C}$ is the $\mathbb{C}$-algebra of all functions $\xi : MorC \to \mathbb{C}$ with the usual structure of vector space over $\mathbb{C}$ and the multiplication (convolution) defined by: $(\xi * \eta)(\alpha) = \sum_{(\beta,\gamma)\in\langle\alpha\rangle} \xi(\beta) \cdot \eta(\gamma)$, where $\langle\alpha\rangle = \{(\beta, \gamma) \in MorC \times MorC \mid \alpha = \beta\gamma\}$. The *identity element* of $A(C)$ is δ, where $\delta(\alpha) = 1$ if α is an identity morphism and $\delta(\alpha) = 0$ otherwise. A *Möbius category* is a decomposition-finite category C satisfying the following condition: an incidence function $\xi \in A(C)$ has a convolution inverse if and only if $\xi(\alpha) \neq 0$ for each identity morphism α of C. An equivalent characterization of Möbius categories is the following: a Möbius category is a decomposition-finite category with finite length (*i.e.*, $l(\alpha) = \sup\{n \in \mathbb{N} \mid$ there exist $\alpha_1, \alpha_2, \ldots, \alpha_n$ non-identities such that $\alpha = \alpha_1\alpha_2 \cdots \alpha_n\}$ is finite for any morphism α). In a Möbius category the identity morphisms are indecomposable morphisms of length 0, and the non-identity indecomposable morphisms are morphisms of length 1. A Möbius category is *graded* if $l(\beta\gamma) = l(\beta) + l(\gamma)$ whenever $\beta\gamma$ makes sense. A *Möbius category of binomial type* is a graded category in which $l(\alpha) = l(\alpha') = n$ implies $\binom{\alpha}{k} = \binom{\alpha'}{k}$ for any non-negative integer $k \leq n$, where $\binom{\alpha}{k}$ denotes the non-negative integer $|\{(\beta, \gamma) \mid \beta\gamma = \alpha$ and $l(\gamma) = k\}|$. The Möbius category C of binomial type is called *full* if $l : MorC \to \mathbb{N}$ is onto. The Möbius function μ of a Möbius category is the convolution inverse of the zeta function ζ defined by $\zeta(\alpha) = 1$ for each morphism α. We have the following basic equivalence: $\xi = \eta * \zeta$ if and only if $\eta = \xi * \mu$ which is called the Möbius inversion formula.

Now, a reduced standard division category $C_F(S)$ is a Möbius category if and only if the inverse monoid S is combinatorial (aperiodic) and *locally finite* (that is, the poset of idempotents $(E(S), \leq)$ is locally finite); see [24, Theorem 3.3]. It is straightforward to see that for a locally finite combinatorial bisimple inverse monoid S the Möbius category $C_F(S)$ is a monoid (a category which has precisely one object) namely the $\mathcal{R}$-class of S containing the identity. Here is a categorical (and combinatorial) interpretation of Clifford's theory of bisimple inverse monoids: every bisimple inverse monoid could be described in terms of a right cancellative monoid in which the set of principal left ideals is closed under finite intersections. In particular, if S is the bicyclic semigroup, then the incidence algebra of $C_F(S)$ is (isomorphic to) the algebra of arithmetical functions with Cauchy product. If S is the multiplicative analogue of the bicyclic semigroup, then the incidence algebra of $C_F(S)$ is (isomorphic to) the algebra of arithmetical functions with Dirichlet product. So, the classical Möbius function is the Möbius function of the multiplicative analogue of the bicyclic semigroup. If S is a fundamental simple inverse ω-semigroup, then $C_F(S)$ is free on a non-trivial cyclic directed graph and so $C_F(S)$ is a Möbius category of full binomial type. In this case, a reduced incidence algebra of $C_F(S)$ is isomorphic to the algebra of formal power series $\mathbb{C}[[X]]$, and the Möbius inversion formula for $C_F(S)$ is one of special kind; see [25, Theorem 4.3]. The Möbius inversion formula of the free monogenic inverse monoid is a two-dimensional analogue of the Möbius inversion formula for the bicyclic semigroup, and in [27], a Dirichlet analogue of the free monogenic inverse monoid via Möbius inversion is given. Now, any abstract Möbius-division category (C, I) (*i.e.*, an abstract division category which is a Möbius category) is isomorphic to a reduced standard division category $C_F(S)$ of a locally finite combinatorial inverse monoid S. This inverse monoid is the Leech [11] inverse monoid $S = \{(\alpha, \beta) \in MorC \times MorC \mid Dom\alpha = Dom\beta = I; Codom\alpha = Codom\beta\}$ with the multiplication defined by $(\alpha, \beta) \cdot (\alpha', \beta') = (p\alpha, q\beta')$, where $[\beta, \alpha', p, q]$ is a pushout.

Zeros are special objects in semigroup theory and in Möbius-division category theory. If S is an inverse monoid with zero, then $(E(S), \leq)$ is *locally finite* (that is, every segment of $(E(S), \leq)$ is finite) if and only if $E(S)$ is finite. So, for a combinatorial inverse monoid S with zero, the reduced standard division category $C_F(S)$ relative to an idempotent transversal F of the $\mathcal{D}$-classes of S with $1 \in F$ is a Möbius category if and only if $E(S)$ is finite. This is a very restrictive condition. In the case where $E(S)$ is not finite we shall omit the terminal object 0 of the reduced standard division category $C_F(S)$ and we shall use Lawson's [6] CRM category theory. One of the key ideas in Lawson's constructions [6, 7, 8] is to treat the zero as a distinguished element. So, Lawson's constructions are a slight generalization of [11], useful in our study via Möbius categories.

Throughout this paper, we shall deal with inverse monoids with zero for which the set of idempotents is not finite. Section 2 of this paper begins by defining (abstract) quasi-Möbius-division categories. Up to isomorphism, the only quasi-Möbius-division categories are the reduced standard division categories of 0-locally finite combinatorial inverse monoids. Some algebraic connections between quasi-Möbius-division categories and combinatorial inverse monoids are presented. The section ends with two evaluations of the Möbius function of a 0-locally finite combinatorial inverse monoid. Section 3 contains two examples: the polycyclic monoid and the McAlister monoid over a non-empty set Σ. Via their quasi-Möbius-division categories and their Möbius functions, the step of generalizations from the bicyclic semigroup to polycyclic monoids and from the free monogenic inverse monoid to McAlister's monoids is then nothing but the passing from the additive monoid of non-negative integers $\mathbb{N}$ to the free monoid Σ^*. These generalizations require quasi-Möbius-division categories instead of Möbius-division categories and therefore the resulting combinatorial inverse monoids are monoids with zero. Our description of the elements of the resulting McAlister's monoid in terms of triples differs from Lawson's [4, 5] and Munn's [21] descriptions. We assume that the reader is familiar with the basic theory of inverse semigroups and categories.

We use [2] and [4] as standard references for the algebraic theory of inverse semigroups, in particular with regard to division categories, polycyclic and McAlister semigroups.

