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ANNIHILATION OF MOTIVIC COHOMOLOGY GROUPS IN CYCLIC 2-EXTENSIONS

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Dedicated to professor John Labute on the occasion of his retirement.

RÉSUMÉ. Supposons que E/F est une extension abélienne de corps de nombres dont le groupe de Galois est G. La conjecture généralisée de Coates-Sinnott prédit que pour $n \geq 2$ il existe un idéal de Stickelberg supérieur, défini de façon naturelle via les valeurs de L-fonctions associées à E/F et évaluées en $1-n$, qui annule le K-groupe supérieur $K_{2n-2}(o_E)$ de l'anneau des entiers algébriques o_E de E. Dans cet article, nous nous concentrons sur la partie 2-primaire de cette conjecture. Nous utilisons tout d'abord les résultats sur l'information 2-primaire fournie par les L-valeurs spéciales pour suggérer un ajustement de la partie 2-primaire de la conjecture en remplaçant les K-groupes par les groupes de cohomologie motivique $H^2_{\mathcal{M}}(o_E, \mathbb{Z}(n))$, qui, selon la conjecture de Bloch-Kato, contiennent les K-groupes d'indice fini égal à une puissance de 2.

Nous démontrons aussi que la formule de Kolster publiée dans Math. Ann., pour le nombre de classes relatif supérieur, combinée avec des idées de B. Smith, implique que la 2-partie de la conjecture de Coates-Sinnott, laquelle dans le cas présent est la même pour les deux formulations, est vérifiée pour une extension abélienne totalement complexe E d'un corps totalement réel F, dont le groupe de Galois est cyclique d'ordre une puissance de 2.

ABSTRACT. Suppose that E/F is an abelian extension of number fields with Galois group G. The generalized Coates-Sinnott Conjecture predicts that for $n \geq 2$ a natural higher Stickelberger ideal, defined using values of the associated L-functions evaluated at $1-n$, annihilates the higher K-group $K_{2n-2}(o_E)$ of the ring of integers o_E in E. We concentrate in this paper on the 2-primary part of the conjecture. We first use results on the 2-primary information provided by the special L-values to suggest an adjustment of the 2-primary part of the conjecture by replacing the K-groups by motivic cohomology groups $H^2_{\mathcal{M}}(o_E, \mathbb{Z}(n))$, which under the Bloch-Kato Conjecture contain the K-groups with finite 2-power index.

We also show that the higher relative class number formula of Kolster, and published in Math. Ann., combined with ideas of B. Smith implies that the 2-part of the Coates-Sinnott Conjecture, which in this situation is the same in both formulations, holds for a totally complex abelian extension E of a totally real field F with cyclic Galois group of 2-power order. We have to assume the validity of the 2-part of the Main Conjecture in Iwasawa theory, which holds *e.g.* if E is absolute abelian.

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1. Introduction

Let E/F be an abelian Galois extension of number fields with Galois group G, and let S be a finite set of primes in F containing the primes ramified in E and the infinite primes. For each irreducible character χ of G we denote by $L^S_{E/F}(\chi, s)$ the associated Artin L-function, *i.e.*, the Artin L-function of χ with Euler factors attached to prime ideals in S removed. It is well-known that there exists a function $\theta^S_{E/F}(s)$ with values in the complex group ring $\mathbb{C}[G]$, such that

$$\bar{\chi}\big(\theta^S_{E/F}(s)\big) = L^S_{E/F}(\chi, s)$$

for all characters χ of G. Here $\bar{\chi}$ denotes the complex conjugate character of χ. By a result of Klingen-Siegel, $\theta^S_{E/F}(1-n)$ is contained in $\mathbb{Q}[G]$ for all $n \geq 1$, and it was shown by Deligne-Ribet that suitable multiples of $\theta^S_{E/F}(1-n)$ are actually contained in the integral group ring $\mathbb{Z}[G]$. More precisely,

$$\mathrm{Ann}_{\mathbb{Z}[G]}\big(W_n(E)\big) \cdot \theta^S_{E/F}(1-n) \subseteq \mathbb{Z}[G].$$

Here $W_n(E)$ is equal to the Galois cohomology group

$$H^0(E, \mathbb{Q}/\mathbb{Z}(n)),$$

where $\mathbb{Q}/\mathbb{Z}(n)$ may be identified with the group μ_∞ of all roots of unity on which an element of the absolute Galois group of $\mathbb{Q}$ acts via its n-th power. The ideal

$$\mathrm{Ann}_{\mathbb{Z}[G]}(W_n(E))) \cdot \theta^S_{E/F}(1-n)$$

is called the *n-th higher Stickelberger ideal* and denoted by $\mathrm{Stick}^S_{E/F}(n)$. We drop the superscript S if the set S is minimal. The classical Stickelberger Theorem states that

$$\mathrm{Stick}_{E/\mathbb{Q}}(1) \subseteq \mathrm{Ann}_{\mathbb{Z}[G]}\big(Cl(o_E)\big),$$

and Brumer conjectured that the same result holds for arbitrary abelian extensions E/F. Now, in terms of algebraic K-theory, the class group $Cl(o_E)$ is equal to the torsion part in $K_0(o_E)$, and another generalization of Stickelberger's theorem, involving higher Quillen K-groups, was suggested by Coates-Sinnott [4] in the case $F = \mathbb{Q}$ and extended to arbitrary base fields by Sands [12] and V. Snaith [17].

