Ann. Sci. Math. Québec **32** (2008), no 2, 211–219

ANNIHILATION OF THE TAME KERNEL FOR A FAMILY OF CYCLIC CUBIC EXTENSIONS

LLOYD D. SIMONS

Dedicated to John Labute on the occasion of his retirement.

RÉSUMÉ. Soit E/k une extension galoisienne de corps de nombres totalement réels avec $\mathrm{Gal}(E/k) = S_3$, et soit F/k la sous-extension quadratique intermédiaire. Pour tout nombre premier $l \neq 2, 3$, nous montrons que l'élément de Stickelberger (sous sa version intégrale) $w_2(E)\theta_{E/F}(-1)$ annule la ℓ-partie de $\mathrm{K}_2(\mathcal{O}_E)$, ce qui fournit une évidence supplémentaire en faveur de la véracité de la conjecture généralisée de Coates-Sinnott.

ABSTRACT. Let E/k be any S_3-extension of totally real number fields, and let F/k be the quadratic subextension. For any prime $\ell \neq 2, 3$, we show that the integralized Stickelberger element $w_2(E)\,\theta_{E/F}(-1)$ annihilates the ℓ-part of $\mathrm{K}_2(\mathcal{O}_E)$, providing evidence for the generalized Coates-Sinnott Conjecture.

1. Introduction

Let E/F be a finite abelian extension of number fields with Galois group G. For each character $\chi \in \widehat{G}$, let $L_{E/F}(s, \chi)$ be the Artin L-function associated to χ, and define the *generalized Stickelberger element* to be

$$\theta_{E/F}(s) = \sum_{\chi \in \widehat{G}} L_{E/F}(s, \chi)\mathbf{e}_{\bar{\chi}}.$$

Here, $\mathbf{e}_{\bar{\chi}}$ denotes the group ring idempotent

$$\mathbf{e}_{\bar{\chi}} \;=\; \frac{1}{|\,G\,|} \sum_{g \in G} \bar{\chi}(g) g.$$

The values $\theta_{E/F}(s)$ are generically in the complex group ring $\mathbb{C}[G]$, but it was shown by Klingen and Siegel that when n is a non-negative integer, the values $L_{E/F}(-n, \chi)$ are rational numbers, so that $\theta_{E/F}(-n) \in \mathbb{Q}[G]$. Moreover, as a consequence of a theorem of Deligne and Ribet, one has a bound on the denominators of these values. Indeed, let $\mathcal{G}$ be the absolute Galois group of E, and let μ_∞ be the group of all roots of unity. For m a positive integer, denote by $\mu_\infty^{\otimes m}$ the tensor product of m copies of the $\mathcal{G}$-module μ_∞, with diagonal action by $\mathcal{G}$. The $\mathcal{G}$-fixed points under this action form

Reçu le 3 juin 2008 et, sous forme définitive, le 4 décembre 2008.

a finite cyclic group of order $w_m(E)$, and Deligne-Ribet (and independently Cassou-Nougès) proved that $w_{n+1}(E)\theta_{E/F}(-n) \in \mathbb{Z}[G]$. For the purposes of this paper, we set $n = 1$ and we can interpret $w_2(E)$ as the number of roots of unity which lie in the compositum of all quadratic extensions of E. In particular, $w_2(\mathbb{Q}) = 24$.

Arithmetic interest in these integralized Stickelberger elements (and more generally, in the n-th *Stickelberger ideal* $(\theta_{E/F}(-n)\mathbb{Z}[G]) \cap \mathbb{Z}[G]$) began at the end of the nineteenth century when Stickelberger showed that, for $n = 0$, these elements annihilate the class groups of cyclotomic extensions of $\mathbb{Q}$ (see [12], chapter 6). Brumer conjectured that a similar statement should be true for abelian CM-extensions of totally real number fields. Now, if $\mathcal{O}_E$ denotes the ring of integers of a number field E, then the class group of E is the torsion subgroup of the algebraic K-group $K_0\mathcal{O}_E$. One can thus rephrase Stickelberger's Theorem (and Brumer's Conjecture) as the annihilation of this K-group. In light of conjectured connections between values of ζ-functions and orders of algebraic K-groups, Coates and Sinnott [1] conjectured that for $E/\mathbb{Q}$ abelian, the algebraic K-group $K_{2n}\mathcal{O}_E$ should be annihilated by suitably integralized higher Stickelberger elements $\beta\theta_{E/\mathbb{Q}}(-n)$ (Coates and Sinnott proved their conjecture for $n = 1$). The reader is referred to [3], [5], or [8] for recent generalizations. The main result of this paper, which we now state, gives a large family of examples of this "Stickelberger phenomenon".

