Ann. Sci. Math. Québec **32** (2008), no 2, 189–197

FABULOUS PRO-p-GROUPS

JOHN LABUTE

To John Tate and Jean-Pierre Serre for their direction and inspiration.

RÉSUMÉ. Soit p un premier impair. Un pro-p-groupe G est dit fabuleux si, en plus d'être un pro-p-groupe quadratique, G est aussi doux et fab. Les seuls exemples connus sont des groupes de Galois de corps de nombres qui sont des p-extensions non ramifiées en dehors d'un ensemble fini S de premiers de caractéristiques résiduelles différentes de p. Nous ne connaissons pas un seul exemple d'un pro-p-groupe fabuleux ayant une présentation explicite. Cet article se veut une tentative pour trouver de tels exemples.

ABSTRACT. Let p be an odd prime. A pro-p-group G is said to be fabulous if it is a mild quadratic pro-p-group that is also fab. The only known examples appear as Galois groups of maximal p-extensions number fields unramified outside a finite set S of primes with residual characteristics different from p. We do not have a single example of a fabulous pro-p-group having an explicit presentation. This paper is an attempt to find such examples.

1. Introduction

Let p be an odd prime. We call a quadratic pro-p-group fabulous if it is fab and mild. These groups appear often as the Galois group $G_S(p)$ of the maximal p-extension of a number field K that is unramified outside a finite set S of primes with residual characteristics different from p (the tame case); *cf.* [6], [14], [9], [10], [12]. They also appear in the case of restricted ramification and prescribed decomposition in the mixed case; *cf.* [16], [15], [11], even for function fields in [8], [12].

In view of the importance of these groups for the Fontaine-Mazur Conjecture, *cf.* [2], it would be desirable to have some kind of classification of these groups. However, up to now, we do not even have an explicit presentation for a single fabulous group.

2. Definitions

Definition 1. A pro-p-group G is said to be *fab* if $H^{ab} = H/[H,H]$ is finite for every closed subgroup H of G of finite index or, equivalently, the factors of the derived series of G are all finite.

Reçu le 30 avril 2008 et, sous forme définitive, le 17 octobre 2008.

Examples of fab pro-p-groups are finite p-groups or pro-p-groups G that are p-adic analytic with $\mathrm{Lie}(G) = [\mathrm{Lie}(G), \mathrm{Lie}(G)]$; for example, an open pro-p-subgroup of $SL_n(\mathbb{Z}_p)$. The groups $G_S(p)$ are fab for a number field K in the tame case since the ramification is tame at the primes of S. We do not have a single example of an infinite non-analytic fab pro-p-group having an explicit presentation.

A fab pro-p-group G is a finitely generated group with minimal number of generators $d = \dim_{\mathbb{F}_p} G/G^p[G,G]$ and minimal number of relators $r \geq d$. We have

$$d = d(G) = \dim H^1(G) \quad \text{and} \quad r = r(G) = \dim H^2(G),$$

where $H^i(G) = H^i(G, \mathbb{Z}/p/Z)$. Since $p \neq 2$, the cup product

$$H^1(G) \otimes H^1(G) \to H^2(G)$$

yields a linear map

$$\phi : \bigwedge^2 H^1(G) \to H^2(G).$$

Definition 2. A finitely generated pro-p-group G is said to be *quadratic* if the linear map ϕ defined above is surjective.

The group G is quadratic if and only if the dual map

$$\phi^* : H^2(G)^* \to \left(\bigwedge^2 H^1(G)\right)^* = \bigwedge^2 H^1(G)^*$$

is injective. Let $V = H^1(G)^*$ and let L be the Lie algebra which is universal for linear mappings of V into Lie algebras over $\mathbb{F}_p$. If $\{\xi_1, \ldots, \xi_d\}$ is a basis for V, then L is the free Lie algebra over $\mathbb{F}_p$ on $\xi_1, \ldots, \xi_d$. Then $\bigwedge^2 H^1(G)^*$ can be identified with L_2, the degree 2 component of the graded Lie algebra L.

