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MOTIVIC GALOIS THEORY FOR 1-MOTIVES

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To professor John Labute on the occasion of his 70th birthday.

RÉSUMÉ. Soit $\mathcal{T}_1(k)$ la catégorie Tannakienne engendrée par les 1-motifs définis sur un corps k de caractéristique 0 et soit $\mathcal{G}_{\mathrm{mot}}\big(\mathcal{T}_1(k)\big)$ son groupe fondamental, *i.e.*, le groupe de Galois motivique $\mathcal{G}_{\mathrm{mot}}\big(\mathcal{T}_1(k)\big)$ des 1-motifs. Nous exhibons quatre suites exactes courtes de groupes affines sous-$\mathcal{T}_1(k)$-schémas de $\mathcal{G}_{\mathrm{mot}}\big(\mathcal{T}_1(k)\big)$, corrélés les uns aux autres via les inclusions et les projections ; ce sont les versions motiviques de suites exactes courtes bien connues de la théorie de Hodge. De plus, étant donné un 1-motif M, nous calculons explicitement la plus grosse sous-catégorie Tannakienne contenue dans celle qui est engendrée par M, et dont le groupe fondamental est commutatif.

ABSTRACT. Let $\mathcal{T}_1(k)$ be the Tannakian category generated by 1-motives defined over a field k of characteristic 0 and let $\mathcal{G}_{\mathrm{mot}}\big(\mathcal{T}_1(k)\big)$ be its fundamental group, *i.e.*, the motivic Galois group $\mathcal{G}_{\mathrm{mot}}\big(\mathcal{T}_1(k)\big)$ of 1-motives. We find four short exact sequences of affine group sub-$\mathcal{T}_1(k)$-schemes of $\mathcal{G}_{\mathrm{mot}}\big(\mathcal{T}_1(k)\big)$, correlated one to each other by inclusions and projections, which are the motivic version of well-known short exact sequences in Hodge theory. Moreover, given a 1-motive M, we compute explicitly the biggest Tannakian subcategory of the one generated by M, whose fundamental group is commutative.

1. Introduction

Let k be a field of characteristic 0 embeddable in $\mathbb{C}$. Fix an algebraic closure $\overline{k}$ of k. Denote by $\mathcal{T}_1(k)$ the Tannakian category generated by 1-motives defined over k (in an appropriate category of mixed realizations). The tensor product of $\mathcal{T}_1(k)$ allows us to define the notion of Hopf algebras in the category $\mathrm{Ind}\mathcal{T}_1(k)$ of Ind-objects of $\mathcal{T}_1(k)$. The category of affine group $\mathcal{T}_1(k)$-schemes is the opposite of the category of Hopf algebras in $\mathrm{Ind}\mathcal{T}_1(k)$. The *motivic Galois group* $\mathcal{G}_{\mathrm{mot}}\big(\mathcal{T}_1(k)\big)$ *of* 1-*motives* is the affine group $\mathcal{T}_1(k)$-scheme $\mathrm{Sp}\,(\Lambda)$, whose Hopf algebra Λ is endowed for each object X of $\mathcal{T}_1(k)$ with a morphism $X \to \Lambda \otimes X$ functorial in X, and is universal for these properties.

The weight filtration W_* of 1-motives induces an increasing filtration W_* of 3 steps on the motivic Galois group $\mathcal{G}_{\mathrm{mot}}\big(\mathcal{T}_1(k)\big)$. In section 2 we recover each of these 3 steps as intersection of some normal group sub-$\mathcal{T}_1(k)$-schemes that we compute explicitly.

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The explicit computation of these normal group sub-$\mathcal{T}_1(k)$-schemes will provide four exact short sequences of group sub-$\mathcal{T}_1(k)$-schemes of $\mathcal{G}_{\mathrm{mot}}\big(\mathcal{T}_1(k)\big)$, which are correlated one to each other by inclusions and projections, and which also involve the filtration W_* of $\mathcal{G}_{\mathrm{mot}}\big(\mathcal{T}_1(k)\big)$ (Theorem 3.6).

One of these short exact sequences is

$$0 \longrightarrow H_{\mathcal{T}_1(k)}\big(\mathrm{Gr}_0^{\mathrm{W}}\mathcal{T}_1(k)\big) \longrightarrow \mathcal{G}_{\mathrm{mot}}\big(\mathcal{T}_1(k)\big) \xrightarrow{\pi} \mathrm{Gal}(\overline{k}/k) \longrightarrow 0, \tag{1.1}$$

where $H_{\mathcal{T}_1(k)}\big(\mathrm{Gr}_0^{\mathrm{W}}\mathcal{T}_1(k)\big)$ is the normal group sub-$\mathcal{T}_1(k)$-scheme which acts trivially on $\mathrm{Gr}_0^{\mathrm{W}}\mathcal{T}_1(k)$. If $e : \mathcal{T}_1(k) \rightarrow \mathcal{T}_1(\overline{k})$ is the base extension functor, we have that the $\mathcal{T}_1(\overline{k})$-scheme $e\,H_{\mathcal{T}_1(k)}\big(\mathrm{Gr}_0^{\mathrm{W}}\mathcal{T}_1(k)\big)$ is canonically isomorphic to $\mathcal{G}_{\mathrm{mot}}\big(\mathcal{T}_1(\overline{k})\big)$. If τ is an element of $\mathrm{Gal}(\overline{k}/k)$, then $\pi^{-1}(\tau)$ is $\underline{\mathrm{Hom}}^{\otimes}(\mathrm{Id}, \tau \circ \mathrm{Id})$, where Id and $\tau \circ \mathrm{Id}$ have to be regarded as functors on $\mathcal{T}_1(\overline{k})$ (Corollary 3.8). For each embedding $\sigma : k \rightarrow \mathbb{C}$, the fibre functor ω_σ, dubbed "Hodge realization", furnishes the $\mathbb{Q}$-pro-algebraic group

$$\omega_\sigma \mathcal{G}_{\mathrm{mot}}\big(\mathcal{T}_1(k)\big) \;=\; \underline{\mathrm{Aut}}^{\otimes}_{\mathbb{Q}}(\omega_\sigma),$$

which is the *Hodge realization of the motivic Galois group of* $\mathcal{T}_1(k)$ (*cf.* 8.13.1 of [7]). Hence the short exact sequence (1.1) is *the geometrical origin, i.e., the motivic version* of the short exact sequence of $\mathbb{Q}$-algebraic groups

$$0 \longrightarrow \underline{\mathrm{Aut}}^{\otimes}_{\mathbb{Q}}\big(\omega_{\overline{\sigma}\,|\mathcal{T}_1(\overline{k})}\big) \longrightarrow \underline{\mathrm{Aut}}^{\otimes}_{\mathbb{Q}}(\omega_\sigma) \longrightarrow \mathrm{Gal}(\overline{k}/k) \longrightarrow 0,$$

where $\overline{\sigma} : \overline{k} \rightarrow \mathbb{C}$ is the embedding of $\overline{k}$ in $\mathbb{C}$ which extends $\sigma : k \rightarrow \mathbb{C}$.

This last sequence is the restriction to 1-motives of the sequence found by P. Deligne (see 6.23 of Part II of [5]) and U. Jannsen (see 4.7 of [9]). Remark that in this article we restrict ourselves to 1-motives because we are interested in motivic (and hence geometric) results and until now we know concretely only 1-motives. Also the equality

$$\pi^{-1}(\tau) \;=\; \underline{\mathrm{Hom}}^{\otimes}(\mathrm{Id}, \tau \circ \mathrm{Id})$$

is *the motivic version* of the one found by P. Deligne and U. Jannsen (*loc. cit.*).

In Section 3 we restrict ourselves to the Tannakian subcategory $\langle M \rangle^{\otimes}$ of $\mathcal{T}_1(k)$ generated by a 1-motive M defined over k. The motivic Galois group $\mathcal{G}_{\mathrm{mot}}(M)$ of M is the fundamental group of the Tannakian category $\langle M \rangle^{\otimes}$. Using the main result of [2], we compute the derived group of the unipotent radical $\mathrm{W}_{-1}\big(\mathrm{Lie}\,\mathcal{G}_{\mathrm{mot}}(M)\big)$ of the Lie algebra of $\mathcal{G}_{\mathrm{mot}}(M)$ (Proposition 4.2). Moreover we construct explicitly the biggest Tannakian subcategory of $\langle M \rangle^{\otimes}$ which has a commutative motivic Galois group: more precisely, starting from the 1-motive M, we construct a sub-1-motive M^{ab} of M whose motivic Galois group $\mathcal{G}_{\mathrm{mot}}(M^{ab})$ is the biggest commutative group sub-$\mathcal{T}_1(k)$-scheme of $\mathcal{G}_{\mathrm{mot}}(M)$ (Theorem 4.4).