Theorem 1.1. *Let E/k be any S_3-extension of totally real number fields, and let F be the fixed field of the subgroup of order 3. Then for any prime $\ell \neq 2, 3$, the integralized Stickelberger element $w_2(E)\theta_{E/F}(-1)$ annihilates the Sylow ℓ-subgroup of the "tame kernel" $\mathrm{K}_2(\mathcal{O}_E)$.*

2. A decomposition lemma

Let E be a cyclic extension of the number field F with Galois group $G = \langle\sigma\rangle$ of prime order p. For M a finite $\mathbb{Z}[G]$-module and ℓ a prime different from p, denote by $M(\ell)$ the ℓ-Sylow subgroup of M. As usual, M^G is the subset of M of all elements which are fixed by the action of G. Denote by

$$f_{E/F\,*} : \mathrm{K}_2(\mathcal{O}_F) \to \mathrm{K}_2(\mathcal{O}_E)$$

(or just f_* if it causes no confusion) the map induced by the inclusion $F \subset E$. Finally, denote by

$$f^*_{E/F} : \mathrm{K}_2(\mathcal{O}_E) \to \mathrm{K}_2(\mathcal{O}_F)$$

(or just f^* when there is no risk of confusion) the transfer map. The following lemma captures most of the algebraic K-theory that we will need for the proof of the theorem. For more details on the definitions and basic properties from algebraic K-theory, the reader is referred to [6].

Lemma 2.1. *For each $\ell \neq p$, we have*

$$f_*\mathrm{K}_2(\mathcal{O}_F)(\ell) \;=\; \mathrm{K}_2(\mathcal{O}_E)(\ell)^G \;=\; \left(\sum_{i=0}^{p-1}\sigma^i\right)\mathrm{K}_2(\mathcal{O}_E)(\ell),$$

and the decomposition

$$K_2(\mathcal{O}_E)(\ell) \;=\; (1-\sigma)K_2(\mathcal{O}_E)(\ell) \oplus f_* K_2(\mathcal{O}_F)(\ell)$$

as $\mathbb{Z}[G]$*-modules.*

Proof. The composition

$$f^* \circ f_* : \mathrm{K}_2(\mathcal{O}_F)(\ell) \to \mathrm{K}_2(\mathcal{O}_F)(\ell)$$

is multiplication by p, which is an isomorphism on the ℓ-parts (and we call the inverse map $1/p$). It follows that f_* is injective and f^* is surjective. Furthermore,

$$f_* \circ f^* : \mathrm{K}_2(\mathcal{O}_E)(\ell) \to \mathrm{K}_2(\mathcal{O}_E)(\ell)$$

is multiplication by the group ring norm element

$$\sum_{i=0}^{p-1} \sigma^i.$$

If $x \in \mathrm{K}_2(\mathcal{O}_E)(\ell)^G$, then $\sum_{i=0}^{p-1} \sigma^i x = px$, so that $\sum_{i=0}^{p-1} \sigma^i (1/p)x = x$. Thus, f_* maps $\mathrm{K}_2(\mathcal{O}_F)(\ell)$ isomorphically onto $\mathrm{K}_2(\mathcal{O}_E)(\ell)^G$. It also shows that the Tate cohomology group $\widehat{\mathrm{H}}^0(G, \mathrm{K}_2(\mathcal{O}_E)(\ell))$ is trivial, so

$$\widehat{\mathrm{H}}^1(G, \mathrm{K}_2(\mathcal{O}_E)(\ell)) \;=\; \mathrm{Im}(\sigma - 1)\Big/\ker\left(\sum_{i=0}^{p-1} \sigma^i\right)$$

is also trivial. Now,

$$\ker\left(\sum_{i=0}^{p-1} \sigma^i\right) \;=\; \ker(f_* \circ f^*) \;=\; \ker(f^*),$$

which shows that the sequence

$$1 \longrightarrow (1-\sigma)\mathrm{K}_2(\mathcal{O}_E)(\ell) \longrightarrow \mathrm{K}_2(\mathcal{O}_E)(\ell) \xrightarrow{f^*} \mathrm{K}_2(\mathcal{O}_F)(\ell) \longrightarrow 1$$