Let $\mathfrak{r}$ be the ideal of L generated by the image W of ϕ^*. Then $\mathfrak{r}/[\mathfrak{r}, \mathfrak{r}]$ is a module over $\mathfrak{g} = L/\mathfrak{r}$ via the adjoint representation. The Lie algebra $\mathfrak{g} = L/\mathfrak{r}$ is called the *holonomy Lie algebra* of G; it is an invariant of G. If U is the enveloping algebra of $\mathfrak{g}$, then $M = \mathfrak{r}/[\mathfrak{r}, \mathfrak{r}]$ is a finitely generated U-module. If M is a free U-module on the image of one (and hence any) basis $\{\rho_1, \ldots, \rho_m\}$ for W, then the Lie algebra $\mathfrak{g}$ is said to be *mild*, in which case the sequence $\rho_1, \ldots, \rho_m$ is said to be *strongly free*. If $c_n = \dim_{\mathbb{F}_p} \mathfrak{g}$, the formal power series

$$P(t) = \sum_{n \geq 0} c_n t^n$$

is called the *Poincaré series* of the graded algebra $\mathfrak{g}$. This Lie algebra is mild if and only if $1/P(t) = 1 - dt + mt^2$ (cf. [6, Prop 3.2]), in which case $m \leq d^2/4$ since the radius of convergence of $P(t)$ is greater than 0 and less than or equal to 1.

Definition 3. A quadratic pro-p-group G is said to be *mild* if its holonomy Lie algebra is mild.

Conversely, let $\rho_1, \ldots, \rho_m$ be a sequence of homogeneous elements of degree 2 in the free $\mathbb{F}_p$-Lie algebra L on $\xi_1, \ldots, \xi_d$ and let $\mathfrak{r}$ be the ideal of L generated by

$\rho_1, \ldots, \rho_m$. In order to construct a quadratic group G whose holonomy Lie algebra is $\mathfrak{g}$, let

$$\rho_k = \sum_{i<j} \bar{a}_{ijk}[\xi_i, \xi_j]$$

with $\bar{a}_{ijk} \in \mathbb{F}_p$. Let F be the free pro-p-group on $x_1, \ldots, x_d$ and let R be the normal subgroup of F generated by $r_1, \ldots, r_m$ where

$$r_k = \prod_{j=1}^{d} x_j^{p\,a_{kj}} \prod_{i<j} [x_i, x_j]^{a_{ijk}} u_k$$

with $a_{kj} \in \mathbb{Z}_p$, $a_{ijk} \in \mathbb{Z}_p$ a lift of $\bar{a}_{ijk}$ to $\mathbb{Z}_p$ and $u_k \in \mathbb{F}_3$, the third term of the lower p-central series (F_n) of F defined by $F_1 = F$, $F_{n+1} = F_n^p[F, F_n]$. Let $\mathfrak{L}(F)$ be the graded Lie algebra associated to the lower p-central series of F. It is a Lie algebra over $\mathbb{F}_p[\pi]$ where the action of the variable π is induced by the p-th power map in F and the Lie bracket is induced by the commutator operation. Note that the n-th homogeneous component $\mathfrak{L}_n(F) = F_n/F_{n+1}$ is denoted additively.

Since $\mathfrak{L}(F)$ is the free Lie algebra over $\mathbb{F}_p[\pi]$ on $\xi_1, \ldots, \xi_d$, where ξ_i is the image of x_i in $V = \mathfrak{L}_1 = F/F^p[F, F]$, we can identify the $\mathbb{F}_p$-Lie subalgebra of $\mathfrak{L}(F)$ generated by $\xi_1, \ldots, \xi_d$ with the free lie algebra L over $\mathbb{F}_p$ on these elements. We also have $\mathfrak{L}(F)/\pi\mathfrak{L}(F) = L$.

Then $G = F/R$ has holonomy Lie algebra $\mathfrak{g}$. To see this, we use the fact that under the identification of $H^2(G)^*$ with $R/R^p[R, F]$ via the transpose of the transgression map associated to the exact sequence

$$1 \to R \to F \to G \to 1,$$

the image of r_k under ϕ is ρ_k; *cf.* [5, Prop. 3]. This map is bijective since $R \subseteq F_2$ implies that the inflation map $H^1(G) \to H^1(F)$ is bijective. Note that G is quadratic if and only if the sequence $\rho_1, \ldots, \rho_m$ is linearly independent, in which case $m = r(G)$. Note also that the group G depends on the parameters $u_1, \ldots, u_m$ but that the holonomy Lie algebra is the same for all choices of these parameter. We call these groups *twists* of the group corresponding to the choice $u_1 = \cdots = u_m = 1$.