In this paper k is a field of characteristic 0 embeddable in $\mathbb{C}$, and we fix an algebraic closure $\overline{k}$ of k.

2. Preliminaries about Tannakian theory

Let $\mathcal{T}$ be a Tannakian category over k, *i.e.*, a rigid abelian k-linear tensor category over k which possesses a fibre functor over a non empty k-scheme (see 3.7 of [5] or see 2.8 and 2.1 of [7]). A *Tannakian subcategory of* $\mathcal{T}$ is a strictly full abelian subcategory $\mathcal{T}'$ of $\mathcal{T}$ which is closed under the formation of subquotients, direct sums, tensor products and duals. Note that $\mathcal{T}'$ is endowed with the restriction to $\mathcal{T}'$ of the fibre functor of $\mathcal{T}$. The tensor product of $\mathcal{T}$ allows us to define the notion of Hopf algebras in the category $\mathrm{Ind}\mathcal{T}$ of Ind-objects of $\mathcal{T}$ indexed by a filtered small category. The *category of affine group* $\mathcal{T}$*-schemes* is the opposite of the category of Hopf algebras in $\mathrm{Ind}\mathcal{T}$. We denote $\mathrm{Sp}(A)$ the affine group $\mathcal{T}$-scheme defined by the Hopf algebra A. The *fundamental group* $\pi(\mathcal{T})$ of a Tannakian category $\mathcal{T}$ is the affine group $\mathcal{T}$-scheme $\mathrm{Sp}(\Lambda)$, whose Hopf algebra Λ is endowed for each object X of $\mathcal{T}$ with a morphism $\lambda_X : X^\vee \otimes X \to \Lambda$ functorial in X, and is universal for these properties. These morphisms, which can be rewritten on the form $X \to X \otimes \Lambda$, define *an action of the fundamental group* $\pi(\mathcal{T})$ *on each object of* $\mathcal{T}$. By 6.4 of [6], to any exact and k-linear $\otimes$-functor $u : \mathcal{T}_1 \to \mathcal{T}_2$ between Tannakian categories over k, corresponds a morphism of affine group $\mathcal{T}_2$-schemes

$$U : \pi(\mathcal{T}_2) \longrightarrow u\,\pi(\mathcal{T}_1).$$

As in Theorem 4.3.2(g) of Part II of [11], we have the following dictionary between the functor u and the morphism U:

(1) Firstly, U is faithfully flat if and only if u is fully faithful and every subobject of $u(X_1)$, for X_1 an object of $\mathcal{T}_1$, is isomorphic to the image of a subobject of X_1.

(2) Secondly, U is a closed immersion if and only if every object of $\mathcal{T}_2$ is isomorphic to a subquotient of an object of the form $u(X_1)$, for X_1 an object of $\mathcal{T}_1$ (for the definition of closed immersion and of faithfully flat morphism of affine group $\mathcal{T}$-schemes (see §5 of [6] or see 7.5–7.12 of [7]).

An immediate consequence of 8.17 of [7] is the following dictionary between Tannakian subcategories of $\mathcal{T}$ and normal affine group sub-$\mathcal{T}$-schemes of $\pi(\mathcal{T})$.

Tannakian correspondence: *There is a bijection between the Tannakian subcategories of* $\mathcal{T}$ *and the normal affine group sub-*$\mathcal{T}$*-schemes of* $\pi(\mathcal{T})$*, which has the following properties:*

• *It associates to each Tannakian subcategory* $\mathcal{T}'$ *of* $\mathcal{T}$*, the kernel* $H_{\mathcal{T}}(\mathcal{T}')$ *of the morphism of affine group* $\mathcal{T}$*-schemes* $I : \pi(\mathcal{T}) \to i\pi(\mathcal{T}')$ *corresponding to the inclusion functor* $i : \mathcal{T}' \to \mathcal{T}$*. In particular we have the exact sequence of affine group* $\mathcal{T}$*-schemes*

$$0 \longrightarrow H_{\mathcal{T}}(\mathcal{T}') \longrightarrow \pi(\mathcal{T}) \longrightarrow i\,\pi(\mathcal{T}') \longrightarrow 0.$$

• *It associates to each normal affine group sub-*$\mathcal{T}$*-scheme* H *of* $\pi(\mathcal{T})$*, the Tannakian subcategory* $\mathcal{T}(H)$ *of objects of* $\mathcal{T}$ *on which the action of* $\pi(\mathcal{T})$ *induces a trivial action of* H*.*

Recall that a sequence

$$0 \longrightarrow H \stackrel{\epsilon}{\longrightarrow} G \stackrel{\eta}{\longrightarrow} G' \longrightarrow 0$$

of group $\mathcal{T}$-schemes is exact if η is faithfully flat and the sequence

$$0 \longrightarrow H(\mathrm{Sp}A) \stackrel{\epsilon}{\longrightarrow} G(\mathrm{Sp}A) \stackrel{\eta}{\longrightarrow} G'(\mathrm{Sp}A)$$

is exact for any $\mathcal{T}$-scheme $\mathrm{Sp}(A)$.

If $\mathcal{T}$ is the Tannakian category of Artin motives, we recover the classical Galois correspondence between field extensions and normal subgroups of the Galois group. (For neutral Tannakian categories, P. Deligne has pointed out to the author that we can see this correspondence as a reformulation of 4.3.2(b) and 4.3.2(g) of Part II of [11]). Moreover he pointed out the following example of non neutral Tannakian category to which we can apply this Tannakian correspondence: the Tannakian category of F-isocristals on the algebraic closure $\mathbb{F}$ of $\mathbb{F}_p$ and the Tannakian subcategories of objects all of whose slopes, multiplied by d, for some fixed integer d, are integers (*cf.* 3.3.3.1 of Part VI of [11]).

Like in Galois theory, the Tannakian correspondence inverts inclusions.

Lemma 2.1. (i) *If $\mathcal{T}_1$ and $\mathcal{T}_2$ are two Tannakian subcategories of $\mathcal{T}$ such that $\mathcal{T}_1 \subseteq \mathcal{T}_2$, then $H_{\mathcal{T}}(\mathcal{T}_1) \supseteq H_{\mathcal{T}}(\mathcal{T}_2)$.*

(ii) *If H_1 and H_2 are two normal subgroups of $\pi(\mathcal{T})$ such that $H_1 \subseteq H_2$, then $\mathcal{T}(H_1) \supseteq \mathcal{T}(H_2)$.*

Let ω be a fibre functor of the Tannakian category $\mathcal{T}$ over a k-scheme S, namely an exact k-linear $\otimes$-functor from $\mathcal{T}$ to the category of quasi-coherent sheaves over S. This defines a $\otimes$-functor, again denoted by ω, from $\mathrm{Ind}\mathcal{T}$ to the category of quasi-coherent sheaves over S. If the fundamental group $\pi(\mathcal{T})$ is the group $\mathcal{T}$-scheme $\mathrm{Sp}(\Lambda)$, we define $\omega\big(\pi(\mathcal{T})\big) = \mathrm{Spec}\big(\omega(\Lambda)\big)$. According to 8.13.1 of [7], the spectrum $\mathrm{Spec}\big(\omega(\Lambda)\big)$ is the affine group S-scheme $\underline{\mathrm{Aut}}_S^{\otimes}(\omega)$ representing the functor which associates to each S-scheme T, $u : T \to S$, the group of automorphisms of $\otimes$-functors of the functor

$$\begin{array}{rcl} \omega_T : \mathcal{T} & \longrightarrow & \big\{\text{locally free sheaves of finite rank over } T\big\} \\ X & \longmapsto & u^*\omega(X). \end{array}$$

From the formalism of 5.11 of [6], we have the following dictionary:

• To give oneself the group $\mathcal{T}$-scheme $\pi(\mathcal{T}) = \mathrm{Sp}\,(\Lambda)$ is the same thing as to give oneself, for each fibre functor ω over a k-scheme S, the group S-scheme $\underline{\mathrm{Aut}}_S^{\otimes}(\omega)$, in a functorial way with respect to ω, and in a compatible way with respect to the base changes $S' \to S$.

• Let $u : \mathcal{T}_1 \to \mathcal{T}_2$ be a k-linear $\otimes$-functor between Tannakian categories over k. To give oneself the corresponding morphism

$$U : \pi(\mathcal{T}_2) \longrightarrow u\pi(\mathcal{T}_1)$$

of group $\mathcal{T}_2$-schemes, is the same thing as to give oneself, for each fibre functor ω of $\mathcal{T}_2$ over a k-scheme S, a morphism of group S-schemes

$$\underline{\mathrm{Aut}}_S^{\otimes}(\omega) \longrightarrow \underline{\mathrm{Aut}}_S^{\otimes}(\omega \circ u),$$

in a functorial way with respect to ω.