is exact, and is split by $f_* \circ (1/p)$. □

3. Rewriting $\theta_{E/F}(s)$

The main result of [7] shows that if $\mathcal{E}/F$ is a multiquadratic extension of totally real number fields (and if the Birch-Tate conjecture is true), then $w_2(\mathcal{E})\theta_{\mathcal{E}/F}(-1)$ annihilates $\mathrm{K}_2(\mathcal{O}_\mathcal{E})$ except possibly for some rare exceptions. Critical to the proof of this theorem is the simple fact that the Artin L-function, for the non-principal character of a quadratic extension, is just the ratio of two Dedekind zeta functions, and thus (using Birch-Tate) can be directly related to orders of K-groups. In this section and in the next section, we rewrite the Stickelberger element with a similar goal in mind.

Let E/F be a cyclic extension of order a prime p, and let G be its Galois group. Fix a generator σ of G, and let ξ be a primitive p-th root of unity. Define the character

$\chi \in \widehat{G}$ by $\chi(\sigma) = \xi$. Then χ generates the dual group $\widehat{G}$ and we can write

$$\begin{aligned}\theta_{E/F}(s) \;=\; \sum_{i=0}^{p-1} L_{E/F}(s,\chi^i)\mathbf{e}_{\bar{\chi}^i} &= \frac{1}{p}\sum_{i=0}^{p-1}\sum_{j=0}^{p-1} L_{E/F}(s,\chi^i)\bar{\chi}^i(\sigma^{-j})\sigma^j \\ &= \frac{1}{p}\sum_{i=0}^{p-1}\sum_{j=0}^{p-1} L_{E/F}(s,\chi^i)\xi^{ij}\sigma^j \\ &= \zeta_F(s)\frac{1}{p}\sum_{j=0}^{p-1}\sigma^j + \frac{1}{p}\sum_{i=1}^{p-1}\sum_{j=0}^{p-1} L_{E/F}(s,\chi^i)\xi^{ij}\sigma^j \\ &= \zeta_F(s)\frac{1}{p}\sum_{j=0}^{p-1}\sigma^j + \frac{1}{p}\sum_{j=0}^{p-1}\sum_{i=1}^{p-1} L_{E/F}(s,\chi^i)\xi^{ji}\sigma^j.\end{aligned}$$

Let $K = \mathbb{Q}(\xi)$ and let Δ be the Galois group of $K/\mathbb{Q}$. If m is a negative integer and $g \in G$, Deligne and Ribet showed that the partial zeta functions $\zeta_{E/F}(m,g)$ are rational. It follows that

$$L_{E/F}(m,\chi) \;=\; \sum_{g\in G}\zeta_{E/F}(m,g)\chi(g)$$

is in K. If the action of $\delta_i \in \Delta$ on K is determined by $\xi^{\delta_i} = \xi^i$, for $i = 1, 2, \ldots, p-1$, then $L_{E/F}(m,\chi)^{\delta_i} = L_{E/F}(m,\chi^i)$. As a consequence, one has

$$\sum_{i=1}^{p-1} L_{E/F}(s,\chi^i)\xi^{ji} \;=\; \sum_{i=1}^{p-1}(L_{E/F}(m,\chi)\xi^j)^{\delta_i} \;=\; \mathrm{Tr}_{K/\mathbb{Q}}(L_{E/F}(m,\chi)\xi^j).$$

We finally arrive at the formula

$$\theta_{E/F}(m) \;=\; \zeta_F(m)\frac{1}{p}\sum_{j=0}^{p-1}\sigma^j + \frac{1}{p}\sum_{j=0}^{p-1}\mathrm{Tr}_{K/\mathbb{Q}}(L_{E/F}(m,\chi)\xi^j)\sigma^j.$$

Remark 3.1. The sum

$$\sum_{j=0}^{p-1}\mathrm{Tr}_{K/\mathbb{Q}}(L_{E/F}(m,\chi)\xi^j)\sigma^j$$

is in the augmentation ideal of $\mathbb{Q}[G]$ since (upon setting $\sigma = 1$) we have

$$\sum_{j=0}^{p-1}\mathrm{Tr}_{K/\mathbb{Q}}\left(L_{E/F}(m,\chi)\xi^j\right) \;=\; \mathrm{Tr}_{K/\mathbb{Q}}\left(L_{E/F}(m,\chi)\sum_{j=0}^{p-1}\xi^j\right) \;=\; 0.$$