Proposition 4. *If G is mild (in which case G is quadratic), then G is of cohomological dimension 2 and*

$$\mathfrak{L}(G) = \langle \xi_1, \ldots, \xi_d \mid \sigma_1, \ldots, \sigma_m \rangle,$$

with $\sigma_k = \sum_j a_{kj}\pi + \rho_k$. Moreover, G is not p-adic analytic if $d > 2$ since $m \le d^2/4$.

For the first statement, *cf.* [6, Theorem 4.1], and *cf.* [13, p. 68, Exercise (c)], for the second.

There is no general algorithm for determining whether the above finitely presented pro-p-group G is mild or not. However, we do have sufficient conditions which yield a rich supply of mild groups; *cf.* [6, Theorem 3.3]. The following invariant formulation of these conditions for quadratic groups is due to Alexander Schmidt; *cf.* [12, Theorem 6.2].

Proposition 5. *If $H^2(G) \neq 0$ and $H^1(G) = U_1 \oplus U_2$ with the cup-product ϕ trivial on $U_2 \wedge U_2$ and $\phi(U_1 \wedge U_2) = H^2(G)$, then G is mild.*

This is equivalent to saying that $m > 1$ and that the presentation can be chosen so that the generating set for F can be divided into two disjoint sets by a partition A, B of $\{1, \ldots, m\}$ with the associated holonomy relators $\rho_1, \ldots, \rho_m$ satisfying

$$\rho_k = \sum_{i \in A} a_{ijk}[\xi_i, \xi_j]$$

and, setting

$$\rho'_k = \sum_{i \in A, j \in B} a_{ijk}[\xi_i, \xi_j],$$

we have that $\rho'_1, \ldots, \rho'_m$ is a linearly independent sequence. For example, the pro-p-group

$$G = \langle x_1^p[x_1, x_2],\ x_2^p[x_2, x_3],\ x_3^p[x_3, x_4],\ x_4^p[x_4, x_1] \rangle$$

is a mild quadratic non-analytic pro-p-group with $d(G) = r(G) = 4$ since the associated holonomy relators

$$[\xi_1, \xi_2],\ [\xi_2, \xi_3],\ [\xi_3, \xi_4],\ [\xi_4, \xi_1]$$

satisfy this with $A = \{1, 3\}$, $B = \{2, 4\}$; here $\rho'_k = \rho_k$.

However, an algorithm for mildness exists when $d = m = 4$; *cf.* [3]. To state this algorithm here we will use the quadratic form $u \mapsto u \wedge u$ on $\bigwedge^2 V$ when V is 4-dimensional so that $\bigwedge^4 V = \mathbb{F}_p$ (setting $\xi_1 \wedge \xi_2 \wedge \xi_3 \wedge \xi_4 = 1$). The associated bilinear form is $b(u, v) = u \wedge v$. If $\xi_1, \ldots, \xi_4$ is a basis of V, then the elements $\xi_i \wedge \xi_j$, with $i < j$, ordered lexicographically form a basis for $\bigwedge^2 V$ and the matrix of b with respect to this basis is

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Proposition 6. *Let V be a 4-dimensional vector space over $\mathbb{F}_p$ and let W be a four dimensional subspace of $\bigwedge^2 V$ spanned by $\rho_1, \ldots, \rho_4$. Then the sequence $\rho_1, \ldots, \rho_4$ is strongly free if and only if $W^\perp \cap W = 0$.*