3. Some motivic Galois groups

Let $MR(k)$ be the category of mixed realizations (for absolute Hodge cycles) over k defined by U. Jannsen in 2.1 of [9]. The category $MR(k)$ is a neutral Tannakian category over $\mathbb{Q}$ with fibre functors $\{\omega_\sigma\}_{\sigma:k\to\mathbb{C}}$, the so-called "Hodge realizations".

The *Tannakian category of Artin motives* $\mathcal{T}_0(k)$ *over* k is the Tannakian subcategory of $MR(k)$ generated by realizations of 0-dimensional smooth varieties over k. By fixing an algebraic closure $\overline{k}$ of k, $\mathcal{T}_0(k)$ is equivalent to the category of finite-dimensional $\mathbb{Q}$-representations of the constant, pro-finite affine group $\mathbb{Q}$-scheme $\mathrm{Gal}(\overline{k}/k)$:

$$\begin{array}{lrcl} (3.1) & \mathcal{T}_0(k) & \cong & \mathrm{Rep}_{\mathbb{Q}}(\mathrm{Gal}(\overline{k}/k)) \\ & X & \longmapsto & \mathbb{Q}^{X(\overline{k})}. \end{array}$$

A 1-*motive* $M = [X \stackrel{u}{\longrightarrow} G]$ over k consists of the following:

(1) A group scheme X over k, which is locally, for the étale topology, a constant group scheme defined by a finitely generated free $\mathbb{Z}$-module.

(2) A semi-abelian variety G defined over k, *i.e.*, an extension of an abelian variety A by a torus $Y(1)$, with cocharacter group Y.

(3) A morphism $u : X \to G$ of group k-schemes.

The 1-motives are mixed motives of level less than or equal to 1: the weight filtration W_* on $M = [X \stackrel{u}{\longrightarrow} G]$ is given by

$$\begin{cases} \mathrm{W}_i(M) &=& M \quad \text{for each } i \geq 0, \\ \mathrm{W}_{-1}(M) &=& [0 \longrightarrow G], \\ \mathrm{W}_{-2}(M) &=& [0 \longrightarrow Y(1)], \\ \mathrm{W}_j(M) &=& 0 \quad \text{for each } j \leq -3. \end{cases}$$

If we denote $\mathrm{Gr}_n^{\mathrm{W}} = \mathrm{W}_n/\mathrm{W}_{n-1}$, we have

$$\mathrm{Gr}_0^{\mathrm{W}}(M) = [X \longrightarrow 0], \quad \mathrm{Gr}_{-1}^{\mathrm{W}}(M) = [0 \longrightarrow A] \quad \text{and} \quad \mathrm{Gr}_{-2}^{\mathrm{W}}(M) = [0 \longrightarrow Y(1)].$$

The *Tannakian category* $\mathcal{T}_1(k)$ *of* 1-*motives over* k is the Tannakian subcategory of $MR(k)$ generated by mixed realizations of 1-motives (see 10.1 of [4]). Since the category $MR(k)$ of mixed realizations is $\mathbb{Q}$-linear, in the following we work with iso-1-motives (see p. 104 and p. 106 of [6]) called just 1-motives below. The unit object 1 of $\mathcal{T}_1(k)$ is the 1-motive $\mathbb{Z}(0) = [\mathbb{Z} \longrightarrow 0]$. For each object M of $\mathcal{T}_1(k)$, we denote its dual by

$$M^\vee = \underline{\mathrm{Hom}}\big(M, \mathbb{Z}(0)\big).$$

The Cartier dual of an object M of $\mathcal{T}_1(k)$ is the object

$$M^* = M^\vee \otimes \mathbb{Z}(1).$$

We denote by $\mathrm{W}_{-1}\mathcal{T}_1(k)$ (resp. $\mathrm{Gr}_n^{\mathrm{W}}\mathcal{T}_1(k)$, for $n \leq 0$) the Tannakian subcategory of $\mathcal{T}_1(k)$ generated by all $\mathrm{W}_{-1}M$ (resp. $\mathrm{Gr}_n^{\mathrm{W}}M$, for $n \leq 0$) with M a 1-motive.

Lemma 3.1. (i) *The Tannakian subcategory* $\mathcal{T}_0(k)$ *of* $MR(k)$ *is equivalent (as a tensor category) to the Tannakian subcategory* $\mathrm{Gr}_0^{\mathrm{W}}\mathcal{T}_1(k)$.

(ii) *We have the following anti-equivalence of tensor categories*

$$\mathcal{T}_0(k) \otimes \langle \mathbb{Z}(1) \rangle^{\otimes} \longrightarrow \mathrm{Gr}^{\mathrm{W}}_{-2} \mathcal{T}_1(k)$$

which is defined on the generators by $X \otimes \mathbb{Z}(1) \longmapsto X^{\vee}(1)$ *(see §5 of* [7] *for the definition of the tensor product* $\mathcal{T}_0(k) \otimes \langle \mathbb{Z}(1) \rangle^{\otimes}$ *of two Tannakian categories).*

Proof. Assertion (i) is a consequence of (3.1). According to (i), we can view an object X of $\mathcal{T}_0(k)$ as the character group of a k-torus T. The dual $X^{\vee}$ of X in the Tannakian category $MR(k)$, can be identified with the cocharacter group of T which can be written, according to our notation, as $X^{\vee}(1)$. The anti-equivalence between the category of character groups and the category of cocharacter groups furnishes the desired anti-equivalence (ii). $\square$

If a Tannakian category $\mathcal{T}$ is generated by motives, the fundamental group $\pi(\mathcal{T})$ is called the *motivic Galois group* $\mathcal{G}_{\mathrm{mot}}(\mathcal{T})$ *of* $\mathcal{T}$. Here are some examples of motivic Galois groups:

(1) $\mathcal{G}_{\mathrm{mot}}\big(\mathbb{Z}(0)\big)$ is the affine group $\langle \mathbb{Z}(0) \rangle^{\otimes}$-scheme $\mathrm{Sp}\big(\mathbb{Z}(0)\big)$: it is the trivial group $\{1\}$.

(2) $\mathcal{G}_{\mathrm{mot}}\big(\mathbb{Z}(1)\big)$ is the affine group $\langle \mathbb{Z}(1) \rangle^{\otimes}$-scheme $\mathbb{G}_m$ defined by the $\mathbb{Q}$-scheme $\mathbb{G}_{m/\mathbb{Q}}$ (see 5.6 of [6]).

(3) $\mathcal{G}_{\mathrm{mot}}\big(\mathcal{T}_0(\overline{k})\big)$ is the affine group $\mathcal{T}_0(\overline{k})$-scheme $\mathrm{Sp}\left(1_{\mathcal{T}_0(\overline{k})}\right)$ defined by the $\mathbb{Q}$-scheme $\mathrm{Spec}\,(\mathbb{Q})$ (*cf.* (3.1) with $k = \overline{k}$ and *cf.* 6.3 of [6]).

(4) By Lemma 2.2 (ii), $\mathcal{G}_{\mathrm{mot}}\big(\mathrm{Gr}^{\mathrm{W}}_{-2}\mathcal{T}_1(\overline{k})\big)$ is the affine group $\mathrm{Gr}^{\mathrm{W}}_{-2}\mathcal{T}_1(\overline{k})$-scheme $\mathbb{G}_m$ defined by the $\mathbb{Q}$-scheme $\mathbb{G}_{m/\mathbb{Q}}$ (see 5.6 of [6]).

(5) If k is algebraically closed, the motivic Galois group of motives of CM-type over k is the Serre group (*cf.* §6 of [5] or *cf.* 4.8 of [10]).

(6) According to (3.1) and to 6.3 of [6], we have also the following example:

Lemma 3.2. $\mathcal{G}_{\mathrm{mot}}\big(\mathcal{T}_0(k)\big)$ *is the affine group* $\mathcal{T}_0(k)$*-scheme* $\mathcal{GAL}(\overline{k}/k)$ *which satisfies functorially the following property: for any fibre functor* ω *over* $\mathrm{Spec}\,(\mathbb{Q})$ *of* $\mathcal{T}_0(k)$*, the affine group scheme*

$$\omega\big(\mathcal{GAL}(\overline{k}/k)\big) \;=\; \underline{\mathrm{Aut}}^{\otimes}_{\mathrm{Spec}\,(\mathbb{Q})}(\omega)$$

is canonically isomorphic to $\mathrm{Gal}(\overline{k}/k)$.