Conjecture 1.1. (Coates-Sinnott) *Let E/F be an abelian extension of number fields with Galois group G, and let $n \geq 2$. Then*

$$\mathrm{Stick}_{E/F}(n) \subseteq \mathrm{Ann}_{\mathbb{Z}[G]}\left(K_{2n-2}(o_E)\right).$$

We note that at negative integers $1-n$, that is for $n \geq 2$, the Artin L-function $L^S_{E/F}(\chi, s)$ vanishes unless F is totally real and $\chi(-1) = (-1)^n$. Therefore one usually restricts attention to totally real base fields F, and either $n \geq 2$ even and E totally real or $n \geq 3$ odd and E a CM-field.

Since the higher even K-groups are finite, the Coates-Sinnott Conjecture can be approached prime by prime. In fact, all the known results have been obtained using the étale Chern characters

$$K_{2n-i}(o_E) \otimes \mathbb{Z}_p \longrightarrow H^i_{\text{ét}}\big(o_E[\tfrac{1}{p}], \mathbb{Z}_p(n)\big)$$

from K-theory to étale cohomology, for each $i = 1, 2$ and prime p, along with computations involving étale cohomology groups and the Main Conjecture in Iwasawa theory. Here

$$H^i_{\text{ét}}\big(o_E\left[\tfrac{1}{p}\right], \mathbb{Z}_p(n)\big) = \varprojlim_m H^i_{\text{ét}}\big(o_E\left[\tfrac{1}{p}\right], \mu_{p^m}^{\otimes n}\big)$$

and $H^i_{\text{ét}}\big(o_E\left[\tfrac{1}{p}\right], \bullet\big)$ denotes the i-th étale cohomology group of $\operatorname{spec} o_E\left[\tfrac{1}{p}\right]$ with values in a sheaf $\bullet$. For odd p, these Chern characters are known to be surjective; in fact, they are isomorphisms if the Bloch-Kato Conjecture holds. It is generally believed that the paper [20] with possible additions from [21] contains a proof of the Bloch-Kato Conjecture, but none of these papers has yet been published. The situation for the prime 2 is different. In general, the Chern characters are not isomorphisms in this case. Based on Voevodsky's proof of the Milnor Conjecture [19], the deviation between

$$K_{2n-i}(o_E) \otimes \mathbb{Z}_2 \quad \text{and} \quad H^i_{\text{ét}}\big(o_E\left[\tfrac{1}{2}\right], \mathbb{Z}_2(n)\big)$$

has essentially been determined by Rognes-Weibel [11]. Based on the 2-primary information provided by special values of L-functions, which we recall below, we suggest a motivic version of the Coates-Sinnott Conjecture, which under the Bloch-Kato Conjecture coincides with the p-part of the K-theoretic version for odd primes p, but adjusts the 2-primary part. In some cases our motivic version is in fact slightly stronger than the K-theoretic version.

Results for the 2-primary part of the conjectures are scarce, and have mostly been restricted to the totally real case and $n \geq 2$ an even integer; see [5], [13], [14] and [15]. In this paper we prove the 2-primary part of the conjecture for odd $n \geq 3$ and cyclic extensions E/F of 2-power degree, where F is totally real and E is CM and abelian over $\mathbb{Q}$, combining ideas of B. Smith [16] and a result of the first author [8]. The restriction to absolute abelian fields E is due to the fact that the 2-adic Main Conjecture in Iwasawa Theory so far has only been proven in this case; see [22].

2. The motivic Coates-Sinnott Conjecture

We recall that the motivic cohomology groups of a smooth scheme X over a base B, which we denote by

$$H^*_{\mathcal{M}}\big(X, \mathbb{Z}(n)\big),$$

are defined as the hypercohomology of Bloch's cycle complex $\mathbb{Z}(n)$ for the Zariski topology. If $B = \operatorname{spec} F$ is the spectrum of the field F, then these groups coincide with Bloch's higher Chow groups (*cf.* [1]) and with Voevodsky's motivic cohomology groups (*cf.* [19]), and we simply denote them by $H^*_{\mathcal{M}}\big(F, \mathbb{Z}(n)\big)$. Similarly, for a Dedekind domain o_F with quotient field F and $X = B = \operatorname{spec} o_F$ we will use the notation $H^*_{\mathcal{M}}\big(o_F, \mathbb{Z}(n)\big)$ for the motivic cohomology groups of $\operatorname{spec} o_F$. The relationship between K-theory, motivic cohomology and étale cohomology is described via Chern characters (see, for instance, [9, Chapter 2] for an overview), and we want to describe briefly the profound consequences which the Bloch-Kato Conjecture has for the interplay between the 3 functors. The Bloch-Kato Conjecture states that for any field F and any $n \geq 1$ the Galois symbol

$$K^M_n(F)/p^m \to H^n(F, \mu_{p^m}^{\otimes n})$$

from Milnor K-theory to Galois cohomology is an isomorphism for any p-power p^m, with $p \neq \mathrm{char}(F)$. As we mentioned in the introduction, it is generally understood that the conjecture has now been proved by Rost and Voevodsky. Since, however, the proof has not been properly reviewed yet, we keep referring to it as a conjecture. The special case $p = 2$, *i.e.*, the Milnor Conjecture, has been proved by Voevodsky [19].