This result follows directly from the main result of [3]. Identifying $\bigwedge^2 V$ with L_2 (so that $\xi_i \wedge \xi_j = [\xi_i, \xi_j]$), we obtain for example that

$$\begin{cases} \rho_1 = [\xi_1, \xi_2] + 2[\xi_1, \xi_3] + [\xi_1, \xi_4], \\ \rho_2 = [\xi_2, \xi_3] + [\xi_2, \xi_4], \\ \rho_3 = 2[\xi_3, \xi_1] + 2[\xi_3, \xi_4], \\ \rho_4 = [\xi_4, \xi_2] + 2[\xi_4, \xi_3] \end{cases}$$

form a strongly free sequence. In [3] it is shown that a mild quadratic algebra

$$\mathfrak{g} = \langle \xi_1, \ldots, \xi_4 \mid \rho_1, \ldots, \rho_4 \rangle$$

isomorphic to precisely one of the two mild quadratic algebras

$$\mathfrak{g}_1 = \langle \xi_1, \dots, \xi_4 \mid [\xi_1, \xi_2],\ [\xi_2, \xi_3],\ [\xi_3, \xi_4],\ [\xi_4, \xi_1] \rangle, \text{ and}$$
$$\mathfrak{g}_2 = \langle \xi_1, \dots, \xi_4 \mid [\xi_1, \xi_2],\ [\xi_2, \xi_3] + [\xi_4, \xi_1],\ [\xi_3, \xi_4],\ [\xi_4, \xi_2] + g[\xi_1, \xi_3] \rangle$$

with g a non-square. It is said to be of *type I* (resp. *type II*) if it is isomorphic to $\mathfrak{g}_1$ (resp. $\mathfrak{g}_2$). It is of type I if and only if the quotient $\mathfrak{g}/[\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]]$ has an element whose centralizer is of dimension 5. The relators in our example above are of type I.

Definition 7. A pro-p-group G is said to be *fabulous* if it is quadratic, mild and fab.

The only known examples of non-analytic fabulous pro-p-groups are the tame Galois groups $G_S(p)$. When $K = \mathbb{Q}$ and $S = \{q_1, \dots, q_d\}$, with $q_i \equiv 1 \pmod p$, we have the following presentation of $G_S(p)$ due to Koch; *cf.* [4, Example 11.11]:

$$G_S(p) = \langle x_1, \dots, x_d \mid r_1, \dots, r_d \rangle$$

with $r_i = x_i^{q_i - 1}[x_i^{-1}, y_i^{-1}]$, where $y_i \equiv \prod_{j=1}^d x_j^{\ell_{ij}} \pmod{F_p[F, F]}$. This presentation is only partially known, but ℓ_{ij}, for $i \neq j$, is the residue class modulo p of any integer satisfying

$$q_i = g_i^{c_{ij}} \pmod{q_j}$$

with g_i a fixed primitive root mod q_j. We have

$$r_i = x_i^{q_i - 1} \prod_{j \neq i} [x_i, x_j]^{\ell_{ij}} u_i$$

with $u_i \in F_3$. Thus the holonomy relators $\rho_1, \dots, \rho_d$ are given by

$$\rho_i = \sum_{j \neq i} \ell_{ij} [\xi_i, \xi_j].$$

The elements ℓ_{ij} are called the *linking numbers* of the Koch presentation for $G_S(p)$.

If $p = 3$ and $S = \{7, 13, 31, 43\}$, we find

$$\begin{cases} \rho_1 &= [\xi_1, \xi_2] + 2[\xi_1, \xi_3] + [\xi_1, \xi_4], \\ \rho_2 &= [\xi_2, \xi_3] + [\xi_2, \xi_4], \\ \rho_3 &= 2[\xi_3, \xi_1] + 2[\xi_3, \xi_4], \\ \rho_4 &= [\xi_4, \xi_2] + 2[\xi_4, \xi_3]. \end{cases}$$

We have seen that these relators form a strongly free sequence of type I. Hence $G_S(3)$ is mild, fab and non-analytic. After the change of basis

$$x_1 \mapsto x_1, \quad x_2 \mapsto x_2^2, \quad x_3 \mapsto x_3, \quad x_4 \mapsto x_4^2,$$

we find that the pro-3-group G with generators $x_1, \dots, x_4$ and relators

$$\begin{cases} x_1^3 [x_2, x_1][x_1, x_3][x_1, x_4], \\ x_2^3 [x_2, x_3][x_4, x_2], \\ x_3^3 [x_3, x_1][x_3, x_4], \\ x_4^3 [x_2, x_4][x_4, x_3] \end{cases}$$

has $G_S(3)$ as a twist. However, while G is mild and non-analytic, it is not fab; MAGMA says that it has a subgroup of index 9 which has an infinite abelianization.