Proposition 3.3. (i) $\mathcal{G}_{\mathrm{mot}}\big(\mathrm{Gr}^{\mathrm{W}}_{0}\mathcal{T}_1(k)\big) \;=\; \mathcal{GAL}(\overline{k}/k)$,

(ii) $\mathcal{G}_{\mathrm{mot}}\big(\mathrm{Gr}^{\mathrm{W}}_{-2}\mathcal{T}_1(k)\big) \;=\; i_1\mathcal{GAL}(\overline{k}/k) \times i_2\mathbb{G}_m$*, where the functors*

$$\begin{array}{rcl} i_1 : \mathcal{T}_0(k) = \mathcal{T}_0(k) \otimes \mathrm{Vec}(\mathbb{Q}) & \longrightarrow & \mathcal{T}_0(k) \otimes \langle \mathbb{Z}(1) \rangle^{\otimes} \\ X & \longmapsto & X \otimes 1, \end{array}$$

and

$$\begin{array}{rcl} i_2 : \langle \mathbb{Z}(1) \rangle^{\otimes} = \mathrm{Vec}(\mathbb{Q}) \otimes \langle \mathbb{Z}(1) \rangle^{\otimes} & \longrightarrow & \mathcal{T}_0(k) \otimes \langle \mathbb{Z}(1) \rangle^{\otimes} \\ \mathbb{Z}(1) & \longmapsto & 1 \otimes \mathbb{Z}(1), \end{array}$$

identify respectively $\mathcal{T}_0(k)$ *and* $\langle \mathbb{Z}(1) \rangle^{\otimes}$ *with full subcategories of* $\mathcal{T}_0(k) \otimes \langle \mathbb{Z}(1) \rangle^{\otimes}$*. Here* $\mathrm{Vec}(\mathbb{Q})$ *is the Tannakian category of finite dimensional vector spaces over* $\mathbb{Q}$*.*

Proof. Assertion (i) is clear from Lemmas 3.1(i) and 3.2. Assertion (ii) is a consequence of Lemma 3.1(ii), because according to 2.40.5 of [10] we have

$$\mathcal{G}_{\mathrm{mot}}\big(\mathrm{Gr}^{\mathrm{W}}_{-2}\mathcal{T}_1(k)\big) \;=\; \mathcal{G}_{\mathrm{mot}}\big(\mathcal{T}_0(k)\otimes\langle\mathbb{Z}(1)\rangle^{\otimes}\big) \;=\; i_1\mathcal{G}_{\mathrm{mot}}\big(\mathcal{T}_0(k)\big)\times i_2\mathbb{G}_m.$$

□

The action of the motivic Galois group $\mathcal{G}_{\mathrm{mot}}\big(\mathcal{T}_1(k)\big)$ on each object of $\mathcal{T}_1(k)$ and the weight filtration W_* on objects of $\mathcal{T}_1(k)$ allow us to define an increasing filtration, again denoted by W_*, on $\mathcal{G}_{\mathrm{mot}}\big(\mathcal{T}_1(k)\big)$ (*cf.* §2 of Chapter IV of [11]). Since the action of $\mathcal{G}_{\mathrm{mot}}\big(\mathcal{T}_1(k)\big)$ on each 1-motive M factorizes through the projection

$$\mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k)) \longrightarrow \mathcal{G}_{\mathrm{mot}}(M)$$

given by the inclusion $\langle M\rangle^{\otimes} \longrightarrow \mathcal{T}_1(k)$, in order to describe the filtration W_* on $\mathcal{G}_{\mathrm{mot}}\big(\mathcal{T}_1(k)\big)$, we restrict ourselves to the generators of $\mathcal{T}_1(k)$, given for any $\mathcal{T}_1(k)$-scheme $\mathrm{Sp}\,(B)$ by

- $\mathrm{W}_0\Big(\mathcal{G}_{\mathrm{mot}}\big(\mathcal{T}_1(k)\big)\Big) \;=\; \mathcal{G}_{\mathrm{mot}}\big(\mathcal{T}_1(k)\big),$
- $\mathrm{W}_{-1}\Big(\mathcal{G}_{\mathrm{mot}}\big(\mathcal{T}_1(k)\big)(\mathrm{Sp}\,B)\Big) =$
$$\left\{ g\in\mathcal{G}_{\mathrm{mot}}\big(\mathcal{T}_1(k)\big)(\mathrm{Sp}\,B) \;\middle|\; \begin{array}{l} (g-id)M\subseteq \mathrm{W}_{-1}(M),\\ (g-id)\mathrm{W}_{-1}(M)\subseteq\mathrm{W}_{-2}(M),\\ (g-id)\mathrm{W}_{-2}(M)\;=\;0,\\ \text{for each } 1\text{−motive } M\in\mathcal{T}_1(k)\end{array}\right\},$$
- $\mathrm{W}_{-2}\Big(\mathcal{G}_{\mathrm{mot}}\big(\mathcal{T}_1(k)\big)(\mathrm{Sp}\,B)\Big) =$
$$\left\{ g\in\mathcal{G}_{\mathrm{mot}}\big(\mathcal{T}_1(k)\big)(\mathrm{Sp}\,B) \;\middle|\; \begin{array}{l} (g-id)M\subseteq \mathrm{W}_{-2}(M),\\ (g-id)\mathrm{W}_{-1}(M)\;=\;0,\\ \text{for each } 1\text{−motive } M\in\mathcal{T}_1(k)\end{array}\right\},$$
- $\mathrm{W}_{-3}\Big(\mathcal{G}_{\mathrm{mot}}\big(\mathcal{T}_1(k)\big)\Big) = 0.$

In order to understand better this filtration W_*, we can apply the formalism of 5.11 of [6] which was recalled at the end of §1: via the fibre functors $\{\omega_\sigma\}_{\sigma:k\to\mathbb{C}}$, the "Hodge realizations", we are led to work with the $\mathbb{Q}$-pro-algebraic groups

$$\omega_\sigma\mathcal{G}_{\mathrm{mot}}\big(\mathcal{T}_1(k)\big) \;=\; \underline{\mathrm{Aut}}^{\otimes}_{\mathbb{Q}}(\omega_\sigma)$$

which act on the Hodge realizations $\omega_\sigma(M)$ of 1-motives.

Now we prove that this filtration W_* of $\mathcal{G}_{\mathrm{mot}}\big(\mathcal{T}_1(k)\big)$ can be recovered from the group sub-$\mathcal{T}_1(k)$-schemes

$$H_{\mathcal{T}_1(k)}\big(\mathrm{Gr}^{\mathrm{W}}_i\mathcal{T}_1(k)\big),$$

with $i=-1,-2$, and

$$H_{\mathcal{T}_1(k)}\big(\mathrm{W}_{-1}\mathcal{T}_1(k)\big)$$

of $\mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k))$. These group sub-$\mathcal{T}_1(k)$-schemes are the motivic generalizations of the algebraic $\mathbb{Q}$-groups introduced in §2 of [1].

Lemma 3.4. (1) *On the one hand,*

$$\mathrm{W}_{-1}\Big(\mathcal{G}_{\mathrm{mot}}\big(\mathcal{T}_1(k)\big)\Big) \;=\; H_{\mathcal{T}_1(k)}\big(\mathrm{Gr}^{\mathrm{W}}_{-1}\mathcal{T}_1(k)\big) \cap H_{\mathcal{T}_1(k)}\big(\mathrm{Gr}^{\mathrm{W}}_{-2}\mathcal{T}_1(k)\big).$$

(2) *On the other hand,*

$$\mathrm{W}_{-2}\Big(\mathcal{G}_{\mathrm{mot}}\big(\mathcal{T}_1(k)\big)\Big) \;=\; H_{\mathcal{T}_1(k)}\big(\mathrm{W}_{-1}\mathcal{T}_1(k)\big) \cap H_{\mathcal{T}_1(k)}\big(\mathrm{W}_0/\mathrm{W}_{-2}\mathcal{T}_1(k)\big).$$

Proof. By definition of the filtration W_* on the motivic Galois group $\mathcal{G}_{\mathrm{mot}}\big(\mathcal{T}_1(k)\big)$, we have

$$\mathrm{W}_{-1}\Big(\mathcal{G}_{\mathrm{mot}}\big(\mathcal{T}_1(k)\big)\Big) \;=\; H_{\mathcal{T}_1(k)}\big(\mathrm{Gr}^{\mathrm{W}}_{-1}\mathcal{T}_1(k)\big) \cap H_{\mathcal{T}_1(k)}\big(\mathrm{Gr}^{\mathrm{W}}_{-2}\mathcal{T}_1(k)\big).$$

Moreover according to Lemma 3.1, the Tannakian category $\mathrm{Gr}^{\mathrm{W}}_0\mathcal{T}_1(k)$ of Artin motives is canonically isomorphic to a Tannakian subcategory of $\mathrm{Gr}^{\mathrm{W}}_{-2}\mathcal{T}_1(k)$ and therefore Lemma 2.1 gives

$$H_{\mathcal{T}_1(k)}\big(\mathrm{Gr}^{\mathrm{W}}_0\mathcal{T}_1(k)\big) \supseteq H_{\mathcal{T}_1(k)}\big(\mathrm{Gr}^{\mathrm{W}}_{-2}\mathcal{T}_1(k)\big).$$

The assertion (i) is now clear.