The first consequence of the Bloch-Kato Conjecture is the Quillen-Lichtenbaum Conjecture, namely that for any odd prime p and any number field F, the étale Chern characters

$$K_{2n-i}(F) \otimes \mathbb{Z}_p \to H^i_{\text{ét}}\big(F, \mathbb{Z}_p(n)\big)$$

are isomorphisms for $n \geq 2$ and $i = 1, 2$. The second consequence is that the same result is true for the motivic cohomology groups for *all* primes p:

$$H^i_{\mathcal{M}}\big(F, \mathbb{Z}(n)\big) \otimes \mathbb{Z}_p \cong H^i_{\text{ét}}\big(F, \mathbb{Z}_p(n)\big).$$

There are also global motivic Chern characters

$$K_{2n-i}(F) \to H^i_{\mathcal{M}}\big(F, \mathbb{Z}(n)\big)$$

compatible with the étale Chern characters (*cf.* [10, Chapter III]), which then under the Bloch-Kato Conjecture are isomorphisms up to finite 2-torsion. To obtain the analogous results for the ring of integers o_F one uses the localization sequences in K-theory (due to Quillen), in étale cohomology (due to Soulé) and in motivic cohomology (due to Geisser [7, Corollary 3.4]), which are compatible with the Chern characters. There is *e.g.* an exact commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K_{2n-2}(o_F) & \longrightarrow & K_{2n-2}(F) & \longrightarrow & \oplus_{\mathfrak{p}} K_{2n-1}(k_{\mathfrak{p}}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H^2\big(o_F, \mathbb{Z}(n)\big) & \longrightarrow & H^2\big(F, \mathbb{Z}(n)\big) & \longrightarrow & \oplus_{\mathfrak{p}} H^1\big(k_{\mathfrak{p}}, \mathbb{Z}(n-1)\big) & \longrightarrow & 0
\end{array}$$

What emerges from this discussion is the following result, where the 2-primary information is unconditional and has been proved by Rognes-Weibel in [11].

Proposition 2.1. *Let o_F be the ring of integers in a number field F with r_1 real embeddings, and let $n \geq 2$. Assume that the Bloch-Kato Conjecture holds for odd primes p. Then for $i = 1, 2$ the Chern character*

$$K_{2n-i}(o_F) \to H^i_{\mathcal{M}}\big(o_F, \mathbb{Z}(n)\big)$$

is

$$\begin{cases} \text{an isomorphism} & \text{if } 2n - i \equiv 0, 1, 2, 7 \pmod 8, \\ \text{injective with cokernel} \cong (\mathbb{Z}/2\mathbb{Z})^{r_1} & \text{if } 2n - i \equiv 6 \pmod 8, \\ \text{surjective with kernel} \cong (\mathbb{Z}/2\mathbb{Z})^{r_1} & \text{if } 2n - i \equiv 3 \pmod 8. \end{cases}$$

In the remaining cases ($n \equiv 3 \pmod 4$), there is an exact sequence

$$\begin{aligned} 0 \longrightarrow K_{2n-2}(o_F) \longrightarrow H^2_{\mathcal{M}}(o_F, \mathbb{Z}(n)) \longrightarrow & \quad (\mathbb{Z}/2\mathbb{Z})^{r_1} \\ \longrightarrow & \quad K_{2n-1}(o_F) \longrightarrow H^1_{\mathcal{M}}(o_F, \mathbb{Z}(n)) \longrightarrow 0. \end{aligned}$$

We also note that under the same assumptions,

- $H^0_{\mathcal{M}}\big(o_F, \mathbb{Z}(n)\big) = 0$ for all $n \geq 2$,
- $H^1_{\mathcal{M}}\big(o_F, \mathbb{Z}(n)\big)_{tors} \cong H^0\big(F, \mathbb{Q}/\mathbb{Z}(n)\big)$, and
- for $i \geq 3$,

$$H^i_{\mathcal{M}}\big(o_F, \mathbb{Z}(n)\big) \cong \begin{cases} 0 & \text{if } i+n \text{ is odd,} \\ (\mathbb{Z}/2\mathbb{Z})^{r_1} & \text{if } i+n \text{ is even.} \end{cases}$$

We suggest the following reformulation of the Coates-Sinnott Conjecture.

Conjecture 2.2 (Motivic Coates-Sinnott Conjecture). *Let E/F be an abelian extension of number fields with Galois group G, and let $n \geq 2$. Then*

$$\mathrm{Stick}_{E/F}(n) \subseteq \mathrm{Ann}_{\mathbb{Z}[G]}\Big(H^2_{\mathcal{M}}\big(o_E, \mathbb{Z}(n)\big)\Big).$$

We note that for this conjecture to make sense we have to assume the properties of the motivic cohomology groups listed above, which would follow from the Bloch-Kato Conjecture. However, the 2-primary part of the conjecture always makes sense, and Proposition 2.1 shows that the motivic version is slightly stronger than the K-theoretic version. (We are using only the case $i = 2$ of Proposition 2.1 here.) This stronger version is supported by results of Cornacchia-Østvær [5] in the case that $E/\mathbb{Q}$ is abelian of prime power conductor.