3. Constructing fabulous groups

Let $G^{(n)}$ be the n-th derived group of the group G; we have

$$G^{(0)} = G \quad \text{and} \quad G^{(n+1)} = \big[G^{(n)}, G^{(n)}\big].$$

Proposition 8. *Let G be a pro-p-group. The following statements are equivalent:*

(a) *The group G is a fab group;*

(b) *The factors of the derived series of G are finite;*

(c) *The quotient $G/G^{(n)}$ is finite for all n;*

(d) *Every solvable quotient of G is finite.*

Proof. If (a) holds then H open in G implies that $[H, H]$ is in H. This implies (b) by induction. That (b),(c) and (d) are equivalent is immediate. To prove that (c) implies (a), let H be a closed subgroup of G of finite index. Then $G^{(n)} \subseteq H$ for some n which implies $G^{(n+1)} \subseteq [H, H]$ and hence the finiteness of $H/[H, H]$. □

The n-th derived subalgebra of a Lie algebra L is defined inductively by

$$L^{(0)} = L, \quad \text{and} \quad L^{(n+1)} = \big[L^{(n)}, L^{(n)}\big].$$

Definition 9. A Lie algebra L is said to be *fab* if $L/L^{(n)}$ is finite for all $n \geq 0$.

Let (C_n) be a central series for G; by definition, we have

$$C_1 = G \quad \text{and} \quad [C_m, C_n] \subseteq C_{m+n}.$$

Let $L(G)$ be the Lie algebra associated to this central series. Then $L(G)$ is a graded Lie algebra with n-homogeneous component $L_n(G) = C_n/C_{n+1}$ (denoted additively). If l_n is the canonical map of C_n onto $L_n(G)$, we have $l_n(xy) = l_n(x) + l_n(y)$; if $x \in C_r$, $y \in C_s$, we have $l_{r+s}([x, y]) = [l_r(x), l_r(y)]$.

For any closed normal subgroup H of G, we have

$$L(G/H) = L(G)/\tilde{L}(H),$$

where $\tilde{L}(H)$ is the Lie algebra associated with the central series $(\tilde{H}_n)$ of H defined by $\tilde{H}_n = H \cap C_n$. If K is a closed normal subgroup of H, we also let $\tilde{L}(H/K)$ be the Lie algebra associated to the central series $(\tilde{H}_n K/K)$ of H/K. Then

$$\tilde{L}(H/K) = \tilde{L}(H)/\tilde{L}(K).$$

Proposition 10. *We have $L(G)^{(n)} \subseteq \tilde{L}(G^{(n)})$.*

Proof. By induction on n. This is immediate for $n = 0$. Since $G^{(n+1)}$ is the kernel of the canonical map $G^{(n)} \to G^{(n)}/G^{(n+1)}$ it follows that $\tilde{L}\left(G^{(n+1)}\right)$ is the kernel of the induced homomorphism of $\tilde{L}\left(G^{(n)}\right)$ onto the abelian Lie algebra $\tilde{L}\left(G^{(n)}/G^{(n+1)}\right)$. Thus

$$\big[\tilde{L}(G)^{(n)}, \tilde{L}(G)^{(n)}\big] \subseteq \tilde{L}\big(G^{(n+1)}\big),$$

which implies the result since, by induction,

$$L(G)^{(n+1)} = \big[L(G)^{(n)}, L(G)^{(n)}\big] \subseteq \big[\tilde{L}(G)^{(n)}, \tilde{L}(G)^{(n)}\big].$$

□

Corollary 11. *If $L(G)$ is fab, then G is fab.*

Indeed, $L\big(G/G^{(n)}\big) = L(G)/\tilde{L}\big(G^{(n)}\big)$ is a quotient of $L(G)/L(G)^{(n)}$. However, as we shall see, the converse statement is not true.

A pro-p-group G is said to be of *elementary type* if $G/[G,G] \cong (\mathbb{Z}/p\mathbb{Z})^d$. If G is a mild quadratic group of elementary type then an explicit presentation for the Lie algebra associated to the lower central is known; *cf.* [1].