2. Quasi-Möbius-division categories

Let S be an inverse monoid with zero for which the set of idempotents $E(S)$ is not finite. We say that S is 0-locally finite if every order interval of the poset $(E^*(S), \leq)$ is finite, where $E^*(S) = E(S) \setminus \{0\}$ and $\leq$ denotes the natural partial order on an inverse semigroup. Recall that S is called *combinatorial (aperiodic)* if all subgroups of S are trivial, that is $\mathcal{H} = 1_S$ (Green's $\mathcal{H}$ relation is the equality relation).

By a *quasi-Möbius-division category* C we mean a small category with the following properties:

(a) every morphism of C is an epimorphism;

(b) C has a quasi initial object I (*i.e.*, for each object X in C, $Hom(I, X) \neq \varnothing$);

(c) C has a terminal object 0;

(d) the full subcategory C^* of C obtained by trimming the objects set ObC to $ObC \setminus \{0\}$ is a Möbius category;

(e) if $\alpha, \beta \in MorC^*$ such that $\alpha'\alpha = \beta'\beta$ for some $\alpha', \beta' \in MorC^*$, then α and β have a pushout in C^*.

Theorem 1. *Let S be an inverse monoid with zero and F be an idempotent transversal of the $\mathcal{D}$-classes of S with $1 \in F$. Then the reduced standard division category $C_F(S)$ is a quasi-Möbius-division category if and only if S is 0-locally finite and combinatorial.*

Proof. Suppose that $C_F(S)$ is a quasi-Möbius-division category. If G is a non-trivial subgroup of S then an $\mathcal{H}$-class H_e, with $e \in F$ and $e \neq 0$, is non-trivial. Let $s \in H_e$ such that $s \neq e$. Then $(s, e) : e \to e$ is a non-identity isomorphism of $C^*_F(S)$. It follows that the length of the identity morphism from e to e is not finite, which is a contradiction. Consequently, S is combinatorial.

To show that $(E^*(S), \leq)$ is locally finite it is enough to show that the interval $[e, 1]$ is finite for any $e \in E^*(S)$. If $e \in E^*(S)$ then there exists a necessarily unique $f_e \in F^*$ ($F^* = F \setminus \{0\}$) such that e and f_e are $\mathcal{D}$-related. Since S is combinatorial there exists a necessarily unique $s_e \in S^*$ ($S^* = S \setminus \{0\}$) such that $s_e^{-1}s_e = e$ and $s_e s_e^{-1} = f_e$. If $g \in [e, 1]$ then it is straightforward to check that $(s_e s_g^{-1}, f_g)$ is a morphism of $C^*_F(S)$ from f_g to f_e, that $(s_g, 1)$ is a morphism of $C^*_F(S)$ from 1 to f_g, and that $(s_e, 1)$ is a morphism of $C^*_F(S)$ from 1 to f_e. Moreover, $(s_e, 1) = (s_e s_g^{-1}, f_g) \cdot (s_g, 1)$, that is $((s_e s_g^{-1}, f_g), (s_g, 1))$ belongs to $\langle (s_e, 1) \rangle$. Since, $g' \neq g$ implies $s_{g'} \neq s_g$, it follows that the function $\theta : [e, 1] \to \langle (s_e, 1) \rangle$ defined by

$$\theta(g) = ((s_e s_g^{-1}, f_g), (s_g, 1))$$

is injective. But the Möbius category $C^*_F(S)$ is decomposition-finite and therefore the set $\langle (s_e, 1) \rangle$ is finite. It follows that $[e, 1]$ is finite.

Conversely, suppose that S is 0-locally finite and combinatorial. A reduced standard division category of an inverse monoid is a division category. Thus $C_F(S)$ has a quasi-initial object (the identity element 1 of S) and every morphism of $C_F(S)$ is an epimorphism. The zero element of S is a final object of $C_F(S)$. Since S is combinatorial and $(E^*(S), \leq)$ is locally finite it follows (by a simple investigation of [24, Theorem 3.3]) that $C^*_F(S)$ is a Möbius category.

Let us prove assertion (e). Let $\alpha = (s, e)$ and $\beta = (t, e)$ be two morphisms of $C^*_F(S)$ such that $\alpha'\alpha = \beta'\beta$ for some $\alpha' = (u, ss^{-1})$, $\beta' = (v, tt^{-1}) \in MorC^*_F(S)$. Then $us = vt \neq 0$. Put $x = t^{-1}ts^{-1}$ and $y = s^{-1}st^{-1}$. It is routine to check that the diagram

$$\begin{tikzcd} e \arrow[r, "{(t,e)}"] \arrow[d, "{(s,e)}"'] & tt^{-1} \arrow[d, "{(y,tt^{-1})}"] \\ ss^{-1} \arrow[r, "{(x,ss^{-1})}"'] & xx^{-1} = yy^{-1} \end{tikzcd}$$

is a pushout in $C^*_F(S)$. □

Recall that [26, Theorem 3.3] establishes that, up to isomorphism, the only Möbius-division categories are the reduced standard division categories of combinatorial inverse monoids with the poset of idempotents locally finite. A similar result holds for quasi-Möbius-division categories.

Theorem 2. *Every quasi-Möbius-division category C with a quasi initial object I is isomorphic to a reduced standard division category $C_F(S)$ of a 0-locally finite combinatorial inverse monoid S.*

Proof. We wish to apply the Leech-Lawson construction (Leech [11], Lawson [6], [7]). Let C be a quasi-Möbius-division category with a quasi initial object I. Put

$$L(C) = \left\{ (\alpha, \beta) \in MorC^* \times MorC^* \;\middle|\; \begin{array}{l} Dom\alpha = Dom\beta = I, \\ Codom\alpha = Codom\beta \end{array} \right\} \cup \{0\}.$$

Define a product (called the *pushout product*) on $L(C)$ as follows:

$$(\alpha, \beta) \cdot (\alpha', \beta') = \begin{cases} (p\alpha, q\beta') & \text{if } [\beta, \alpha', p, q] \text{ is a pushout,} \\ 0 & \text{if } \beta, \alpha' \text{ has no pushout,} \end{cases}$$

(and $0 \cdot (\alpha, \beta) = (\alpha, \beta) \cdot 0 = 0 \cdot 0 = 0$). This product is associative and $(1_I, 1_I)$ is the identity element of $L(C)$. We have

$$E(L(C)) = \{(\alpha, \alpha) \mid \alpha \in MorC^*\} \cup \{0\}.$$

The monoid $L(C)$ is an inverse monoid, $(\alpha, \beta)^{-1} = (\beta, \alpha)$ and $0^{-1} = 0$. Since $(\alpha_1, \beta_1)\mathcal{L}(\alpha_2, \beta_2)$ if and only if $\beta_1 = \beta_2$, and $(\alpha_1, \beta_1)\mathcal{R}(\alpha_2, \beta_2)$ if and only if $\alpha_1 = \alpha_2$ (where $\mathcal{L}$ and $\mathcal{R}$ are the Green relations), it follows that $L(C)$ is combinatorial. Next observe that $(\alpha, \alpha) \leq (\beta, \beta)$ if and only if $q\beta = \alpha$ for some $q \in MorC^*$, and because C^* is decomposition finite it follows that $L(C)$ is 0-locally finite. Now, in the 0-locally finite combinatorial inverse monoid $L(C)$, two idempotents (α, α) and (β, β) are $\mathcal{D}$-related if and only if $Codom\alpha = Codom\beta$. We make a choice α_A from $Hom(I, A)$