The original formulation of the Coates-Sinnott Conjecture was motivated by the Lichtenbaum Conjecture, which predicted a relationship between special values of zeta-functions and negative integers, Borel regulators and orders of algebraic K-groups (see *e.g.* [9] for a precise statement), as a generalization of the analytic class number formula. The Lichtenbaum Conjecture was vague with respect to 2-primary contributions, and as was explained in [9] the precise version of the Lichtenbaum Conjecture involving the 2-primary part should again be in terms of motivic cohomology rather than K-theory.

In the formulation of the conjecture one can avoid motivic cohomology groups: as was explained in [9] one can define finitely generated groups $H^i\big(o_F, \mathbb{Z}(n)\big)$, for $i = 1, 2$, in terms of étale cohomology, which have exactly the expected properties of the corresponding motivic cohomology groups: For $i = 2$ one simply defines

$$H^2\big(o_F, \mathbb{Z}(n)\big) = \prod_p H^2_{\text{ét}}\big(o_F\left[\tfrac{1}{p}\right], \mathbb{Z}_p(n)\big),$$

but for $i = 1$, the construction is more involved. In any case there are natural isomorphisms

$$H^i\big(o_F, \mathbb{Z}(n)\big) \otimes \mathbb{Z}_p \cong H^i_{\text{ét}}\big(o_F\left[\tfrac{1}{p}\right], \mathbb{Z}_p(n)\big)$$

for all primes p, all $n \geq 2$ and $i = 0, 1$.

Without invoking the validity of the Bloch-Kato Conjecture one can then formulate the *cohomological version* of the Coates-Sinnott Conjecture by simply replacing the motivic cohomology groups $H^i_{\mathcal{M}}\big(o_F, \mathbb{Z}(n)\big)$ by $H^i\big(o_F, \mathbb{Z}(n)\big)$.

All known results on the validity of the general Coates-Sinnott Conjecture in fact use the cohomological version of the conjecture. The results in this paper are no exception: For the rest of this paper we are working exclusively with étale cohomology groups and the groups $H^i\big(o_F, \mathbb{Z}(n)\big)$. Corresponding results on the motivic version of the Coates-Sinnott Conjecture would then follow *e.g.* from the validity of the Bloch-Kato Conjecture.

3. The setup

From now on, we fix a totally real number field F and a totally complex Galois extension E of F such that the Galois group $G = \mathrm{Gal}(E/F)$ is cyclic of order 2^m for some positive integer m. Let σ be a generator for G and $\tau = \sigma^{2^{m-1}}$ be the unique element of order 2 in G, which must represent complex conjugation for each embedding of E in $\mathbb{C}$. The fixed field E^+ of τ is then totally real, and E/E^+ is a CM-extension. Let χ_1 generate the character group $\hat{G}$ of G, and extend χ_1 to a ring homomorphism $\chi_1 : \mathbb{Z}[G] \to \mathbb{C}$. Set $\zeta_{2^m} = \chi_1(\sigma)$, a primitive 2^m-th root of unity. Then χ_1 induces an isomorphism $\mathbb{Z}[G]/(1+\tau) \cong \mathbb{Z}\left[\zeta_{2^m}\right]$, and the latter is a Dedekind domain which we denote simply by $\mathcal{O}$.

We now fix an odd integer $n = 2k+1$, with $k \geq 1$, and consider the action of τ on $W_{2k+1}(E) = H^0\big(E, \mathbb{Q}/\mathbb{Z}(2k+1)\big)$. A lift of τ may be identified with complex conjugation, whose $(2k+1)$-th power is also complex conjugation. Thus the action of τ on $W_{2k+1}(E)$ may be viewed as complex conjugation acting on roots of unity, hence by inversion. Consequently $W_{2k+1}(E)^{1+\tau} = 1$, *i.e.*, $W_{2k+1}(E)$ is annihilated by $1+\tau$ and may be considered as a module over $\mathbb{Z}[G]/(1+\tau)$. By the isomorphism induced by χ_1 mentioned above, $W_{2k+1}(E)$ becomes a module over the Dedekind domain $\mathcal{O}$. The next section summarizes the facts we will need about finite modules over a Dedekind domain.

4. Finite modules over Dedekind domains

Proposition 4.1. *Suppose that M is a finite module over a Dedekind domain D. Then we have:*

(1) $M \cong \bigoplus_i D/P_i^{e_i}$, *where the ideals P_i are prime ideals which are not necessarily distinct.*

(2) *The annihilator ideal of M, denoted by $\mathrm{Ann}_D(M)$, is the least common multiple of the $P_i^{e_i}$.*

(3) *The Fitting ideal of M, denoted by $\mathrm{Fit}_D(M)$, is the product of all the $P_i^{e_i}$'s, and is contained in $\mathrm{Ann}_D(M)$.*

(4) *The D-module M is cyclic if and only if $\mathrm{Fit}_D(M) = \mathrm{Ann}_D(M)$.*

(5) *The norm (or index) of the Fitting ideal of M equals the order of M :*

$$\mathrm{N}\big(\mathrm{Fit}_D(M)\big) = \big(D : \mathrm{Fit}_D(M)\big) = |M|.$$

Proof. The parts (1)-(4) follow from the structure theorem for finitely generated modules over a Dedekind domain. The fifth part follows from the first and third and the multiplicativity of the norm. $\square$

For P a prime ideal of D, let $M\{P\}$ denote the P-primary part of M; it consists of all elements of M which are annihilated by some power of P. For a rational prime p, we can also consider the p-primary part $M\{p\}$ of M as a finite abelian group; it consists of all elements of M which are annihilated by some power of p. We record some facts about $M\{P\}$ and $M\{p\}$ which follow easily from part (1) of Proposition 4.1.