Proposition 12. *If G is a mild quadratic group of elementary type, then $\mathfrak{L}(G)$ is fab if and only if $\mathfrak{g} = \mathfrak{L}(G)/\pi\mathfrak{L}(G)$ is fab.*

Proof. Since $\pi\mathfrak{L}(G) \subseteq [\mathfrak{L}(G), \mathfrak{L}(G)]$ it follows that $\pi^{2k}\mathfrak{L}(G)^{(k)} \subseteq \mathfrak{L}(G)^{(k+1)}$. If $\mathfrak{g}$ is fab then $M_k = \mathfrak{L}(G)^{(k)}/\mathfrak{L}(G)^{(k+1)}$ is a finitely generated $\mathbb{F}_p[\pi]$-module since $M_k/\pi M_k = \mathfrak{h}^{(k)}/\mathfrak{h}^{(k+1)}$ is finite and hence M_k is finite since it is a torsion module. Conversely, if $\mathfrak{L}(G)$ is fab then $\mathfrak{g}$ is fab since a quotient of a fab Lie algebra is fab. □

If $G = G_S(p)$, with $K = \mathbb{Q}$, $p = 3$ and $S = \{7, 13, 31, 43\}$, its holonomy Lie algebra $\mathfrak{g}$ is of type I and hence isomorphic to the Lie algebra

$$\mathfrak{h} = \langle \xi_1, \ldots, \xi_4 \mid [\xi_1, \xi_2], [\xi_2, \xi_3], [\xi_3, \xi_4], [\xi_4, \xi_1] \rangle.$$

The quotient $\mathfrak{h}/(\xi_2, \xi_4)$ is a free Lie algebra on two generators and hence is not fab. It follows that $\mathfrak{h}$, and hence $\mathfrak{g}$, is not fab. Thus the Lie algebra $\mathfrak{L}(G)$ associated to the lower 3-central series of the fab pro-3-group $G_S(3)$ is not fab. Since $\mathfrak{g}$ is also a quotient of $\mathfrak{L}(G)$ it follows that $\mathfrak{L}(G)$ is not fab which confirms that G is not fab, as we saw using MAGMA.

More generally, if

$$\mathfrak{k} = \langle \xi_1, \ldots, \xi_4 \mid \rho_1, \ldots, \rho_4 \rangle$$

is a quadratic Lie algebra over $\mathbb{F}_p$, with $\rho_1, \ldots, \rho_4$ strongly free, then by [3] it is isomorphic to the Lie algebra $\mathfrak{h}$ above after possibly a quadratic extension. It follows that the Lie algebra $\mathfrak{k}$ is not fab.

The holonomy Lie algebra of the group $G = G_S(3)$, with $S = \{7, 13, 31, 61\}$, has the presentation $\langle \xi_1, \ldots, \xi_4 \mid \rho_1, \ldots, \rho_4 \rangle$ with

$$\begin{cases} \rho_1 &= [\xi_1, \xi_2] + 2[\xi_1, \xi_3] + 2[\xi_1, \xi_4], \\ \rho_2 &= [\xi_2, \xi_3] + 2[\xi_2, \xi_4], \\ \rho_3 &= 2[\xi_3, \xi_1] + [\xi_3, \xi_4], \\ \rho_4 &= [\xi_4, \xi_1] + [\xi_4, \xi_2]. \end{cases}$$

This presentation defines a mild quadratic Lie algebra of type II. The pro-3-group $\tilde{G}$, with presentation $\langle x_1, \ldots, x_4 \mid s_1, \ldots, s_4 \rangle$, where

$$\begin{cases} s_1 &=& x_1^3[x_2, x_1][x_1, x_3][x_4, x_1], \\ s_2 &=& x_2^3[x_2, x_3][x_2, x_4], \\ s_3 &=& x_3^3[x_3, x_1][x_4, x_3], \\ s_4 &=& x_4^3[x_4, x_1][x_2, x_4], \end{cases}$$

has G as a twist. MAGMA reports that $\tilde{G}/\tilde{G}''$ is finite and that every subgroup of $\tilde{G}$ of index 3, 9 or 27 has a finite abelianization as well as all index 81 subgroups tested so far. We do not know if this group is fab or not. Boston [2] has found a similar example of a mild quadratic pro-2-group with 4 generators and 4 relators which is fab as far as MAGMA can tell.