Remark 3.2. If $L_{E/F}(m,\chi) \in \mathbb{Q}$, then we have

$$\sum_{j=0}^{p-1}\mathrm{Tr}_{K/\mathbb{Q}}\left(L_{E/F}(m,\chi)\xi^j\right)\sigma^j \;=\; L_{E/F}(m,\chi)\sum_{j=0}^{p-1}\mathrm{Tr}_{K/\mathbb{Q}}(\xi^j)\sigma^j.$$

It follows that

$$\theta_{E/F}(m) \;=\; \zeta_F(m)\frac{1}{p}\sum_{j=0}^{p-1}\sigma^j + L_{E/F}(m,\chi)\left(1-\frac{1}{p}\sum_{j=0}^{p-1}\sigma^j\right). \tag{1}$$

4. Character relations

For the remainder of this paper, E is a totally real extension of a field k with Galois group $\mathcal{A}$ isomorphic to S_3. Let G be the normal subgroup of order 3 and fix a generator σ of G. Let τ be any element of order 2 in $\mathcal{A}$ and let H be the subgroup generated by τ. Denote by F and k_1, the fields fixed by G and H, respectively. There are three irreducible characters of $\mathcal{A}$:

(1) The principal character ψ_0;
(2) The non-trivial, one dimensional character ψ_1;
(3) The character ψ_2 of an irreducible two-dimensional representation.

From the basic theory of Artin L-functions (see [10]), one has

$$\zeta_E(s) \;=\; \zeta_k(s)L_{E/k}(s,\psi_1)L_{E/k}(s,\psi_2)^2 \;=\; \zeta_F(s)L_{E/k}(s,\psi_2)^2. \tag{2}$$

Let ξ be a primitive third root of unity, and let $\chi \in \widehat{G}$ be the character defined by $\chi(\sigma)=\xi$. One can check that

$$\mathrm{Ind}_G^{\mathcal{A}}\chi \;=\; \mathrm{Ind}_G^{\mathcal{A}}\bar{\chi} \;=\; \psi_2,$$

so that

$$L_{E/k}(s,\psi_2) \;=\; L_{E/F}(s,\chi) \;=\; L_{E/F}(s,\bar{\chi}).$$

Similarly, with χ_1 the non-trivial character of H, one can show that

$$\mathrm{Ind}_H^{\mathcal{A}}\chi_1 = \psi_1+\psi_2,$$

so that

$$L_{E/k_1}(s,\chi_1) \;=\; L_{E/k}(s,\psi_1)L_{E/k}(s,\psi_2) \;=\; \frac{\zeta_F(s)}{\zeta_k(s)}L_{E/F}(s,\chi). \tag{3}$$

Replacing $L_{E/k_1}(s,\chi_1)$ by $\zeta_E(s)/\zeta_{k_1}(s)$, using equations (2) and (3), and rearranging gives

$$L_{E/F}(s,\chi) \;=\; \frac{\zeta_{k_1}(s)}{\zeta_k(s)}. \tag{4}$$

In particular, the values of $L_{E/F}(s,\chi)$ at negative integers m are rational numbers. Letting $\lambda=(1/3)(1+\sigma+\sigma^2)$, we can apply equation (1) to get an explicit expression for $\theta_{E/F}(m)$ in terms of Dedekind ζ-functions, namely,

$$\theta_{E/F}(m) \;=\; \zeta_F(m)\lambda + \frac{\zeta_{k_1}(m)}{\zeta_k(m)}(1-\lambda). \tag{5}$$

In 1970, Birch and Tate (see [9]) conjectured that if F is a totally real number field, then

$$\zeta_F(-1) \;=\; (-1)^{[F:\mathbb{Q}]}\frac{|\,\mathrm{K}_2(\mathcal{O}_F)\,|}{w_2(F)}.$$

This formula is now known to be true (as a consequence of the Main Conjecture in Iwasawa Theory) for F abelian over $\mathbb{Q}$, and for arbitrary totally real number fields F up to a power of 2. Setting $m = -1$ in (5) and using Birch-Tate, we obtain the following expression for the *integralized* Stickelberger element:

$$w_2(E)\theta_{E/F}(-1) \;=\; \frac{w_2(E)|\,\mathrm{K}_2(\mathcal{O}_F)\,|}{w_2(F)}\lambda + w_2(E)\frac{|\,\mathrm{K}_2(\mathcal{O}_{k_1})\,|w_2(k)}{|\,\mathrm{K}_2(\mathcal{O}_k)\,|w_2(k_1)}(1-\lambda).$$