for any $A \in ObC^*$ such that $\alpha_A = 1_I$ if $A = I$. Then,

$$F = \{(\alpha_A, \alpha_A) \mid Dom\alpha_A = I, Codom\alpha_A = A\}_{A \in ObC^*} \cup \{0\}$$

is an idempotent transversal of the $\mathcal{D}$-classes of $L(C)$ with $(1_I, 1_I) \in F$, and the application $G : C \to C_F(L(C))$ defined by

$$\begin{cases} G(0) = 0, \\ G(A) = (\alpha_A, \alpha_A), \\ G(\beta) = ((\alpha_{Codom\beta}, \beta\alpha_{Dom\beta}), (\alpha_{Dom\beta}, \alpha_{Dom\beta})), \\ G(A \to 0) = (\alpha_A, \alpha_A) \to 0, \end{cases}$$

is an isomorphism of categories. □

As usual, by a 0-*E-unitary* (or *E^*-unitary*) *inverse monoid* we mean an inverse monoid S such that for all $s \in S^*(= S \setminus \{0\})$ and $e \in E^*(S)$, $es \in E^*(S)$ implies $s \in E^*(S)$. We have (see also [6, Section 3, Theorem 5]):

Theorem 3. *Let C be a quasi-Möbius-division category. Then C^* is cancellative (that is, every morphism of C^* is both a monomorphism and an epimorphism) if and only if the 0-locally finite combinatorial inverse monoid $L(C)$ is E^*-unitary.*

Proof. Suppose that C is cancellative. Let $(\alpha, \beta) \in L^*(C)$ and $(\gamma, \gamma) \in E^*(L(C))$ be such that $(\gamma, \gamma)(\alpha, \beta) \in E^*(L(C))$. But,

$$(\gamma, \gamma)(\alpha, \beta) = (p\gamma, q\beta),$$

where the diagram

$$\begin{tikzcd} I \arrow[r, "\alpha"] \arrow[d, "\gamma"'] & Codom\alpha \arrow[d, "q"] \\ Codom\gamma \arrow[r, "p"'] & \bullet \end{tikzcd}$$

is a pushout diagram. So, $p\gamma = q\beta$ implies $q\alpha = q\beta$. By cancellativity, $\alpha = \beta$. Hence, $L(C)$ is E^*-unitary.

Conversely, suppose that $L(C)$ is E^*-unitary. Now, we will show that every morphism of C^* is a monomorphism. Let u, α and β be three morphisms of C^* such that $u\alpha = u\beta$. If $w \in Hom(I, Dom\alpha = Dom\beta)$, then the diagram

$$\begin{tikzcd} I \arrow[r, "\alpha w"] \arrow[d, "\gamma"'] & Domu \arrow[d, "u"] \\ Codomu \arrow[r, "1_{Codomu}"'] & Codomu \end{tikzcd}$$

is a pushout diagram, where $\gamma = u\alpha w = u\beta w$. Therefore,

$$(\gamma, \gamma)(\alpha w, \beta w) = (1_{Codomu}\gamma, u\beta w) = (u\beta w, u\beta w).$$

Since $L(C)$ is E^*-unitary we obtain $(\alpha w, \beta w) \in E^*(L(C))$, that is $\alpha w = \beta w$. But in C every morphism in an epimorphism, and therefore $\alpha = \beta$. Thus, u is a monomorphism. □

Before moving on to other matters, let us remark that in a Möbius category the condition

$$|Hom(A,A)| = 1, \quad \text{for any object } A,$$

holds frequently; see [25]. Recall that an inverse semigroup is *completely semisimple* if the natural partial order is equality when restricted to any $\mathcal{D}$-class. If S is a completely semisimple combinatorial inverse monoid with $(E(S), \leq)$ locally finite and F is an idempotent transversal of the $\mathcal{D}$-classes of S with $1 \in F$, then the reduced standard division category $C_F(S)$ is a Möbius category with the above condition; see [24, Theorem 4.1]. We shall say that a Möbius category is *of type* 1 if $|Hom(A,A)| = 1$ for any object A.

Theorem 4. *Let C be a quasi-Möbius-division category. Then the Möbius category C^* is of type 1 if and only if the 0-locally finite combinatorial inverse monoid $L(C)$ is completely semisimple.*

Proof. Suppose that C^* is of type 1. To show that $L(C)$ is completely semisimple, let (α, α) and (β, β) be two idempotent elements of $L(C)$ such that $(\alpha,\alpha)\mathcal{D}(\beta,\beta)$ and $(\alpha, \alpha) \leq (\beta, \beta)$. Then

$$Codom\alpha = Codom\beta \quad \text{and} \quad q\beta = \alpha, \text{ for some } q.$$

Since $|Hom(Codom\beta, Codom\alpha)| = 1$, it follows that $q = 1_{Codom\alpha}$. Consequently, $(\alpha, \alpha) = (\beta, \beta)$. Hence $L(C)$ is completely semisimple.

Conversely, suppose that the 0-locally finite combinatorial inverse monoid $L(C)$ is completely semisimple. If $\gamma \in Hom(A,A)$ for some object A of C^*, then the diagram

$$\begin{tikzcd} I \arrow[r, "u"] \arrow[d, "\gamma u"'] & A \arrow[d, "\gamma"] \\ A \arrow[r, "1_A"'] & A \end{tikzcd}$$

is a pushout for any $u \in Hom(I, A)$. It follows that $(\gamma u, \gamma u)(u, u) = (\gamma u, \gamma u)$, that is $(\gamma u, \gamma u) \leq (u, u)$. Now, it is clear that $(\gamma u, \gamma u)\mathcal{D}(u, u)$. Since $L(C)$ is completely semisimple, we have $(\gamma u, \gamma u) = (u, u)$, that is $\gamma u = u$. Hence $\gamma = 1_A$. We have proved that $|Hom(A,A)| = 1$ for any object A of C^*. So, the Möbius category C^* is of type 1. □

If the inverse monoid S is combinatorial and $(E^*(S), \leq)$ is locally finite then the Möbius category $C_F^*(S)$ is a (Möbius) monoid if $C_F^*(S)$ has precisely one object. An inverse monoid S with zero and two $\mathcal{D}$-classes, $\{0\}$ and S^*, is 0-bisimple. If $C_F^*(S)$ is a Möbius monoid then $MorC_F^*(S) = Hom(1,1) = \{(s,1) \mid ss^{-1} = 1\}$.

Theorem 5. *Let S be an inverse monoid with zero and F be an idempotent transversal of the $\mathcal{D}$-classes of S with $1 \in F$. Then,*

(a) *$C_F^*(S)$ is a Möbius monoid if and only if $(E^*(S), \leq)$ is locally finite and S is both combinatorial and 0-bisimple.*

(b) *If $C_F^*(S)$ is a Möbius monoid then the monoid of morphisms of $C_F^*(S)$ is (isomorphic to) the $\mathcal{R}$-class of S containing the identity.*

Now, let S be a 0-locally finite combinatorial inverse monoid (such that $E(S)$ is not finite). Then $C_F(S)$ is a quasi-Möbius division category and therefore $C_F^*(S)$ is a Möbius category. We say that the Möbius function of $C_F^*(S)$ is the *Möbius function of the* 0*-locally finite combinatorial inverse monoid* S. Specializing to our case, we see that [24, Theorem 3.5] leads to the following result.