Corollary 4.2. *With the assumptions of Proposition 4.1, we have:*

(1) $M = \bigoplus_P M\{P\}$ *and* $M = \bigoplus_p M\{p\}$ *as* D*-modules.*

(2) *As* D*-modules,* $M\{p\} = \bigoplus_{p \in P} M\{P\}$.

(3) *In the isomorphism of Proposition 4.1 (1),* $M\{P\} \cong \bigoplus_{P_i = P} D/P_i^{e_i}$.

(4) *If* N *is a* D*-submodule of* M*, then*

$$(M/N)\{P\} = M\{P\}/N\{P\} \quad \text{and} \quad (M/N)\{p\} = M\{p\}/N\{p\}.$$

Proposition 4.3. *If* $m > 1$*, then the order* $w_{2k+1}(E)$ *of* $W_{2k+1}(E)$ *is exactly divisible by* 2.

Proof. Clearly $-1 \in W_{2k+1}(E)$, so it suffices to show that $\sqrt{-1}$ does not lie in $W_{2k+1}(E)$. But if it did, the Galois group of $E(\sqrt{-1})$ over E would have exponent $2k+1$. Since it clearly has exponent 2, we would necessarily have $\sqrt{-1} \in E$. Then $F(\sqrt{-1})$ would be the unique relative quadratic extension of F in E and hence would live in the unique extension E^+ of degree 2^{m-1}over F in E. Since $F(\sqrt{-1})$ is totally complex, this is absurd. $\square$

Now the prime 2 is totally ramified in $\mathcal{O}$, indeed $(2) = (1 - \zeta_{2^m})^{2^{m-1}}$. From Propositions 4.1 and 4.3, we immediately obtain a corollary.

Corollary 4.4. *If* $m > 1$*, then* $A_{2k+1}(E) = \mathrm{Fit}_{\mathcal{O}}\big(W_{2k+1}(E)\big)$ *is exactly divisible by* $(1 - \zeta_{2^m})$.

5. Higher class groups

We now consider the higher analogs $H^2\big(o_E, \mathbb{Z}(2k+1)\big)$ of the class group of E, whose order is denoted by $h_{2k+1}(E)$. The corestriction

$$H^2\big(o_E, \mathbb{Z}(2k+1)\big) \longrightarrow H^2\big(o_{E^+}, \mathbb{Z}(2k+1)\big)$$

is surjective (*cf.* [8, Proposition 2.8]), and we denote the order of the kernel by

$$h^-_{2k+1}(E) = \frac{h_{2k+1}(E)}{h_{2k+1}(E^+)}.$$

This is the higher relative class number.

There is a natural map

$$\iota_* : H^2\big(o_{E^+}, \mathbb{Z}(2k+1)\big) \longrightarrow H^2\big(o_E, \mathbb{Z}(2k+1)\big),$$

whose kernel and cokernel we simply denote by $\ker_{2k+1}(E)$ and $\operatorname{coker}_{2k+1}(E)$ respectively. This allows us to express

$$|\operatorname{coker}_{2k+1}(E)| = \frac{|\ker_{2k+1}(E)| \cdot h_{2k+1}(E)}{h_{2k+1}(E^+)} = |\ker_{2k+1}(E)| \cdot h^-_{2k+1}(E).$$

We will make use of the higher Q-index, equal to either 1 or 2, which is defined as

$$Q_{2k+1}(E) = \Big[H^1_{\text{ét}}\big(E, \mathbb{Z}_2(2k+1)\big) : H^1_{\text{ét}}\big(E^+, \mathbb{Z}_2(2k+1)\big) \cdot H^0\big(E, \mathbb{Q}_2/\mathbb{Z}_2(2k+1)\big)\Big]$$

and obviously equal to

$$\Big[H^1\Big(o_E, \mathbb{Z}(2k+1)\Big) : H^1\big(o_{E^+}, \mathbb{Z}(2k+1)\big) \cdot H^0\big(E, \mathbb{Q}/\mathbb{Z}(2k+1)\big)\Big].$$

Proposition 5.1. *The order* $|\ker_{2k+1}(E)|$ *divides* $\dfrac{2}{Q_{2k+1}(E)}$.

Proof. We first note that $\ker_{2k+1}(E)$ is of exponent 2, and so is

$$H^1\Big(H, H^1\big(E, \mathbb{Z}(2k+1)\big)\Big),$$

where $H = \langle\tau\rangle$ denotes the subgroup of order 2 generated by τ. It follows from Lemma 3.1 in [8] that

$$\Big|H^1\Big(H, H^1\big(E, \mathbb{Z}(2k+1)\big)\Big)\Big| = \frac{2}{Q_{2k+1}(E)}.$$

Proposition 2.8 in [8] then shows that $H^1\Big(H, H^1\big(E, \mathbb{Z}(2k+1)\big)\Big)$ is isomorphic to $\ker^S_{2k+1}(E)$, the kernel of the map

$$H^2\big(o^S_{E^+}, \mathbb{Z}(2k+1)\big) \longrightarrow H^2\big(o^S_E, \mathbb{Z}(2k+1)\big),$$

whenever S contains the infinite primes, the dyadic primes and the primes ramified in E. The localization sequence in étale cohomology now shows that

$$\ker_{2k+1}(E) \subseteq \ker^S_{2k+1}(E),$$

which proves the claim. $\square$

If we let

$$\delta_{2k+1}(E) = 2^{\epsilon_{2k+1}(E)} = \frac{2}{|\ker_{2k+1}(E)| \cdot Q_{2k+1}(E)},$$

we clearly obtain the following.