Lemma 4.1. *We have $w_2(k_1) = w_2(k)$ and $w_2(E) = w_2(F)$.*

Proof. Both extensions E/F and k_1/k are of degree 3 and are clearly disjoint from the cyclotomic $\mathbb{Z}_3$-extensions of F and k, respectively. □

In light of Lemma 4.1, we may simplify our expression for $w_2(E)\theta_{E/F}(-1)$:

$$w_2(E)\theta_{E/F}(-1) \;=\; |\,\mathrm{K}_2(\mathcal{O}_F)\,|\lambda + w_2(E)\frac{|\,\mathrm{K}_2(\mathcal{O}_{k_1})\,|}{|\,\mathrm{K}_2(\mathcal{O}_k)\,|}(1-\lambda). \tag{6}$$

5. Proof of the theorem

We will show that each of the two pieces of $w_2(E)\theta_{E/F}(-1)$ on the right hand side of (6) annihilates $\mathrm{K}_2(\mathcal{O}_E)(\ell)$. These pieces are not in fact in $\mathbb{Z}[G]$, but are in $\mathbb{Z}_\ell[G]$. Thus, we understand this statement in the following sense: if M is a finite ℓ-power torsion $\mathbb{Z}[G]$- module, we can identify it with its image in the $\mathbb{Z}_\ell[G]$- module $M \otimes \mathbb{Z}_\ell$. We have that $\theta \in \mathbb{Z}[G]$ annilhilates M if and only if

$$\theta \otimes 1 \in \mathbb{Z}[G] \otimes \mathbb{Z}_\ell \cong \mathbb{Z}_\ell[G]$$

annilhilates $M \otimes \mathbb{Z}_\ell$. If now $\theta \otimes 1 = \alpha + \beta$ with $\alpha, \beta \in \mathbb{Z}_\ell[G]$, and if both α and β individually annihilate $M \otimes \mathbb{Z}_\ell$, then θ annihilates M.

We now proceed with the proof of the theorem. Let $x \in \mathrm{K}_2(\mathcal{O}_E)(\ell)$ and use Lemma 2.1 to find a $y \in \mathrm{K}_2(\mathcal{O}_E)(\ell)$ and a $z \in \mathrm{K}_2(\mathcal{O}_F)(\ell)$ so that $x = (1-\sigma)y + f_*(z)$. Then $\lambda(1-\sigma) = 0$ and $|\,\mathrm{K}_2(\mathcal{O}_F)\,|z = 0$, so $|\,\mathrm{K}_2(\mathcal{O}_F)\,|\lambda x = 0$. Furthermore,

$$f_*(z) \in (\mathrm{K}_2(\mathcal{O}_E)(\ell))^G \;=\; \lambda \mathrm{K}_2(\mathcal{O}_E)(\ell),$$

and is thus annihilated by $(1-\lambda)$. All that remains to be shown is that

$$w_2(E)\frac{|\,\mathrm{K}_2(\mathcal{O}_{k_1})\,|}{|\,\mathrm{K}_2(\mathcal{O}_k)\,|}$$

annihilates $(1-\sigma)\mathrm{K}_2(\mathcal{O}_E)(\ell)$. This we accomplish with a pair of lemmas.

Lemma 5.1. *The $\mathbb{Z}[G]$-module $(1-\sigma)K_2(\mathcal{O}_E)(\ell)$ is stable under the action of $H = \mathrm{Gal}(E/k_1)$, and its order is the power of ℓ dividing $\left(\frac{\zeta_{k_1}(-1)}{\zeta_k(-1)}\right)^2$.*

Proof. The idempotent $1-\lambda$ factors as $(1-\sigma)(2+\sigma)/3$, where $(2+\sigma)/3$ is a unit of $\mathbb{Z}_\ell[G]$. Hence

$$(1-\sigma)\mathrm{K}_2(\mathcal{O}_E)(\ell) = (1-\lambda)\mathrm{K}_2(\mathcal{O}_E)(\ell).$$