Again by definition of the filtration W_*, we have

$$\mathrm{W}_{-2}\Big(\mathcal{G}_{\mathrm{mot}}\big(\mathcal{T}_1(k)\big)\Big) \;=\; H_{\mathcal{T}_1(k)}\big(\mathrm{W}_{-1}\mathcal{T}_1(k)\big) \cap H_{\mathcal{T}_1(k)}\big(\mathrm{W}_0/\mathrm{W}_{-2}\mathcal{T}_1(k)\big).$$

In the category of mixed realizations, we have the duality

$$N \longmapsto \underline{\mathrm{Hom}}(N,\mathbb{Z}(1)),$$

which corresponds to the Cartier duality $M^* = \underline{\mathrm{Hom}}(M,\mathbb{Z}(1))$ in the case of realizations of 1-motives. This duality induces an anti-equivalence of tensor categories

$$\mathrm{W}_0/\mathrm{W}_{-2}\mathcal{T}_1(k) \;\longrightarrow\; \mathrm{W}_{-1}\mathcal{T}_1(k)$$

(in Proposition 3.7 of Exposé VIII of [8], Grothendieck proves this anti-equivalence for the generators of these two categories). Therefore we get the assertion (ii). □

Remark 3.5. If we restrict ourselves to the Tannakian subcategory of $\mathcal{T}_1(k)$ generated by a 1-motive M, it is not true that the Tannakian category generated by $M/\mathrm{W}_{-2}M$ is equivalent to the Tannakian category generated by $\mathrm{W}_{-1}M$. Hence Lemma 3.4 must be modified in the following way:

$$\begin{aligned}
\mathrm{W}_{-1}\big(\mathcal{G}_{\mathrm{mot}}(M)\big) &= H_{\mathcal{T}_1(k)}\big(\mathrm{Gr}^{\mathrm{W}}_{-1}M\big) \cap H_{\mathcal{T}_1(k)}\big(\mathrm{Gr}^{\mathrm{W}}_{-2}M\big),\\
\mathrm{W}_{-2}\big(\mathcal{G}_{\mathrm{mot}}(M)\big) &= H_{\mathcal{T}_1(k)}\big(M/\mathrm{W}_{-2}M\big) \cap H_{\mathcal{T}_1(k)}\big(\mathrm{W}_{-1}M\big).
\end{aligned}$$

Before stating the main theorem of this paragraph, we need some notation. Consider the base extension functor

$$\begin{aligned}
e : \mathcal{T}_1(k) &\longrightarrow \mathcal{T}_1(\overline{k})\\
M &\longmapsto M \otimes_k \overline{k}.
\end{aligned} \tag{3.2}$$

According to (3.1), for $k = \overline{k}$, the images through e of the objects of $\mathcal{T}_0(k)$ are in the Tannakian subcategory generated by the unit object $1_{\mathcal{T}_1(\overline{k})}$ of $\mathcal{T}_1(\overline{k})$ and they generate it. Moreover, every object M in $\mathcal{T}_1(\overline{k})$ can be written as a subquotient of $M' \otimes_k \overline{k}$ for some object M' of $\mathcal{T}_1(k)$: in fact, for M' we can take the restriction of scalars $\mathrm{Res}_{k'/k}M_0$

with M_0 a model of M over a finite extension k' of k (see 2.16 of [9] for the definition of $\mathrm{Res}_{k'/k}$ and recall that by Proposition 5 of 7.6 of [3], the restriction of scalars of an abelian variety (resp. a semi-abelian variety, resp. a 1-motive) is again an abelian variety (resp. a semi-abelian variety, resp. a 1-motive)). Therefore the corresponding morphism of affine $\mathcal{T}_1(\overline{k})$-schemes

$$E : \mathcal{G}_{\mathrm{mot}}\big(\mathcal{T}_1(\overline{k})\big) \longrightarrow e\,\mathcal{G}_{\mathrm{mot}}\big(\mathcal{T}_1(k)\big)$$

is a closed immersion.

Let

$$\begin{cases} H & : & \mathcal{G}_{\mathrm{mot}}\big(\mathcal{T}_1(k)\big) \longrightarrow h\,\mathcal{G}_{\mathrm{mot}}\big(\mathrm{Gr}_0^{\mathrm{W}}\mathcal{T}_1(k)\big), \\ I & : & \mathcal{G}_{\mathrm{mot}}\big(\mathcal{T}_1(k)\big) \longrightarrow i\,\mathcal{G}_{\mathrm{mot}}\big(\mathrm{Gr}_*^{\mathrm{W}}\mathcal{T}_1(k)\big), \\ L & : & \mathcal{G}_{\mathrm{mot}}\big(\mathcal{T}_1(k)\big) \longrightarrow l\,\mathcal{G}_{\mathrm{mot}}\big(\mathrm{Gr}_{-1}^{\mathrm{W}}\mathcal{T}_1(k)\big), \end{cases}$$

be the faithfully flat morphisms corresponding respectively to the inclusions

$$\begin{cases} h & : & \mathrm{Gr}_0^{\mathrm{W}}\mathcal{T}_1(k) \longrightarrow \mathcal{T}_1(k), \\ i & : & \mathrm{Gr}_*^{\mathrm{W}}\mathcal{T}_1(k) \longrightarrow \mathcal{T}_1(k), \\ l & : & \mathrm{Gr}_{-1}^{\mathrm{W}}\mathcal{T}_1(k) \longrightarrow \mathcal{T}_1(k). \end{cases}$$

Then we can state the following theorem.

Theorem 3.6. *We have the following diagram of affine group $\mathcal{T}_1(k)$-schemes*

$$\begin{tikzcd}
0 \arrow[r] & H_{\mathcal{T}_1(k)}\big(\mathrm{Gr}_0^{\mathrm{W}}\mathcal{T}_1(k)\big) \arrow[r] & \mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k)) \arrow[r, "H"] & h\,\mathcal{GAL}(\overline{k}/k) \arrow[r] & 0 \\
0 \arrow[r] & H_{\mathcal{T}_1(k)}\big(\mathrm{Gr}_{-2}^{\mathrm{W}}\mathcal{T}_1(k)\big) \arrow[r] \arrow[u] & \mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k)) \arrow[r] \arrow[u, equal] & h\,\mathcal{GAL}(\overline{k}/k) \times \mathbb{G}_m \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & \mathrm{W}_{-1}\mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k)) \arrow[r] \arrow[u] & \mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k)) \arrow[r, "I"] \arrow[u, equal] & i\,\mathcal{G}_{\mathrm{mot}}\big(\mathrm{Gr}_*^{\mathrm{W}}\mathcal{T}_1(k)\big) \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & \mathrm{W}_{-2}\mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k)) \arrow[r] \arrow[u] & \mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k)) \arrow[r, "L"] \arrow[u, equal] & l\,\mathcal{G}_{\mathrm{mot}}(\mathrm{W}_{-1}\mathcal{T}_1(k)) \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

where all horizontal short sequences are exact and where the vertical arrows on the left are inclusions and those on the right are surjections. Moreover we have the following canonical isomorphisms of affine group $\mathcal{T}_1(\overline{k})$-schemes:

$$\begin{cases} e\,H_{\mathcal{T}_1(k)}\big(\mathrm{Gr}_0^{\mathrm{W}}\mathcal{T}_1(k)\big) & \cong & \mathcal{G}_{\mathrm{mot}}\big(\mathcal{T}_1(\overline{k})\big), \\ e\,H_{\mathcal{T}_1(k)}\big(\mathrm{Gr}_{-2}^{\mathrm{W}}\mathcal{T}_1(k)\big) & \cong & H_{\mathcal{T}_1(\overline{k})}\big(\langle \mathbb{Z}(1)\rangle^{\otimes}\big). \end{cases}$$

Proof. We prove the exactness of these four horizontal short sequences by applying the Tannakian correspondence to the following Tannakian subcategories of $\mathcal{T}_1(k)$:

$$\mathcal{T}_0(k), \quad \mathrm{Gr}_{-2}^{\mathrm{W}}\mathcal{T}_1(k), \quad \mathrm{Gr}_*^{\mathrm{W}}\mathcal{T}_1(k) \quad \text{and} \quad \mathrm{W}_{-1}\mathcal{T}_1(k).$$

The first and the second exact sequence are a consequence of Proposition 3.3. The third and the fourth exact sequence are a consequence of Lemma 3.4 (i) and (ii) respectively.