Theorem 6. *Let S be a 0-locally finite combinatorial inverse monoid (such that $E(S)$ is not finite). The Möbius function μ of S is given in either of the following ways:*

(a) $\mu(s,e) = \mu_{Q^*(e)}\left([(s,e),(e,e)]_{Q^*(e)}\right)$, *where $\mu_{Q^*(e)}$ is the Möbius function of the poset of quotient objects $Q^*(e)$ of e in the Möbius category $C_F^*(S)$;*

(b) $\mu(s,e) = \mu_{E^*(eSe)}([s^{-1}s,e]_{E^*(eSe)})$, *where $\mu_{E^*(eSe)}$ is the Möbius function of the poset $E^*(eSe)$.*

3. Examples

In this section we will study two examples: the polycyclic monoid and the McAlister monoid. Both are 0-locally finite combinatorial inverse monoids whose sets of idempotents are not finite. If S is a 0-locally finite combinatorial inverse monoid and $E(S)$ is not finite, then we say that $C_F^*(S)$ is the *Möbius category of* S. We say that the incidence algebra of $C_F^*(S)$ is the *incidence algebra of* S, and the Möbius function of $C_F^*(S)$ is the *Möbius function of* S. (Up to isomorphism, $C_F^*(S)$ is uniquely determined.)

3.1. The free monoid as the Möbius monoid of full binomial type of the polycyclic monoid

Let Σ^* be the free monoid on a non-empty set Σ. For a string $u = x_1x_2\ldots x_m$, its length m is denoted by $|u|$. The empty string is denoted by 1 and $|1| = 0$. If $w = uv$, then u is a *prefix* of w and v is a *suffix* of w. If $w = uzv$, then z is a *factor* of w. A prefix and a suffix of w are factors of w.

The monoid Σ^* is a Möbius monoid. Every morphism of Σ^* is both a monomorphism and an epimorphism. If the non-empty set Σ is not a singleton, then the Möbius monoid Σ^* is not a division monoid because it is not a category with pushouts: it is easy to see that the coangle in Σ^*

$$\begin{tikzcd} \bullet \arrow[r, "v"] \arrow[d, "u"'] & \bullet \\ \bullet & \end{tikzcd}$$

has an embedding in a commutative square if and only if u and v are *suffix-comparable* (that is, one is a suffix of the other). But if $v = u'u$ for some $u' \in \Sigma^*$, or $u = v'v$ for some $v' \in \Sigma^*$, then one of the corresponding diagrams

$$\begin{tikzcd} \bullet \arrow[r, "v"] \arrow[d, "u"'] & \bullet \arrow[d, "1"] \\ \bullet \arrow[r, "u'"'] & \bullet \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \bullet \arrow[r, "v"] \arrow[d, "u"'] & \bullet \arrow[d, "v'"] \\ \bullet \arrow[r, "1"'] & \bullet \end{tikzcd}$$

is a pushout. It follows that the category Σ_0^*, obtained from Σ^* (with $|\Sigma| > 1$) by adjoining a terminal object 0, is a quasi-Möbius-division category. By Theorem 2, there is a 0-locally finite combinatorial inverse monoid $L(\Sigma_0^*)$ such that a reduced standard division category $C_F(L(\Sigma_0^*))$ relative to an idempotent transversal F of the $\mathcal{D}$-classes of $L(\Sigma_0^*)$, with $1 \in F$, is isomorphic to Σ_0^*. We shall denote $L(\Sigma_0^*)$ as simply P_Σ. The monoid P_Σ is given by (see the construction of $L(C)$ in the proof of Theorem 2, and the above pushout diagrams)

$$P_\Sigma = (\Sigma^* \times \Sigma^*) \cup \{0\},$$

with the (pushout) product defined by

$$(u,v)\cdot(u',v') = \begin{cases} (u, u''v') & \text{if } u' \text{ is a suffix of } v \text{ and } v = u''u' \text{ for some string } u'', \\ (v''u, v') & \text{if } v \text{ is a suffix of } u' \text{ and } u' = v''v \text{ for some string } v'', \\ 0 & \text{otherwise,} \end{cases}$$

(and $0 \cdot (u,v) = (u,v) \cdot 0 = 0 \cdot 0 = 0$). This monoid is called the *polycyclic monoid over* Σ. Polycyclic monoids were introduced by Nivat and Perrot [22]. The following corollary is a consequence of Theorems 2, 3 and 5 of the previous section):

Corollary 1. *The polycyclic monoid P_Σ ($|\Sigma| > 1$) is a 0-locally finite, combinatorial, E^*-unitary, 0-bisimple inverse monoid.*

Remark 1. If Σ is a singleton, then Σ^* is isomorphic to the monoid of non-negative integers with respect to addition. This monoid is a Möbius-division monoid that does not require a terminal object attachment. So, if Σ is a singleton, the zero is omitted and P_Σ (with $|\Sigma| = 1$) is the bicyclic semigroup. The polycyclic monoids are natural generalizations of the bicyclic semigroup. In the case $|\Sigma| = 1$, the incidence algebra of the Möbius-division monoid Σ^* is the algebra of arithmetical functions with Cauchy product.

It remains the assumption $|\Sigma| > 1$. The set

$$E(P_\Sigma) = \{(u,u) \mid u \in \Sigma^*\} \cup \{0\}$$

is the set of idempotents of the polycyclic monoid P_Σ. Two non-zero idempotents (u,u), (v,v) are $\mathcal{D}$-related and therefore,

$$F = \{(1,1), 0\}$$

is an idempotent transversal of the $\mathcal{D}$-classes of P_Σ.

Corollary 2. (a) *The Möbius monoid $C_F^*(P_\Sigma)$ is isomorphic to the free monoid Σ^*.*

(b) *The free monoid Σ^* is isomorphic to the $\mathcal{R}$-class of the polycyclic monoid P_Σ which contain the identity of P_Σ.*

Theorem 7. *The Möbius category (monoid) of P_Σ is of full binomial type and the reduced incidence algebra*

$$R = \{\xi \in A(\Sigma^*) \mid \xi(u) = \xi(v) \text{ if } |u| = |v|\}$$

is isomorphic to the algebra of formal power series $\mathbb{C}[[X]]$.

Proof. For the Möbius monoid Σ^*, the length function $l : \Sigma^* \to \mathbb{N}$ is given by $l(u) = |u|$. Then $\binom{u}{k} = 1$ if $k \leq l(u)$; $l(uv) = l(u) + l(v)$; and l is onto. Thus the Möbius monoid of the polycyclic monoid P_Σ is a Möbius category of full binomial type.