Since τ commutes with $1-\lambda$, the first claim of the lemma is true. Now on the one hand, by Lemma 2.1, we have

$$|\,(1-\sigma)\mathrm{K}_2(\mathcal{O}_E)(\ell)\,| \;=\; \frac{|\,\mathrm{K}_2(\mathcal{O}_E)(\ell)\,|}{|\,\mathrm{K}_2(\mathcal{O}_F)(\ell)\,|}.$$

On the other hand, Birch-Tate and Lemma 4.1 give

$$\frac{|\,\mathrm{K}_2(\mathcal{O}_E)\,|}{|\,\mathrm{K}_2(\mathcal{O}_F)\,|} \;=\; \frac{|\,\zeta_E(-1)\,|}{|\,\zeta_F(-1)\,|}\frac{w_2(F)}{w_2(E)} \;=\; \frac{|\,\zeta_E(-1)\,|}{|\,\zeta_F(-1)\,|}.$$

Now equations (2) and (4) give

$$\frac{|\,\zeta_E(-1)\,|}{|\,\zeta_F(-1)\,|} \;=\; \left(\frac{|\,\zeta_{k_1}(-1)\,|}{|\,\zeta_k(-1)\,|}\right)^2,$$

and taking ℓ-parts completes the proof of the lemma. □

Lemma 2.1 can be applied to the quadratic extension E/k_1; in particular one can conclude that the group of τ-fixed points of $\mathrm{K}_2(\mathcal{O}_F)(\ell)$ is precisely the image of the map

$$f_{E/k_1*} : \mathrm{K}_2(\mathcal{O}_{k_1})(\ell) \longrightarrow \mathrm{K}_2(\mathcal{O}_E)(\ell).$$

Consider now the composition

$$\gamma = (1-\lambda)\circ f_{E/k_1*} : \mathrm{K}_2(\mathcal{O}_{k_1})(\ell) \longrightarrow (1-\sigma)\mathrm{K}_2(\mathcal{O}_E)(\ell).$$

Lemma 5.2. *The image of the map γ is precisely the group of τ-fixed points in $(1-\sigma)K_2(\mathcal{O}_E)(\ell)$, and the kernel of γ is $f_{k_1/k*}(K_2(\mathcal{O}_k)(\ell))$.*

Proof. Certainly the image of γ is fixed by τ. Conversely, if $x \in (1-\sigma)\mathrm{K}_2(\mathcal{O}_E)(\ell)$ is fixed by τ, then $x = f_{E/k_1*}(y)$ for some $y \in \mathrm{K}_2(\mathcal{O}_{k_1})(\ell)$. But $(1-\lambda)x = x$, proving the surjectivity claim. Now given any $z \in \mathrm{K}_2(\mathcal{O}_k)(\ell)$, we have

$$f_{E/k_1*}\circ f_{k_1/k*}(z) = f_{E/F*}\circ f_{F/k*}(z) \in \lambda\mathrm{K}_2(\mathcal{O}_E)(\ell),$$

which is killed by $1-\lambda$, so $f_{k_1/k*}(z) \in \ker(\gamma)$.

Conversely, if $w \in \ker(\gamma)$, then $f_{E/k_1*}(w)$ is killed by $1-\sigma$ and hence, must be of the form $f_{E/F*}(u)$ for some $u \in \mathrm{K}_2(\mathcal{O}_F)(\ell)$. But u must be fixed by τ (since $f_{E/F*}(u)$ is fixed by τ and $f_{E/F*}$ is injective on ℓ-parts). Another application of Lemma 2.1 (for the extension F/k this time) shows that $u = f_{F/k*}(v)$ for some $v \in \mathrm{K}_2(\mathcal{O}_k)(\ell)$ and $f_{k_1/k*}(v) = w$ from the injectivity of f_{E/k_1*}. □

We can now complete the proof of the theorem. The action of τ decomposes the group $(1-\sigma)\mathrm{K}_2(\mathcal{O}_E)(\ell)$ into a direct sum of τ-fixed points (the *plus* part) and points on which τ acts as -1 (the *minus* part). From Lemma 5.2 above, the order of the plus part is

$$\frac{|\,\mathrm{K}_2(\mathcal{O}_{k_1})(\ell)\,|}{|\,f_{k_1/k*}\,(\mathrm{K}_2(\mathcal{O}_k)(\ell))\,|} \;=\; \frac{|\,\mathrm{K}_2(\mathcal{O}_{k_1})(\ell)\,|}{|\,\mathrm{K}_2(\mathcal{O}_k)(\ell)\,|},$$

which, by Birch-Tate and Lemma 4.1, is the ℓ-part of $|\,\zeta_{k_1}(-1)/\zeta_k(-1)\,|$. By Lemma 5.1, the minus part must have the same order. In particular, $|\,\zeta_{k_1}(-1)/\zeta_k(-1)\,|$ annihilates $(1-\sigma)\mathrm{K}_2(\mathcal{O}_E)(\ell)$, and the theorem is proved.