In order to prove that the left vertical arrows are inclusions and that the right vertical arrows are surjections, it is enough to apply Lemma 2.1. As we have observed, the images through the base extension functor e of the objects of $\mathcal{T}_0(k)$ generate the Tannakian subcategory $\langle 1_{\mathcal{T}_1(\overline{k})} \rangle^{\otimes}$ of $\mathcal{T}_1(\overline{k})$. Therefore the $\mathcal{T}_1(\overline{k})$-scheme $eH_{\mathcal{T}_1(k)}\big(\mathrm{Gr}_0^{\mathrm{W}} \mathcal{T}_1(k)\big)$ is canonically isomorphic to $\mathcal{G}_{\mathrm{mot}}\big(\mathcal{T}_1(\overline{k})\big)$.

According to Lemma 3.1 (ii), $\mathrm{Gr}_{-2}^{\mathrm{W}} \mathcal{T}_1(\overline{k})$ is equivalent as a tensor category to the Tannakian subcategory $\langle \mathbb{Z}(1) \rangle^{\otimes}$ of $\mathcal{T}_1(\overline{k})$ generated by the k-torus $\mathbb{Z}(1)$. Hence the objects of $\mathrm{Gr}_{-2}^{\mathrm{W}} \mathcal{T}_1(k)$ are exactly those objects of $\mathcal{T}_1(k)$ on which, after extension of scalars, the sub-$\mathcal{T}_1(\overline{k})$-scheme $H_{\mathcal{T}_1(\overline{k})}\big(\langle \mathbb{Z}(1) \rangle^{\otimes}\big)$ of $\mathcal{G}_{\mathrm{mot}}\big(\mathcal{T}_1(\overline{k})\big)$ acts trivially. Hence we get the second canonical isomorphism

$$e\, H_{\mathcal{T}_1(k)}\big(\mathrm{Gr}_{-2}^{\mathrm{W}} \mathcal{T}_1(k)\big) \cong H_{\mathcal{T}_1(\overline{k})}\big(\langle \mathbb{Z}(1) \rangle^{\otimes}\big).$$

□

As a corollary, we get the motivic version of 6.23(a), 6.23(c) of Part II of [5] and of 4.7(c), 4.7(e) of [9] . But before stating this corollary, we need the following lemma.

Lemma 3.7. *Let*

$$u_1 : \mathcal{T}_1 \to \mathcal{T}_2 \quad \text{and} \quad u_2 : \mathcal{T}_2 \to \mathcal{T}_3$$

be two exact and k-linear $\otimes$-functors between Tannakian categories over k. Denote by

$$U_1 : \pi(\mathcal{T}_2) \longrightarrow u_1 \pi(\mathcal{T}_1) \quad \text{and} \quad U_2 : \pi(\mathcal{T}_3) \to u_2 \pi(\mathcal{T}_2)$$

the morphisms of affine group $\mathcal{T}_2$-schemes and $\mathcal{T}_3$-schemes defined respectively by u_1 and u_2. Then the morphism of affine group $\mathcal{T}_3$-schemes corresponding to $u_2 \circ u_1$ is

$$U = u_2 U_1 \circ U_2 : \pi(\mathcal{T}_3) \longrightarrow u_2 \pi(\mathcal{T}_2) \longrightarrow u_2 u_1 \pi(\mathcal{T}_1).$$

Moreover, we have the following:

(i) *if the composition $u_2 \circ u_1$ is the constant functor $1_{\mathcal{T}_3}$, then U takes values in the trivial group, i.e., $U : \pi(\mathcal{T}_3) \longrightarrow \mathrm{Sp}\,(1_{\mathcal{T}_3})$;*

(ii) *if $\mathcal{T}_1 = \mathcal{T}_3$ and $u_2 \circ u_1 = id$, then $U = id$.*

Proof. The morphism of group $\mathcal{T}_2$-schemes

$$U_1 : \pi(\mathcal{T}_2) \longrightarrow u_1 \pi(\mathcal{T}_1)$$

provides a morphism of group $\mathcal{T}_3$-schemes

$$u_2 U_1 : u_2 \pi(\mathcal{T}_2) \longrightarrow u_2 u_1 \pi(\mathcal{T}_2).$$

Denote by

$$U : \pi(\mathcal{T}_3) \longrightarrow u_2 u_1 \pi(\mathcal{T}_1)$$

the morphism of group $\mathcal{T}_3$-schemes corresponding to the functor

$$u_2 \circ u_1 : \mathcal{T}_1 \longrightarrow \mathcal{T}_3.$$

According to the formalism 5.11 of [6], having the morphisms $U,\ \ u_2 U_1$ and U_2 of $\mathcal{T}_3$-schemes is respectively the same thing as having, for each fibre functor ω of $\mathcal{T}_3$ over a k-scheme S, the morphisms of group S-schemes

$$\begin{cases} \underline{\mathrm{Aut}}_S^{\otimes}(\omega) & \longrightarrow \quad \underline{\mathrm{Aut}}_S^{\otimes}\big(\omega\circ(u_2\circ u_1)\big), \\ \underline{\mathrm{Aut}}_S^{\otimes}(\omega\circ u_2) & \longrightarrow \quad \underline{\mathrm{Aut}}_S^{\otimes}\big((\omega\circ u_2)\circ u_1\big), \\ \underline{\mathrm{Aut}}_S^{\otimes}(\omega) & \longrightarrow \quad \underline{\mathrm{Aut}}_S^{\otimes}\big(\omega\circ u_2\big). \end{cases}$$

Hence we observe that $U = u_2 U_1 \circ U_2$. The remaining assertions are clear: in particular, if $u_2 \circ u_1$ is the constant functor $1_{\mathcal{T}_3}$, we have

$$\underline{\mathrm{Aut}}_S^{\otimes}\big(\omega_{|\langle 1_{\mathcal{T}_3}\rangle^{\otimes}}\big) \;=\; \mathrm{Spec}\,(k)$$

for each fibre functor ω of $\mathcal{T}_3$ over a k-scheme S. $\square$

Suppose that

$$F_1, F_2 : \mathcal{T}_1(\overline{k}) \longrightarrow \mathcal{T}_1(\overline{k})$$

are two functors. We define $\underline{\mathrm{Hom}}^{\otimes}(F_1,F_2)$ to be the functor which associates to each $\mathcal{T}_1(\overline{k})$-scheme $\mathrm{Sp}\,(B)$, the set of morphisms of $\otimes$-functors from

$$(F_1)_{\mathrm{Sp}\,(B)} : X \mapsto F_1(X)\otimes B$$

to

$$(F_2)_{\mathrm{Sp}\,(B)} : X \mapsto F_2(X)\otimes B.$$

Here $(F_1)_{\mathrm{Sp}\,(B)}$ and $(F_2)_{\mathrm{Sp}\,(B)}$ are $\otimes$-functors from $\mathcal{T}_1(\overline{k})$ to the category of modules over $\mathrm{Sp}\,(B)$. According to 8.11 of [7], this functor $\underline{\mathrm{Hom}}^{\otimes}(F_1,F_2)$ is representable. Moreover, each element τ of $\mathrm{Gal}(\overline{k}/k)$ defines a functor

$$\tau : \mathcal{T}_1(\overline{k}) \longrightarrow \mathcal{T}_1(\overline{k})$$

in the following way: the category $\mathcal{T}_1(\overline{k})$ is generated by motives of the form $e(M)$ with $M\in\mathcal{T}_1(k)$, and so it is enough to define $\tau e(M)$. We put $\tau e(M) = M\otimes_k \tau\overline{k}$.

Consider the inclusions of Tannakian categories

$$\mathcal{T}_0(k) \xrightarrow{\;j\;} \mathrm{Gr}_*^{\mathrm{W}}\mathcal{T}_1(k) \xrightarrow{\;i\;} \mathcal{T}_1(k).$$

We obtain the corresponding faithfully flat morphisms of group $\mathcal{T}_1(k)$-schemes

$$\mathcal{G}_{\mathrm{mot}}\big(\mathcal{T}_1(k)\big) \xrightarrow{\;I\;} i\,\mathcal{G}_{\mathrm{mot}}\left(\mathrm{Gr}_*^{\mathrm{W}}\mathcal{T}_1(k)\right) \xrightarrow{\;iJ\;} ij\,\mathcal{GAL}(\overline{k}/k),$$

where

$$iJ : i\,\mathcal{G}_{\mathrm{mot}}\big(\mathrm{Gr}_*^{\mathrm{W}}\mathcal{T}_1(k)\big) \longrightarrow ij\,\mathcal{G}_{\mathrm{mot}}\left(\mathrm{Gr}_0^{\mathrm{W}}\mathcal{T}_1(k)\right)$$

is defined by the morphism

$$J : \mathcal{G}_{\mathrm{mot}}\big(\mathrm{Gr}_*^{\mathrm{W}}\mathcal{T}_1(k)\big) \longrightarrow j\,\mathcal{G}_{\mathrm{mot}}\big(\mathrm{Gr}_0^{\mathrm{W}}\mathcal{T}_1(k)\big)$$

corresponding to the inclusion j.