If $\xi \in R$, then we write $\xi(n)$ for $\xi(u)$ if $|u| = n$. Then, $\Phi : R \to \mathbb{C}[[X]]$ defined by

$$\Phi(\xi) = \sum_{n \geq 0} \xi(n) X^n$$

is an algebra isomorphism. □

Now, for two non-zero idempotents (u, u) and (v, v) of the polycyclic monoid P_Σ, we have $(u, u) \leq (v, v)$ if and only if v is a suffix of u. It follows that in the locally finite partial ordered set $(E^*(P_\Sigma), \leq)$, the interval $[(u, u), (1, 1)]$ is a chain for any $u \in \Sigma^*$. But, $\Sigma^* \cong \{s = (u, v) \in P_\Sigma \mid ss^{-1} = (1, 1)\} = \{(1, v) \mid (1, v) \in P_\Sigma\}$. By Theorem 6 (b), it follows that

$$\mu(v) = \mu(1, v) = \mu_{E^*(P_\Sigma)}([(v, v), (1, 1)]_{E^*(P_\Sigma)})$$

and therefore

$$\mu(v) = \begin{cases} 1 & \text{if } v = 1, \\ -1 & \text{if } |v| = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 8 (The Möbius inversion formula for P_Σ). *Let $\xi, \eta : \Sigma^* \to \mathbb{C}$ such that $\xi(1) = \eta(1)$. Then*

$$\xi(x_1 x_2 \cdots x_m) = \eta(1) + \eta(x_1) + \eta(x_1 x_2) + \cdots + \eta(x_1 x_2 \cdots x_m)$$

for any string $x_1 x_2 \cdots x_m \in \Sigma^*$ if and only if

$$\eta(x_1 x_2 \cdots x_m) = \xi(x_1 x_2 \cdots x_m) - \xi(x_1 x_2 \cdots x_{m-1}).$$

Proof. The theorem follows from the basic equivalence: $\xi = \eta * \zeta$ if and only if $\eta = \xi * \mu$. □

Notice that the Möbius-division category $C_F(S)$ (as a reduced standard division category) of the bicyclic semigroup S is the monoid (as a category with one object) of the non-negative integers with the usual addition $(\mathbb{N}, +)$ and with the Möbius function

$$\mu(n) = \begin{cases} 1 & \text{if } n = 0, \\ -1 & \text{if } n = 1, \\ 0 & \text{otherwise,} \end{cases}$$

with $n \in \mathbb{N}$; see [24].

Using the free monoid Σ^* (as a quasi-Möbius-division category if 0 is adjoined) instead of $(\mathbb{N}, +)$, the resulting 0-locally finite combinatorial inverse monoid (as the Leech-Lawson monoid) is the polycyclic monoid P_Σ with the Möbius function obtained before Theorem 8.

3.2. The McAlister monoid

Following [25], the Möbius-division category $C_F(S)$ (as a reduced standard division category) of the free monogenic inverse monoid S is given by

- $ObC_F(S) = \mathbb{N}$;
- $Hom(m,n) = \begin{cases} \{(a,n,b) \in \mathbb{N}^3 \mid a+m+b=n\} & \text{if } m \leq n, \\ \varnothing & \text{otherwise;} \end{cases}$
- The composition of two morphisms $(a,n,b) : m \to n$ and $(a',p,b') : n \to p$ is given by $(a',p,b') \cdot (a,n,b) = (a'+a, p, b'+b)$; and the Möbius function μ of $C_F(S)$ is the following one:

$$\mu(a,n,b) = \begin{cases} 1 & \text{if } a=b=0 \text{ or } a=b=1, \\ -1 & \text{if } (a=0, b=1) \text{ or } (a=1, b=0), \\ 0 & \text{otherwise.} \end{cases}$$

Using Σ^* instead of $\mathbb{N}$, we consider the (quasi-Möbius-division) category C_Σ defined by

- $ObC_\Sigma = \Sigma^*$, with a terminal object 0 adjoined;
- $Hom(u,v) = \begin{cases} \{(a,v,b) \mid v = u'uu'', |u'| = a, |u''| = b\} & \text{if } u \text{ is a factor of } v, \\ \varnothing & \text{otherwise;} \end{cases}$
- The composition of two morphisms $(a,v,b) : u \to v$ and $(a',w,b') : v \to w$ is given by $(a',w,b') \cdot (a,v,b) = (a'+a, w, b'+b)$.

The resulting 0-locally finite combinatorial inverse monoid $L(C_\Sigma)$ (the Leech-Lawson monoid of C_Σ) and the Möbius function will be computed below.

It is straightforward to see that the small category C_Σ^* (the full subcategory of C_Σ obtained by trimming the objects set ObC_Σ to Σ^*) is a decomposition-finite category with finite length. So the category C_Σ^* is a Möbius category of type 1.

Theorem 9. *The Möbius category C_Σ^* is graded but it is not of binomial type.*

Proof. First, $l(a,v,b) = a+b = |v| - |u|$ is the length of a morphism (a,v,b) of C_Σ^* from u to v. It follows that $l((a',w,b')(a,v,b)) = l(a',w,b') + l(a,v,b)$ whenever the composition $(a',w,b')(a,v,b)$ makes sense.

If $v = u'uu''$, with $|u'| = |u''| = 1$, then $(1,v,1) = (0,v,1)(1,u'u,0)$ and $(1,v,1) = (1,v,0)(0,uu'',1)$ are the non-identity indecomposable factorizations of $(1,v,1) \in Hom(u,v)$. Moreover, if $w = u'u''u$, with $|u'| = |u''| = 1$, then $(2,w,0) = (1,w,0)(1,u''u,0)$ is the unique non-identity indecomposable factorization of $(2,w,0) \in Hom(u,w)$. Hence $\binom{\alpha}{1} = 2$ and $\binom{\beta}{1} = 1$, where $\alpha = (1,v,1)$

and $\beta = (2, w, 0)$. But $l(\alpha) = l(\beta)$. Thus, the Möbius category C^*_Σ is graded but it is not of binomial type. $\square$

Our further efforts to compute the Möbius function μ of C^*_Σ will be greatly simplified by the use of Theorem 6 (a). The poset $Q^*(u)$ of quotient objects of $u \in \Sigma^*$ is the set $\cup_{v\in\Sigma^*} Hom(u, v)$ under the usual quotient ordering:

$$(a, v, b) \leq (a', v', b') \text{ if and only if } (a'', v, b'') \cdot (a', v', b') = (a, v, b)$$

for some morphism (a'', v, b'') of C^*_Σ, *i.e.*, $(a, v, b) \leq (a', v', b')$ if and only if

(i) $a' \leq a$ and $b' \leq b$,

(ii) v' is a factor of v,

(iii) $v = u'v'w'$, where $\begin{cases} |u'| = a - a', \\ |w'| = b - b', \end{cases}$

i.e.,

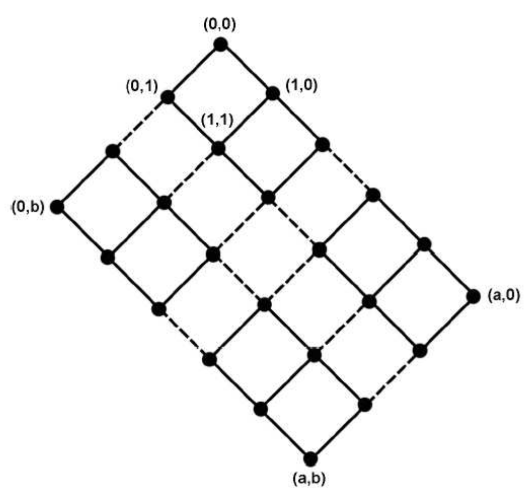
$$(a, v, b) \leq (a'v'b') \text{ if and only if } (a - a', v, b - b') \text{ is a morphism of } C^*_\Sigma \text{ from } v' \text{ to } v,$$

where both morphisms (a, v, b) and (a', v', b') have the same domain u.