Corollary 5.2. *We have*

$$\delta_{2k+1}(E) \cdot |\operatorname{coker}_{2k+1}(E)| = \frac{2}{Q_{2k+1}(E)} \cdot h^-_{2k+1}(E),$$

with $\delta_{2k+1}(E)$ *equal to* 1 *or* 2.

As usual, the cohomological norm

$$1+\tau : H^2\big(o_E, \mathbb{Z}(2k+1)\big) \longrightarrow H^2\big(o_E, \mathbb{Z}(2k+1)\big)$$

factors through $H^2\big(o_{E^+}, \mathbb{Z}(2k+1)\big)$ via the corestriction and ι_*. Consequently, we obtain the following.

Proposition 5.3. *The* $\mathbb{Z}[G]$*-module* $\operatorname{coker}_{2k+1}(E)$ *is annihilated by* $1+\tau$*, and therefore becomes an* $\mathcal{O}$*-module via* χ_1*. The norm of its Fitting ideal satisfies*

$$\mathrm{N}\Big(\mathrm{Fit}_{\mathcal{O}}\big(\operatorname{coker}_{2k+1}(E)\big)\Big) = \big|\operatorname{coker}_{2k+1}(E)\big|.$$

Proof. This follows from the discussion above and Proposition 4.1 (5). □

6. L-functions

For each χ in the character group $\hat{G}$ of G, let $L_{E/F}(s,\chi)$ be the associated Artin L-function. Let S denote the set of primes of F which ramify in E. Note that under our assumptions, S contains all of the infinite primes of F, of which there are $[F:\mathbb{Q}]$.

To relate the higher relative class number to the value of the Artin L-function, we use the higher relative class number formula from [8]. To apply the main result from [8] we have to restrict now to an absolute abelian field E. This restriction comes from the Main Conjecture in Iwasawa theory for the prime 2, which so far has only been proven for abelian fields; see [22].

Theorem 6.1. *Assume now in addition that* E *is abelian over* $\mathbb{Q}$*. Then*

$$L_{E/E^+}\big(-2k, \chi_1|_{\langle\tau\rangle}\big) = (-1)^{k[E^+:\mathbb{Q}]}\cdot 2^{[E^+:\mathbb{Q}]}\cdot \frac{2}{Q_{2k+1}(E)}\cdot\frac{h^-_{2k+1}(E)}{w_{2k+1}(E)}.$$

For our present purposes, we record another form of this formula.

Corollary 6.2. *We have*

$$L^S_{E/E^+}\big(-2k, \chi_1|_{\langle\tau\rangle}\big) = L_{E/E^+}\big(-2k, \chi_1|_{\langle\tau\rangle}\big) = \pm 2^{[E^+:\mathbb{Q}]}\cdot\delta_{2k+1}(E)\cdot\frac{|\operatorname{coker}_{2k+1}(E)|}{w_{2k+1}(E)}.$$

Proof. First,

$$L^S_{E/E^+}\big(-2k, \chi_1|_{\langle\tau\rangle}\big) = L_{E/E^+}\big(-2k, \chi_1|_{\langle\tau\rangle}\big)$$

since the Euler factors for the ramified primes are trivial in the case of a relative quadratic extension. The second equality follows from Theorem 6.1 upon substituting the identity of Corollary 5.2. □

7. Higher Stickelberger ideals

We now consider the Stickelberger ideal

$$\mathrm{Stick}_{E/F}(1+2k) = \mathrm{Ann}_{Z[G]}\big(W_{2k+1}(E)\big)\,\theta_{E/F}(-2k).$$

Lemma 7.1. *We have* $(1+\tau)\,\theta_{E/F}(-2k) = 0$.

Proof. This follows from the fact that the natural projection of $\theta_{E/F}(-2k)$ to $\mathbb{Z}[\mathrm{Gal}(E^+/F)]$ equals $\theta_{E^+/F}(-2k)$, by the inflation property of Artin L-functions. So $(1+\tau)\theta_{E/F}(-2k)$ is the co-restriction of $\theta_{E^+/F}(-2k)$. But $\theta_{E^+/F}(-2k) = 0$ since E^+ is totally real and $-2k$ is even. □

The image of the Stickelberger ideal in $\mathcal{O}$, obtained by applying the surjective homomorphism χ_1, is clearly the ideal

$$\chi_1\big(\mathrm{Stick}_{E/F}(1+2k)\big) = \mathrm{Fit}_{\mathcal{O}}\big(W_{2k+1}(E)\big) L^S_{E/F}\big(-2k, \overline{\chi_1}\big) \subseteq \mathcal{O}.$$

In particular, for $w_{2k+1}(E) = |W_{2k+1}(E)|$, we have $w_{2k+1}(E)\theta_{E/F}(-2k) \in \mathbb{Z}[G]$, and $w_{2k+1}(E)L^S_{E/F}\big(-2k, \overline{\chi_1}\big) \in \mathcal{O}$.