Denote by

$$H : \mathcal{G}_{\mathrm{mot}}\big(\mathcal{T}_1(k)\big) \longrightarrow h\,\mathcal{G}_{\mathrm{mot}}\big(\mathrm{Gr}_0^{\mathrm{W}}\mathcal{T}_1(k)\big)$$

the faithfully flat morphism corresponding to the inclusion

$$h : \mathrm{Gr}_0^{\mathrm{W}}\mathcal{T}_1(k) \longrightarrow \mathcal{T}_1(k).$$

In particular, by Lemma 3.7 we have that $H = iJ\circ I$. The functor "take the graded"

$$\mathrm{gr}_*^{\mathrm{W}} : \mathcal{T}_1(k) \longrightarrow \mathrm{Gr}_*^{\mathrm{W}}\mathcal{T}_1(k)$$

corresponds to the closed immersion of affine group $\mathrm{Gr}_*^{\mathrm{W}}\mathcal{T}_1(k)$-schemes

$$\mathrm{Gr}_*^{\mathrm{W}} : \mathcal{G}_{\mathrm{mot}}\big(\mathrm{Gr}_*^{\mathrm{W}}\mathcal{T}_1(k)\big) \longrightarrow \mathrm{gr}_*^{\mathrm{W}}\mathcal{G}_{\mathrm{mot}}\big(\mathcal{T}_1(k)\big),$$

which identifies the motivic Galois group of $\mathrm{Gr}_*^{\mathrm{W}}\mathcal{T}_1(k)$ with the quotient $\mathrm{Gr}_0^{\mathrm{W}}$ of $\mathcal{G}_{\mathrm{mot}}\big(\mathcal{T}_1(k)\big)$.

Corollary 3.8. (i) *We have the following diagram of affine group $\mathcal{T}_1(k)$-schemes in which all the short sequences are exact:*

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & \mathrm{W}_{-1}\mathcal{G}_{\mathrm{mot}}\left(\mathcal{T}_1(k)\right) \arrow[r, equal] \arrow[d] & \mathrm{W}_{-1}\mathcal{G}_{\mathrm{mot}}\left(\mathcal{T}_1(k)\right) \arrow[d] & & \\
0 \arrow[r] & H_{\mathcal{T}_1(k)}\big(\mathrm{Gr}_0^{\mathrm{W}}\mathcal{T}_1(k)\big) \arrow[r] \arrow[d] & \mathcal{G}_{\mathrm{mot}}\left(\mathcal{T}_1(k)\right) \arrow[r, "H"] \arrow[d, "I"'] & h\,\mathcal{GAL}(\overline{k}/k) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & i\,H_{\mathrm{Gr}_*^{\mathrm{W}}\mathcal{T}_1(k)}\big(\mathrm{Gr}_0^{\mathrm{W}}\mathcal{T}_1(k)\big) \arrow[r] \arrow[d] & i\,\mathcal{G}_{\mathrm{mot}}\big(\mathrm{Gr}_*^{\mathrm{W}}\mathcal{T}_1(k)\big) \arrow[r, "iJ"] \arrow[d] & h\,\mathcal{GAL}(\overline{k}/k) \arrow[r] & 0. \\
 & 0 & 0 & &
\end{tikzcd}$$

(ii) *The morphism*

$$\mathrm{Gr}_*^{\mathrm{W}} : \mathcal{G}_{\mathrm{mot}}\left(\mathrm{Gr}_*^{\mathrm{W}}\mathcal{T}_1(k)\right) \longrightarrow \mathrm{gr}_*^{\mathrm{W}}\,\mathcal{G}_{\mathrm{mot}}\big(\mathcal{T}_1(k)\big)$$

of affine group $\mathrm{Gr}_*^{\mathrm{W}}\mathcal{T}_1(k)$-schemes is a section of the morphism $\mathrm{gr}_*^{\mathrm{W}}I$.

(iii) *For any affine $\mathcal{T}_1(k)$-scheme $\mathrm{Sp}\,(B)$ and for any $\tau \in h\,\mathcal{GAL}(\overline{k}/k)\big(\mathrm{Sp}\,(B)\big) = \mathrm{Gal}(\overline{k}/k)$, we have*

$$e\,H^{-1}_{\mathrm{Sp}\,(B)}(\tau) = \underline{\mathrm{Hom}}^{\otimes}(\mathrm{Id}, \mathrm{Id}\circ\tau)\big(\mathrm{Sp}\,(B)\big)$$

in $\mathcal{G}_{\mathrm{mot}}\big(\mathcal{T}_1(\overline{k})\big)\big(\mathrm{Sp}\,(B)\big)$, regarding Id and $\mathrm{Id}\circ\tau$ as functors on $\mathcal{T}_1(\overline{k})$. In an analogous way,

$$e\,(iJ)^{-1}_{\mathrm{Sp}\,(B)}(\tau) = \underline{\mathrm{Hom}}^{\otimes}(\mathrm{Id}, \mathrm{Id}\circ\tau)\big(\mathrm{Sp}\,(B)\big)$$

in $i\,\mathcal{G}_{\mathrm{mot}}\big(\mathrm{Gr}_*^{\mathrm{W}}\mathcal{T}_1(\overline{k})\big)\big(\mathrm{Sp}\,(B)\big)$, regarding Id and $\mathrm{Id}\circ\tau$ as functors on $\mathrm{Gr}_*^{\mathrm{W}}\mathcal{T}_1(\overline{k})$.

Proof. (i). We only have to prove the exactness of the last horizontal short sequence for which we apply the Tannakian correspondence to the category $\mathrm{Gr}_0^{\mathrm{W}}\mathcal{T}_1(k)$ viewed as a subcategory of $\mathrm{Gr}_*^{\mathrm{W}}\mathcal{T}_1(k)$.

(ii). Since the composition

$$\mathrm{gr}_*^{\mathrm{W}} \circ i : \mathrm{Gr}_*^{\mathrm{W}}\mathcal{T}_1(k) \longrightarrow \mathcal{T}_1(k) \longrightarrow \mathrm{Gr}_*^{\mathrm{W}}\mathcal{T}_1(k)$$

is the identity, from Lemma 3.7 we have

$$\mathrm{gr}_*^{\mathrm{W}}I \circ \mathrm{Gr}_*^{\mathrm{W}} = id : \mathcal{G}_{\mathrm{mot}}\left(\mathrm{Gr}_*^{\mathrm{W}}\mathcal{T}_1(k)\right) \to \mathrm{gr}_*^{\mathrm{W}}\,\mathcal{G}_{\mathrm{mot}}\big(\mathcal{T}_1(k)\big) \to \mathcal{G}_{\mathrm{mot}}\left(\mathrm{Gr}_*^{\mathrm{W}}\mathcal{T}_1(k)\right).$$

(iii). By 8.11 of [7], the fundamental group $\mathcal{G}_{\mathrm{mot}}\big(\mathcal{T}_1(k)\big)$ represents the functor $\underline{\mathrm{Aut}}^{\otimes}(\mathrm{Id})$ which associates to each $\mathcal{T}_1(k)$-scheme $\mathrm{Sp}\,(B)$ the group of automorphisms of $\otimes$-functors of the functor

$$\begin{array}{rcl}\mathrm{Id}_{\mathrm{Sp}\,(B)} : \mathcal{T}_1(k) & \longrightarrow & \{\text{modules over } \mathrm{Sp}\,(B)\} \\ X & \longmapsto & X \otimes B.\end{array}$$

Hence if g is an element of $\mathcal{G}_{\mathrm{mot}}\big(\mathcal{T}_1(k)\big)\big(\mathrm{Sp}\,B\big) = \underline{\mathrm{Aut}}^{\otimes}(\mathrm{Id})(\mathrm{Sp}\,B)$, for each pair of objects M and N of $\mathcal{T}_1(k)$ and for each morphism $f : M \to N$ of $\mathcal{T}_1(k)$, we have the commutative diagram

$$\begin{tikzcd} M \otimes B \arrow[r, "g_M"] \arrow[d, "f\otimes id_B"'] & M \otimes B \arrow[d, "f\otimes id_B"] \\ N \otimes B \arrow[r, "g_N"] & N \otimes B. \end{tikzcd}$$

Let M and N be two objects of $\mathcal{T}_1(k)$. Since $\mathrm{Hom}_{\mathcal{T}_1(\overline{k})}\big(e(M), e(N)\big)$ is an object of $\mathrm{Rep}_{\mathbb{Q}}\,\big(\mathrm{Gal}(\overline{k}/k)\big)$, it can be regarded as an Artin motive over k. Moreover, the elements of $\mathrm{Hom}_{\mathcal{T}_1(k)}(M, N)$ are exactly the elements of $\mathrm{Hom}_{\mathcal{T}_1(\overline{k})}\big(e(M), e(N)\big)$ which are invariant under the action of $\mathrm{Gal}(\overline{k}/k)$.