By Theorem 6 (a),

$$\mu(a, v, b) = \mu_{Q^*(u)}([(a, v, b), (0, u, 0)]_{Q^*(u)}).$$

The interval $[(a, v, b), (0, u, 0)]_{Q^*(u)}$ is described by the Hasse diagram:



A straightforward computation of Möbius function of the above lattices gives the Möbius function μ of the Möbius category C^*_Σ (via Theorem 6 (a)):

$$\mu(a, v, b) = \begin{cases} 1 & \text{if } a = b = 0 \text{ or } a = b = 1, \\ -1 & \text{if } (a = 0, b = 1) \text{ or } (a = 1, b = 0), \\ 0 & \text{otherwise.} \end{cases}$$

Since the incidence functions ζ, δ and μ of the Möbius category C_Σ^* do not depend on strings, the basic equivalence $\xi = \eta * \zeta$ if and only if $\eta = \xi * \mu$ for strings-independent incidence functions ξ and η leads to a classic case of Möbius inversion:

Given functions $\xi, \eta : \mathbb{N} \times \mathbb{N} \to \mathbb{C}$, the relation

$$\xi(a,b) = \sum_{i=0}^{a} \sum_{j=0}^{b} \eta(i,j)$$

holds for all $a, b \geq 0$ if and only if

$$\eta(a,b) = \begin{cases} \xi(a,b) - \xi(a-1,b) - \xi(a,b-1) + \xi(a-1,b-1) & \textit{if } a,b \geq 1, \\ \xi(a,0) - \xi(a-1,0) & \textit{if } a \geq 1 \textit{ but } b = 0, \\ \xi(0,b) - \xi(0,b-1) & \textit{if } a = 0 \textit{ and } b \geq 1, \\ \xi(0,0) & \textit{if } a = b = 0. \end{cases}$$

We now change focus somewhat and take up the study of $L(C_\Sigma)$ (that is, the Leech-Lawson monoid of C_Σ). We will use Lawson's [5] superscript "-1" defined by

$$wv^{-1} = \begin{cases} u & \text{if } w = uv, \\ 1 & \text{otherwise,} \end{cases} \qquad \text{and} \qquad u^{-1}w = \begin{cases} v & \text{if } w = uv, \\ 1 & \text{otherwise,} \end{cases}$$

and we denote by $P(w)$ (or $S(w)$) the set of all prefixes (or suffixes, respectively) of w.

It is straightforward to see that the empty string 1 is a quasi-initial object of C_Σ and every morphism of C_Σ is an epimorphism (moreover C_Σ^* is cancellative). However, if Σ is not a singleton then C_Σ^* is not a division category. The coangle

$$\begin{tikzcd} u \arrow[r, "{(c,w,d)}"] \arrow[d, "{(a,v,b)}"'] & w \\ v & \end{tikzcd}$$

has no embedding in a commutative square if there is no common factor x of v and w with certain properties. We distinguish four cases.

Case (1) We have $a \geq c$, $b \leq d$, and there exists $x \in \Sigma^*$ with the following properties:

(i) $x \in S(v)$ and $x \in P(w)$;

(ii) u is a factor of x, $x = u'uu''$ and $|u'| = c$, $|u''| = b$.

(Then we have $v(x^{-1}w) = (vx^{-1})w$ and we denote this string by y; $|y| = a + |u| + d$.)

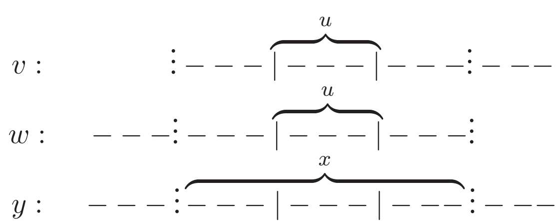
$$\begin{array}{ll} v: & - - - \vdots - - - \overbrace{| - - - |}^{u} - - - \vdots \\ w: & \qquad \vdots - - - \overbrace{| - - - |}^{u} - - - \vdots - - - \\ y: & - - - \vdots \overbrace{- - - | - - - | - - -}^{x} \vdots - - - \end{array}$$

Case (2) We have $a \leq c, b \geq d$, and there exists $x \in \Sigma^*$ with the following properties:

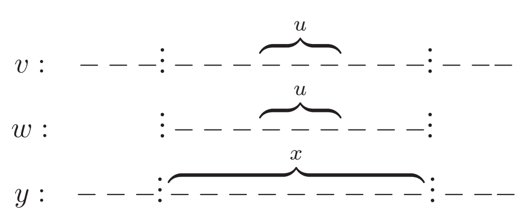
(i) $x \in P(v)$ and $x \in S(w)$;

(ii) u is a factor of x, $x = u'uu''$ and $|u'| = a$, $|u''| = d$.

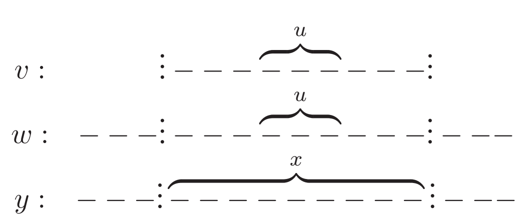
(Then we have $(wx^{-1})v = w(x^{-1}v)$ and we denote this string by y; $|y| = c + |u| + b$.)



Case (3) We have $a \geq c, b \geq d$ and, for $x = w$, x is a factor of v such that $v = v'xv''$, where $|v'| = a - c$ and $|v''| = b - d$. (Then we denote v by y.)



Case (4) We have $a \leq c, b \leq d$ and, for $x = v$, x is a factor of w such that $w = w'xw''$, where $|w'| = c - a$ and $|w''| = d - b$. (Then we denote w by y.)



Only in these four cases, the above coangle has an embedding in a commutative square and the corresponding pushout diagrams are as below.

(1)
$$\begin{tikzcd} u \arrow[r, "{(c,w,d)}"] \arrow[d, "{(a,v,b)}"'] & w \arrow[d, "{(a-c,y,0)}"] \\ v \arrow[r, "{(0,y,d-b)}"'] & y \end{tikzcd}$$

(2)
$$\begin{tikzcd} u \arrow[r, "{(c,w,d)}"] \arrow[d, "{(a,v,b)}"'] & w \arrow[d, "{(0,y,b-d)}"] \\ v \arrow[r, "{(c-a,y,0)}"'] & y \end{tikzcd}$$

(3)
$$\begin{tikzcd} u \arrow[r, "{(c,w,d)}"] \arrow[d, "{(a,v,b)}"'] & w \arrow[d, "{(a-c,v,b-d)}"] \\ v \arrow[r, "{(0,v,0)}"'] & y = v \end{tikzcd}$$

(4)
$$\begin{tikzcd} u \arrow[r, "{(c,w,d)}"] \arrow[d, "{(a,v,b)}"'] & w \arrow[d, "{(0,w,0)}"] \\ v \arrow[r, "{(c-a,w,d-b)}"'] & y = w \end{tikzcd}$$

The following result follows at once.