We now derive an expression for the index of $\chi_1\big(\mathrm{Stick}_{E/F}(1+2k)\big)$ in $\mathcal{O}$. For the purposes of the computation, we extend the norm as usual by multiplicativity to fractional ideals of $\mathcal{O}$.

We have

$$\begin{aligned}\mathrm{N}\Big(\chi_1\big(\mathrm{Stick}_{E/F}(1+2k)\big)\Big) &= \mathrm{N}\Big(\mathrm{Fit}_{\mathcal{O}}\big(W_{2k+1}\big) L^S_{E/F}\big(-2k, \overline{\chi_1}\big)\Big) \\ &= \mathrm{N}\Big(\mathrm{Fit}_{\mathcal{O}}\big(W_{2k+1}\big)\Big) \cdot \mathrm{N}\Big(L^S_{E/F}\big(-2k, \overline{\chi_1}\big)\Big),\end{aligned}$$

by multiplicativity of the norm. By Proposition 4.1 (5) and a property of the norm, we have

$$\begin{aligned}\mathrm{N}\Big(\mathrm{Fit}_{\mathcal{O}}\big(W_{2k+1}\big)\Big) \cdot \mathrm{N}\Big(L^S_{E/F}\big(-2k, \overline{\chi_1}\big)\Big) &= w_{2k+1}(E) \cdot \mathrm{N}\Big(L^S_{E/F}\big(-2k, \overline{\chi_1}\big)\Big) \\ &= w_{2k+1}(E) \cdot \prod_{j=1}^{2^{m-1}} L^S_{E/F}\big(-2k, {\chi_1}^{1+2j}\big)\end{aligned}$$

as odd powers of χ_1 are its conjugates. Using properties of Artin L-functions, we deduce that

$$\begin{aligned}& w_{2k+1}(E) \cdot \prod_{j=1}^{2^{m-1}} L^S_{E/F}\big(-2k, {\chi_1}^{1+2j}\big) \\ &\quad = w_{2k+1}(E) \cdot L^S_{E/E^+}\big(-2k, \chi_1|_{\langle\tau\rangle}\big) \\ &\quad = 2^{[E^+:\mathbb{Q}]} \cdot \delta_{2k+1}(E) \cdot |\mathrm{coker}_{2k+1}(E)| \\ &\quad = \mathrm{N}\Big(\big(1-\zeta_{2^m}\big)^{[E^+:\mathbb{Q}]+\epsilon_{2k+1}(E)}\Big) \cdot \mathrm{N}\Big(\mathrm{Fit}_{\mathcal{O}}\big(\mathrm{coker}_{2k+1}(E)\big)\Big) \\ &\quad = \mathrm{N}\Big(\big(1-\zeta_{2^m}\big)^{[E^+:\mathbb{Q}]+\epsilon_{2k+1}(E)}\, \mathrm{Fit}_{\mathcal{O}}\big(\mathrm{coker}_{2k+1}(E)\big)\Big)\end{aligned}$$

where the second, the third and the fourth equalities follow respectively from Corollary 6.2, Proposition 5.3 and the multiplicativity of the norm.

Let us record the implications of this computation.

Proposition 7.2. *We have*

$$\mathrm{N}\Big(\chi_1\big(\mathrm{Stick}_{E/F}(1+2k)\big)\Big) = \mathrm{N}\Big(\big(1-\zeta_{2^m}\big)^{[E^+:\mathbb{Q}]+\epsilon_{2k+1}(E)}\, \mathrm{Fit}_{\mathcal{O}}\big(\mathrm{coker}_{2k+1}(E)\big)\Big).$$

Hence $\chi_1\big(\mathrm{Stick}_{E/F}(1+2k)\big)$ *is exactly divisible by*

$$\big(1-\zeta_{2^m}\big)^{[E^+:\mathbb{Q}]+\epsilon_{2k+1}(E)+c},$$

where $(1-\zeta_{2^m})^c$ *is the exact power of* $1-\zeta_{2^m}$ *appearing in* $\mathrm{Fit}_{\mathcal{O}}\left(\mathrm{coker}_{2k+1}(E)\right)$. *If* $\mathfrak{p}$ *is the unique prime ideal of* $\mathcal{O}$ *dividing some odd rational prime* p, *then the* $\mathfrak{p}$*-primary parts of*

$$\chi_1\big(\mathrm{Stick}_{E/F}(1+2k)\big) \quad \text{and} \quad \mathrm{Fit}_{\mathcal{O}}\big(\mathrm{coker}_{2k+1}(E)\big)$$

are equal.

Proof. This is clear from the fact that $(1-\zeta_{2^m})$ (respectively $\mathfrak{p}$) is the unique prime above 2 (respectively p) in $\mathcal{O}$, and the multiplicativity of the norm. □

Theorem 7.3. *Suppose that* $p=2$ *or* p *is an odd prime with only one prime lying above it in* $\mathcal{O}$. *Then* $\mathrm{Stick}_{E/F}(1+2k)$ *annihilates* $H^2_{\text{ét}}\big(o_E\left[\frac{1}{p}\right], \mathbb{Z}_p(2k+1)\big)$.