6. Closing comments

The version of the generalized Coates-Sinnott Conjecture presented in this paper is rather simplified. More common is to specify S to be a finite set of primes containing the primes of k which ramify in the extension E/k. One can replace the rings of integers, the associated tame kernels, the zeta functions, and $\theta_{E/F}(m)$ itself by S-versions. This allows the following more precise annihilation statement.

For a prime ideal $\wp \notin S$, let $\sigma_\wp \in \mathrm{Gal}(E/F)$ be the Artin symbol. Then the (integral, again by Deligne-Ribet) group ring element

$$(\mathcal{N}\wp^2 - \sigma_\wp)\,\theta^S_{E/F}(-1)$$

(conjecturally) annihilates the K-group $\mathrm{K}_2(\mathcal{O}^S_E)$.

We have not carried out the details, but an S-version of our main theorem should be true with essentially the same proof. Similarly, there should be no great problems in extending the techniques involved in this paper to prove the annihilation of the ℓ-parts of $K_{2n}(\mathcal{O}_E)$ by integralized higher Stickelberger elements $w_{n+1}(E)\theta_{E/F}(-n)$ for $\ell \neq 2, 3$. Our approach fails for the 2 and 3-parts because Lemmas 2.1 and 5.2 are no longer valid, but there is hope that combining the techniques of this paper with the methods of [7] may at least manage the 2-part.

Acknowledgements. The author wishes to thank Jonathan Sands for many helpful discussions, and the referee for pointing out the simple proof of Lemma 4.1. Also special thanks to John Labute for initially piquing my interest in the connections between algebraic K-theory and algebraic number theory.

REFERENCES

[1] J. Coates and W. Sinnott, *An analogue of Stickelberger's Theorem for the higher K-groups*, Invent. Math. **24** (1974), 149–161.

[2] P. Deligne and K. Ribet, *Values of abelian L-functions at negative integers over totally real fields*, Invent. Math. **59** (1980), no. 3, 227–286.

[3] C. Greither, *Arithmetic annihilators and Stark-type conjectures*, Contemp. Math. **358** (2004), 55–78.

[4] M. Kolster,*K-theory and arithmetic*, Contemporary developments in algebraic K-theory, ICTP Lect. Notes **15** (2003), 191–258.

[5] M. Kolster and J. Sands, *Annihilation of motivic cohomology groups in cyclic 2-extensions*, preprint.

[6] J. Milnor, *Introduction to algebraic K-theory*, Annals of Mathematics Studies **72**, Princeton U. Press, Princeton, 1971, xiii + 184 pp.

[7] J. Sands and L.D. Simons, *Values at $s = -1$ of L-functions for multi-quadratic extensions of number fields and the annihilation of the tame kernel*, J. Lond. Math. Soc. (2) **76** (2007), no. 3, 545–555.

[8] V. Snaith, *Relative K_0, annihilators, Fitting ideals and the Stickelberger phenomena*, Proc. London Math. Soc. (3) **90** (2005), no. 3, 545–590.

[9] J. Tate, *Symbols in arithmetic*, Actes du Congrès International des Mathématiciens, Tome 1, Gauthier-Villars, Paris, 1971, 201–211.

[10] J. Tate, *Les conjectures de Stark sur les fonctions L d'Artin en $s = 0$*, Lecture notes edited by Dominique Bernardi and Norbert Schappacher, Progress in Mathematics, Boston, 1984, 143 pp.

[11] L.C. Washington, *Introduction to cyclotomic fields*, Graduate Texts in Mathematics **83**, Springer-Verlag, New York, 1982, xi + 389 pp.

[12] A. Wiles, *The Iwasawa conjecture for totally real fields*, Ann. of Math.(2) **131** (1990), no. 3, 493–540.

LLOYD D. SIMONS, SAINT MICHAEL'S COLLEGE, COLCHESTER, VERMONT, USA.
lsimons@smcvt.edu