Theorem 10. *The category C_Σ is a quasi-Möbius-division category.*

Now, if u is the empty string, then the above four cases become:

Case (1) We have $a \geq c$, $b \leq d$, and there exists $x \in \Sigma^*$ such that $x \in S(v)$, $x \in P(w)$ and $|x| = c + b$.

Case (2) We have $a \leq c$, $b \geq d$, and there exists $x \in \Sigma^*$ such that $x \in P(v)$, $x \in S(w)$ and $|x| = a + d$.

Case (3) We have $a \geq c$, $b \geq d$, and w is a factor of v such that $v = v'wv''$, where $|v'| = a - c$ and $|v''| = b - d$.

Case (4) We have $a \leq c$, $b \leq d$, and v is a factor of w such that $w = w'vw''$, where $|w\prime| = c - a$ and $|w''| = d - b$.

Eliminating redundant relations, we obtain:

Case (1) We have $a \geq c$, and there exists $x \in \Sigma^*$ such that $x \in S(v)$, $x \in P(w)$ and $|v| - |x| = a - c$.

Case (2) We have $a \leq c$, and there exists $x \in \Sigma^*$ such that $x \in P(v)$, $x \in S(w)$, and $|w| - |x| = c - a$.

Case (3) We have $a \geq c$, and w is a factor of v such that $v = v'wv''$, where $|v'| = a - c$.

Case (4) We have $a \leq c$, and v is a factor of w such that $w = w'vw''$, where $|w'| = c - a$.

A morphism from the quasi initial object 1 to v is a triple (a, v, b) with $b = |v| - a$ and $a \in \{0, 1, \ldots, |v|\}$ (actually a pair (a, v)) and therefore we have

$$L(C_\Sigma) = \left\{ ((a', v, b'), (a, v, b)) \;\middle|\; \begin{array}{l} a', a \in \{0, 1, \ldots, |v|\}, v \in \Sigma^*, \\ b' = |v| - a' \text{ and } b = |v| - a \end{array} \right\} \cup \{0\}$$

(in short,

$$L(C_\Sigma) = \{(a', a, v) \mid v \in \Sigma^* \text{ and } a', a \in \{0, 1, \ldots, |v|\}\} \cup \{0\}).$$

We will use the pushout product definition from the proof of Theorem 2. If the inner square of the diagram

$$\begin{tikzcd} 1 \arrow[r, shift left, "{(c',w,d')}"] \arrow[r, shift right, "{(c,w,d)}"'] \arrow[d, shift right, "{(a',v,b')}"'] \arrow[d, shift left, "{(a,v,b)}"] & w \arrow[d, "{(\gamma,y,\delta)}"] \\ v \arrow[r, "{(\alpha,y,\beta)}"'] & y \end{tikzcd}$$

is a pushout, then

$$((a', v, b'), (a, v, b)) \cdot ((c, w, d), (c', w, d')) = ((\alpha + a', y, \beta + b'), (\gamma + c', y, \delta + d')).$$

It follows that the product in $L(C_\Sigma)$ is defined by

$$((a', v, b'), (a, v, b)) \cdot ((c, w, d), (c', w, d')) =$$

$$\begin{cases} ((a', v(x^{-1}w), d-b+b'), (a-c+c', v(x^{-1}w), d')) & \text{if } a \geq c \text{ and } x \in S(v) \cap P(w) : \\ & |v|-|x| = a-c; \\ ((a'-a+c, w(x^{-1}v), b'), (c', w(x^{-1}v), b-d+d')) & \text{if } a \leq c \text{ and } x \in P(v) \cap S(w) : \\ & |w|-|x| = c-a; \\ ((a', v, b'), (a-c+c', v, b-d+d')) & \text{if } a \geq c \text{ and } v = v'wv'' \\ & \text{with } |v'| = a-c; \\ ((a'-a+c, w, d-b+b'), (c', w, d')) & \text{if } a \leq c \text{ and } w = w'vw'' \\ & \text{with } |w'| = c-a; \\ 0 & \text{otherwise}; \end{cases}$$

(and $0 \cdot ((a', v, b'), (a, v, b)) = ((a', v, b'), (a, v, b)) \cdot 0 = 0 \cdot 0 = 0$).

Thus, in the above short description of the elements of $L(C_\Sigma)$ (and now we shall denote $L(C_\Sigma)$ as simply M_Σ), we have

$$M_\Sigma = \{(a', a, v) \mid v \in \Sigma^* \text{ and } a', a \in \{0, 1, \ldots, |v|\}\} \cup \{0\},$$

and the product is given as follows:

$$(a', a, v) \cdot (c, c', w) =$$

$$\begin{cases} (a', a - c + c', v(x^{-1}w)) & \text{if } a \geq c \text{ and } x \in S(v) \cap P(w) : |v| - |x| = a - c; \\ (a' - a + c, c', w(x^{-1}v)) & \text{if } a \leq c \text{ and } x \in P(v) \cap S(w) : |w| - |x| = c - a; \\ (a', a - c + c', v) & \text{if } a \geq c \text{ and } v = v'wv'' \text{ with } |v'| = a - c; \\ (a' - a + c, c', w) & \text{if } a \leq c \text{ and } v = w'vw'' \text{ with } |w'| = c - a; \\ 0 & \text{otherwise}; \end{cases}$$

(and $0 \cdot (a', a, v) = (a', a, v) \cdot 0 = 0 \cdot 0 = 0$).

Consequently, we have

$$(a', a, v) \cdot (c, c', w) =$$
$$\begin{cases} (a', a - c + c', v(x^{-1}w)) & \text{if } a \geq c, \text{ and } x \in P(w) : v = v'xv'' \\ & \text{with } (v'' = 1 \text{ or } x = w) \text{ and } |v'| = a - c; \\ (a' - a + c, c', w(x^{-1}v)) & \text{if } a \leq c, \text{ and } x \in P(v) : w = w'xw'' \\ & \text{with } (w'' = 1 \text{ or } x = v) \text{ and } |w'| = c - a; \\ 0 & \text{otherwise}; \end{cases}$$

(and $0 \cdot (a', a, v) = (a', a, v) \cdot 0 = 0 \cdot 0 = 0$).

Basic structural properties of M_Σ that can be obtained by routine verifications are listed below.