Question 1. Suppose that G is a quadratic pro-p-group of elementary type and suppose that its holonomy Lie algebra is a mild quadratic algebra with 4 generators and 4 relators which is of type II. Is G fab?

Question 2. Can one find a strongly free sequence over $\mathbb{F}_p$ consisting of d quadratic Lie polynomials $\rho_1, \ldots, \rho_d$ in $d \leq m$ variables $\xi_1, \ldots, \xi_d$ such that the Lie algebra

$$\mathfrak{h} = \langle \xi_1, \ldots, \xi_d \mid \rho_1, \ldots, \rho_d \rangle$$

is mild and fab?

If the answer to this question is yes, then one can produce an explicitly presented quadratic pro-p-group G whose holonomy Lie algebra is $\mathfrak{h}$. The classification of mild quadratic Lie algebras is not known when $m = d \geq 5$. In this case we do not know even if there is more than one isomorphism class over the algebraic closure of $\mathbb{F}_p$.

Question 3. If $G_S(p)$ is quadratic and mild, can one find an explicit twist G of $G_S(p)$ such that G is fab? This would be the case if G were isomorphic to $G_S(p)$.

If the answer to any of these questions is yes, the group G in question is then a fabulous group which is non-analytic since $d(G) \geq 4$.

Remark. The above results can be extended to the case $p = 2$ when the cup-product is alternating; *cf.* [6, p. 175]. If not, the situation is technically quite different since the map $x \mapsto x^2$ in a pro-2-group G does not induce a linear operator on $\mathfrak{L}(G)$. This case will be treated in [7].

REFERENCES

[1] G. Abramov, *Nilpotent class field theory*, Ph.D. Thesis, Humboldt U., Berlin (1998).

[2] N. Boston, *Reducing the Fontaine-Mazur conjecture to group theory*, Progress in Galois theory, 39–50, Dev. Math., **12**, Springer, New York, 2005.

[3] M. R. Bush and J. Labute, *Mild pro-p-groups with* 4 *generators*, J. Algebra **308** (2007), no. 2, 828–839.

[4] H. Koch, *Galois theory of p-extensions*, Translated from the 1970 German original by Franz Lemmermeyer, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2002, xiv+190 pp.

[5] J. Labute, *Classification of Demushkin groups*, Canad. J. Math. **19** (1967), 106–132.

[6] J. Labute, *Mild pro-p-groups and Galois groups of p-extensions of* $\mathbb{Q}$, J. reine angew. Math. **596** (2006), 155–182.

[7] J. Labute and J. Mináč, *Mild pro-2-groups and 2-extensions of $\mathbb{Q}$ with restricted ramification*, work in progress.

[8] L. Salle, *Mild pro-p-groups as Galois groups over global fields*, to appear in J. Number Theory.

[9] A. Schmidt, *Circular sets of prime numbers and p-extensions of the rationals*, J. reine angew. Math. **596** (2006), 115–130.

[10] A. Schmidt, *Rings of integers of type $K(\pi, 1)$*, Doc. Math. **12** (2007), 441–471 (electronic).

[11] A. Schmidt, *On the $K(\pi, 1)$ property for rings of integers in the mixed case*, preprint.

[12] A. Schmidt, *Über Pro-p-Fundamentalgruppen markierter arithmetischer Kurven*, to appear in J. reine angew. Math., Engl. Transl.: arXiv:0806.1863.

[13] J-P. Serre, *Cohomologie galoisienne*, (French) Fifth edition, Lecture Notes in Mathematics, **5**, Springer-Verlag, Berlin, 1994, x+181 pp.

[14] D. Vogel, *Circular sets of primes of imaginary quadratic number fields*, preprint.

[15] D. Vogel, *p-extensions with restricted ramification – the mixed case*, in Preprints der Forschergruppe Lgebraische Zykel L-funktionen Regensberg/Freiberg Nr. **11**, 2007: www.mathematik.uni-r.de/FGALgZyk.

[16] K. Wingberg, *Arithmetical Koch groups*, preprint.

J. Labute, Dept. of Mathematics and Statistics, McGill U., Burnside Hall, 805 Sherbrooke Street West, Montreal, QC, H3A 2K6, Canada
labute@math.mcgill.ca