Proof. Fix an arbitrary $\sigma\in G$, and extend it to $\tilde\sigma$ on $E\big(W_{2k+1}(E)\big)$. Let n_σ be an integer such that $\tilde\sigma(\omega)=\omega^{n_\sigma}$ for a generator ω of $W_{2k+1}(E)$. Then $\sigma-n_\sigma^{2k+1}\in\mathbb{Z}[G]$ annihilates $W_{2k+1}(E)$ and indeed such elements generate $\mathrm{Ann}_{\mathbb{Z}[G]}\big(W_{2k+1}(E)\big)$, by Lemma 2.5 in [3]. Thus as σ and the corresponding values of n_σ vary, the elements $(\sigma-n_\sigma^{2k+1})\theta_{E/F}(-2k)$ generate $\mathrm{Stick}_{E/F}(1+2k)$, and it suffices to show that the elements $(\sigma-n_\sigma^{2k+1})\theta_{E/F}(-2k)$ annihilate $H^2_{\text{ét}}\big(o_E\left[\frac{1}{p}\right], \mathbb{Z}_p(2k+1)\big)$.

Since

$$(\sigma-n_\sigma^{2k+1})\theta_{E/F}(-2k)\in \mathrm{Stick}_{E/F}(1+2k)\subseteq\mathbb{Z}[G]$$

and

$$(\sigma-n_\sigma^{2k+1})\theta_{E/F}(-2k)(1+\tau)=0$$

by Lemma 7.1, it follows that $(\sigma-n_\sigma^{2k+1})\theta_{E/F}(-2k)=\beta(1-\tau)$ for some $\beta\in\mathbb{Z}[G]$. Applying the homomorphism χ_1 gives

$$2\chi_1(\beta)=\big(\chi_1(\sigma)-n_\sigma^{2k+1}\big)\chi_1\big(\theta_{E/F}(-2k)\big)\in\chi_1\big(\mathrm{Stick}_{E/F}(1+2k)\big).$$

Using Proposition 7.2 and noting that the ideal

$$(1-\zeta_{2^m})^{[E^+:\mathbb{Q}]}=(1-\zeta_{2^m})^{[E^+:F][F:\mathbb{Q}]}=(1-\zeta_{2^m})^{2^{m-1}[F:\mathbb{Q}]}=(2^{[F:\mathbb{Q}]}),$$

we find that

$$\mathfrak{v}_{\mathfrak{p}}\big(2\chi_1(\beta)\big)\ge\mathfrak{v}_{\mathfrak{p}}\big(2\,\mathrm{Fit}_{\mathcal{O}}(\mathrm{coker}_{2k+1}(E)\big)\ge\mathfrak{v}_{\mathfrak{p}}\big(2\,\mathrm{Ann}_{\mathcal{O}}(\mathrm{coker}_{2k+1}(E)\big).$$

Consequently $\chi_1(\beta)$ annihilates $\mathrm{coker}_{2k+1}(E)\{\mathfrak{p}\}$ which equals $\mathrm{coker}_{2k+1}(E)\{p\}$ by Corollary 4.2 (2). At the same time, $\chi_1(\beta)$ by definition acts as β on $\mathrm{coker}_{2k+1}(E)$. Using Corollary 4.2 (4), we find that

$$\begin{aligned} H^2_{\text{ét}}\big(o_E\left[\tfrac{1}{p}\right],\mathbb{Z}_p(2k+1)\big)^{\beta} &= H^2_{\text{ét}}\big(o_E\left[\tfrac{1}{p}\right],\mathbb{Z}_p(2k+1)\big)^{\chi_1(\beta)} \\ &\subseteq \iota_*\Big(H^2_{\text{ét}}\big(o_E^+\left[\tfrac{1}{p}\right],\mathbb{Z}_p(2k+1)\big)\Big). \end{aligned}$$

Now τ fixes E^+ and thus acts trivially on $\iota_*\Big(H^2_{\text{ét}}\big(o_E^+\left[\frac{1}{p}\right],\mathbb{Z}_p(2k+1)\big)\Big)$. So the image of

$$\iota_*\Big(H^2_{\text{ét}}\big(o_E^+\left[\tfrac{1}{p}\right],\mathbb{Z}_p(2k+1)\big)\Big)$$

under $1-\tau$ is trivial, and therefore so is the submodule

$$H^2_{\text{ét}}\big(o_E\left[\tfrac{1}{p}\right],\mathbb{Z}_p(2k+1)\big)^{\beta(1-\tau)}=H^2_{\text{ét}}\big(o_E\left[\tfrac{1}{p}\right],\mathbb{Z}_p(2k+1)\big)^{(\sigma-n_\sigma^{2k+1})\theta_{E/F}(-2k)}.$$

This completes the proof. □

Remark 7.4. We note that the same result would hold by a similar argument for other rational primes p if we knew that $\mathrm{Stick}_{E/F}(1+2k)$ and $\mathrm{Fit}_{\mathcal{O}}\big(\mathrm{coker}_{2k+1}(E)\big)$ had the same $\mathfrak{p}$-primary parts for each $\mathfrak{p}$ dividing p. One may ask whether this is always true, as Smith [16] does in the case of $k = 0$. A modification of our argument shows that for each rational prime p, $\mathrm{Stick}_{E/F}(1 + 2k)$ does annihilate $H^2_{\text{ét}}\big(o_E\left[\frac{1}{p}\right], \mathbb{Z}_p(2k + 1)\big)\{\mathfrak{p}\}$ for at least one prime $\mathfrak{p}$ dividing p in $\mathcal{O}$.

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