Theorem 11. *In the inverse monoid M_Σ,*

(a) *the inverse of (a', a, v) is (a, a', v);*

(b) $E(M_\Sigma) = \{(a, a, v) \in \mathbb{N}^2 \times \Sigma^* \mid a \leqslant |v|\} \cup \{0\}$ *is the set of idempotents;*

(c) $(a', a, v)^{-1} \cdot (a', a, v) = (a, a, v)$ *and* $(a', a, v) \cdot (a', a, v)^{-1} = (a', a', v)$;

(d) $(a', a, v)\mathcal{L}(b', b, w)$ *if and only if* $a = b$ *and* $v = w$;

(e) $(a', a, v)\mathcal{R}(b', b, w)$ *if and only if* $a' = b'$ *and* $v = w$;

(f) $(a', a, v)\mathcal{H}(b', b, w)$ *if and only if* $a = b$*,* $a' = b'$ *and* $v = w$;

(g) $(a', a, v)\mathcal{D}(b', b, w)$ *if and only if* $v = w$;

(h) $\psi : \Sigma^* \to M_\Sigma$ *defined by* $\psi(v) = (0, |v|, v)$ *is an embedding.*

The following result follows from Theorems 2, 3 and 4 of the previous section.

Theorem 12. *The monoid M_Σ (with $|\Sigma| > 1$) is a 0-locally finite, combinatorial, completely semisimple, E^*-unitary inverse monoid.*

This monoid M_Σ is just the McAlister monoid over Σ. In the case $|\Sigma| = 1$ the McAlister monoid (without zero) is the free monogenic inverse monoid. In [5], Lawson introduces McAlister semigroups in terms of triples of strings. Our description of the elements of the McAlister monoid is also a description in terms of triples but differing from those in [5] and [21]. In Lawson's description (see [5] and [4, Section 9.4]),

$$M'_\Sigma = \{(u, v, w) \in {\Sigma^*}^3 \mid u \in P(v), w \in S(v)\} \cup \{0\}$$

together with the multiplication given by

$$(x, y, z) \cdot (u, v, w) =$$
$$\begin{cases} ([(uz)y^{-1}]x, [(uz)y^{-1}]y[(uz)^{-1}v], w[v^{-1}(uz)]) & \text{if } \Sigma^*(uz) \cap \Sigma^* y \neq \varnothing \\ & \text{and } (uz)\Sigma^* \cap v\Sigma^* \neq \varnothing; \\ 0 & \text{otherwise}; \end{cases}$$

(and $0 \cdot (u, v, w) = (u, v, w) \cdot 0 = 0 \cdot 0 = 0$), is the McAlister monoid. It is routine to check that $\varphi : M'_\Sigma \to M_\Sigma$ defined by

$$\varphi(u, v, w) = (|u|, |v| - |w|, v) \quad \text{and} \quad \varphi(0) = 0$$

is an isomorphism of monoids.

If $|\Sigma| = 1$ then the category C_Σ is a Möbius-division category and therefore it does not require a terminal object attachment. In this case the McAlister monoid is given by

$$M_{|\Sigma|=1} = \{(a', a, m) \in \mathbb{N}^3 \mid a', a \leqslant m\},$$

with the multiplication

$$(a', a, m)\cdot(c, c', n) = \begin{cases} (a', a - c + c', n + a - c) & \text{if} \quad a \geqslant c \text{ and } n + a - c \geqslant m; \\ (a' - a + c, c', m + c - a) & \text{if} \quad a \leqslant c \text{ and } m + c - a \geqslant n; \\ (a', a - c + c', m) & \text{if} \quad a \geqslant c \text{ and } n + a - c \leqslant m; \\ (a' - a + c, c', n) & \text{if} \quad a \leqslant c \text{ and } m + c - a \leqslant n; \end{cases}$$

that is

$$(a', a, m) \cdot (c, c', n) = \begin{cases} (a', a - c + c', \max\{n + a - c, m\}) & \text{if} \quad a \geqslant c, \\ (a' - a + c, c', \max\{m + c - a, n\}) & \text{if} \quad a \leqslant c. \end{cases}$$

Consequently $M_{|\Sigma|=1} = \{(a', a, m) \in \mathbb{N}^3 \mid a', a \leqslant m\}$ equipped with the product

$$(a', a, m) \cdot (c, c', n) = ((a', a) \circ (c, c'), \max\{(m, a) \circ (c, n)\}),$$

where $\circ$ denotes the bicyclic multiplication ([4, Section 3.4, Proposition 2]), is an isomorphic copy of the free monogenic inverse monoid.

3.3. Concluding discussion

The two examples of Section 3 are natural generalizations of two fundamental inverse monoids via quasi-Möbius-division categories. The two generalizations via quasi Möbius-division categories involving the free monoid Σ^* (with $|\Sigma| > 1$) have a deep similarity as outlined in the following table:

<table>
<tr><td colspan="2">(1) The starting locally finite combinatorial inverse monoid S:</td></tr>
<tr><td>(A) S = the bicyclic semigroup</td><td>(B) S = the free monogenic inverse monoid</td></tr>
<tr><td colspan="2">(2) The Möbius-division category of S (as a reduced standard division category of S) and the Möbius function of S:</td></tr>
<tr><td>(A) The monoid (as a category with one object) of the non-negative integers with the usual addition $(\boldsymbol{N},+)$:

$\mu(n)=\begin{cases} 1 & \text{if } n=0, \\ -1 & \text{if } n=1, \\ 0 & \text{otherwise.}\end{cases}$</td><td>(B) The truly standard division category $D(\boldsymbol{N})$ of the additive monoid $(\boldsymbol{N},+)$:

- $Ob(D(\boldsymbol{N}))=\boldsymbol{N}$;

- $Hom(m,n)=\begin{cases}\{(a,n,b)\in N^3 \mid a+m+b=n\} & \text{if } m\le n, \\ \phi & \text{otherwise}\end{cases}$
- $(a',p,b')\cdot(a,n,b)=(a'+a,p,b'+b)$;

$\mu(a,n,b)=\begin{cases} 1 & \text{if } a=b=0 \text{ or } a=b=1 \\ -1 & \text{if } (a=0,b=1) \text{ or } (a=1,b=0) \\ 0 & \text{otherwise}\end{cases}$</td></tr>
<tr><td colspan="2">(3) The step of generalization to a quasi Möbius-division category involving the free monoid Σ^* $(|\Sigma|>1)$; the Möbius function :</td></tr>
<tr><td>(A) The free monoid Σ^* adjoined with a terminal object 0;

$\mu(v)=\begin{cases} 1 & \text{if } v=1, \\ -1 & \text{if } |v|=1, \\ 0 & \text{otherwise.}\end{cases}$</td><td>(B) The category C_Σ:

- $Ob\,C_\Sigma=\Sigma^*$ with a terminal object 0 adjoined;
- $Hom(u,v)=\begin{cases}\{(a,v,b) \mid v=u'uu'', a=|u'|, b=|u''|\}, \\ \phi \quad \text{if } u \text{ is not a factor of } v.\end{cases}$
- $(a',w,b')\cdot(a,v,b)=(a'+a,w,b'+b)$;

$\mu(a,v,b)=\begin{cases} 1 & \text{if } a=b=0 \text{ or } a=b=1 \\ -1 & \text{if } (a=0,b=1) \text{ or } (a=1,b=0) \\ 0 & \text{otherwise}\end{cases}$</td></tr>
<tr><td colspan="2">(4) The resulting 0-locally finite combinatorial inverse monoid (its reduced standard division category is the above quasi Möbius-division category):</td></tr>
<tr><td>(A) The polycyclic monoid P_Σ</td><td>(B) The McAlister monoid M_Σ</td></tr>
</table>

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