Let g be an element of

$$\mathcal{G}_{\mathrm{mot}}\big(\mathcal{T}_1(k)\big)\big(\mathrm{Sp}\,B\big) = \underline{\mathrm{Aut}}^{\otimes}(\mathrm{Id})(\mathrm{Sp}\,B),$$

and let $H(g) = \tau$ be an element of

$$h\,\mathcal{GAL}(\overline{k}/k)\big(\mathrm{Sp}\,B\big) = \mathrm{Gal}(\overline{k}/k).$$

This means that g acts via τ on $\mathrm{Hom}_{\mathcal{T}_1(\overline{k})}\big(e(M), e(N)\big)$. Then for any morphism

$$f : e(M) \to e(N)$$

of $\mathcal{T}_1(\overline{k})$, we have the commutative diagram

$$\begin{tikzcd} e(M) \otimes B \arrow[r, "e(g_M)"] \arrow[d, "f\otimes id_B"'] & e(M) \otimes B \arrow[d, "\tau f\otimes id_B"] \\ e(N) \otimes B \arrow[r, "e(g_N)"] & e(N) \otimes B. \end{tikzcd} \tag{3.3}$$

Since M and N are defined over k, $e(M)$ and $e(N)$ are respectively isomorphic to $\tau e(M)$ and $\tau e(N)$ and therefore the upper line of (3.3) defines a morphism

$$e(M) \otimes B \longrightarrow \tau e(M) \otimes B$$

which is functorial in $e(M)$ and B, and which is compatible with tensor products. Moreover we have already observed that the Tannakian category $\mathcal{T}_1(\overline{k})$ is generated by motives of the form $e(M)$ with $M \in \mathcal{T}_1(k)$. We can then conclude that g defines an element of $\underline{\mathrm{Hom}}^{\otimes}(\mathrm{Id}, \mathrm{Id} \circ \tau)$, regarding Id and $\mathrm{Id} \circ \tau$ as functors on $\mathcal{T}_1(\overline{k})$. $\square$

4. Case of a 1-motive

In this section we construct the biggest Tannakian subcategory of the one generated by a 1-motive M over k, whose motivic Galois group in commutative. In order to do that, we need a more symmetric description of 1-motives: according to 10.2.14 of [4], to have a 1-motive $M = [X \xrightarrow{u} G]$ over k is equivalent to have the 7-tuple $(X, Y^\vee, A, A^*, v, v^*, \psi)$ where

- X and $Y^\vee$ are two group k-schemes, which are locally for the étale topology, constant group schemes defined by a finitely generated free $\mathbb{Z}$-module;
- A and A^* are two abelian varieties defined over k, dual to each other;
- $v : X \to A$ and $v^* : Y^\vee \to A^*$ are two morphisms of group k-schemes;
- ψ is a trivialization of the pull-back $(v, v^*)^*\mathcal{P}_A$ by (v, v^*) of the Poincaré biextension $\mathcal{P}_A$ of (A, A^*) by $\mathbb{G}_m$.

The Lie algebra $\mathrm{Lie}\,\mathcal{G}_{\mathrm{mot}}(M)$ of the motivic Galois group of M is an object of the Tannakian category $\langle M\rangle^{\otimes}$ generated by M which is endowed with a structure of Lie algebra, *i.e.*, a skew symmetric application

$$[-,-] : \mathrm{Lie}\,\mathcal{G}_{\mathrm{mot}}(M) \otimes \mathrm{Lie}\,\mathcal{G}_{\mathrm{mot}}(M) \to \mathrm{Lie}\,\mathcal{G}_{\mathrm{mot}}(M)$$

satisfying the Jacobi identity. If $\mathcal{G}_{\mathrm{mot}}(M)$ is $\mathrm{Sp}(\lambda)$, then $\mathrm{Lie}\,\mathcal{G}_{\mathrm{mot}}(M)$ is the dual of I/I^2, where I is the augmentation ideal of the Hopf algebra Λ. The main result of [2] is that the unipotent radical of the Lie algebra of $\mathcal{G}_{\mathrm{mot}}(M)$ is the semi-abelian variety defined by the adjoint action of the Lie algebra

$$\Big(\mathrm{Gr}^{\mathrm{W}}_*\big(\mathrm{W}_{-1}\mathrm{Lie}\,\mathcal{G}_{\mathrm{mot}}(M)\big),\quad [-,-]\Big)$$

on itself. Recall that according to 10.1.3 of [4], the functor "Hodge realization" from the category of 1-motives defined over $\overline{k}$ to the category of $\mathbb{Q}$-mixed Hodge structures H of type $\{(0,0),(-1,0),(0,-1),(-1,-1)\}$ and with the quotient $\mathrm{Gr}^{\mathrm{W}}_{-1}(\mathrm{H})$ polarizable, is fully faithful. The abelian variety B and the torus $Z(1)$ underlying this semi-abelian variety can be computed explicitly.

We recall here briefly their construction: The motive $E = \mathrm{W}_{-1}\big(\underline{\mathrm{End}}(\mathrm{Gr}^{\mathrm{W}}_* M)\big)$ is the direct sum of the abelian variety $E_1 = A \otimes X^\vee + A^* \otimes Y$, which is the component of E of weight -1, and of the torus $E_2 = X^\vee \otimes Y(1)$, which is the component of E of weight -2. It is endowed with a Lie bracket

$$[-,-] : E \otimes E \to E$$

whose only non trivial component is

$$[-,-] : E_{-1} \otimes E_{-1} \longrightarrow E_{-2}.$$

According to Corollary 2.7 of [2], this Lie bracket corresponds to a $\Sigma - X^\vee \otimes Y(1)$-torsor $\mathcal{B}$ living over $A \otimes X^\vee + A^* \otimes Y$ (see 2.2 of [2] for the definition of a Σ-torsor.) As proved in 3.3 of [2], the 1-motives $\mathrm{Gr}^{\mathrm{W}}_* M$ and $\mathrm{Gr}^{\mathrm{W}}_* M^\vee$ are Lie $\big(E, [-,-]\big)$-modules.

In particular, E acts on the components $\mathrm{Gr}_0^{\mathrm{W}} M$ and $\mathrm{Gr}_0^{\mathrm{W}} M^\vee$ through the projections:

$$\left\{ \begin{array}{lcl} \alpha & : & (X^\vee \otimes A) \otimes X \longrightarrow A, \\ \beta & : & (A^* \otimes Y) \otimes Y^\vee \longrightarrow A^*, \\ \gamma & : & \big(X^\vee \otimes Y(1)\big) \otimes X \longrightarrow Y(1). \end{array} \right. \tag{4.1}$$

Denote by $b = (b_1, b_2)$ the k-rational point of the abelian variety $A \otimes X^\vee + A^* \otimes Y$ defining the morphisms

$$v : X \to A \quad \text{and} \quad v^* : Y^\vee \to A^*.$$

Let B be the smallest abelian subvariety (modulo isogenies) of $X^\vee \otimes A + A^* \otimes Y$ containing the point

$$b = (b_1, b_2) \in (X^\vee \otimes A)(k) + (A^* \otimes Y)(k).$$

The restriction $i^*\mathcal{B}$ of the $\Sigma - X^\vee \otimes Y(1)$-torsor $\mathcal{B}$ by the inclusion

$$i : B \to X^\vee \otimes A + A^* \otimes Y$$

is a $\Sigma - X^\vee \otimes Y(1)$-torsor over B. Denote by Z_1 the smallest subgroup of $X^\vee \otimes Y$ such that the torus $Z_1(1)$, that it defines, contains the image of the Lie bracket

$$[-,-] : B \otimes B \longrightarrow X^\vee \otimes Y(1).$$

The direct image $p_* i^* \mathcal{B}$ of the $\Sigma - X^\vee \otimes Y(1)$-torsor $i^*\mathcal{B}$ by the projection

$$p : X^\vee \otimes Y(1) \longrightarrow \big(X^\vee \otimes Y/Z_1\big)(1)$$

is a trivial $\Sigma - \big(X^\vee \otimes Y/Z_1\big)(1)$-torsor over B. We denote by

$$\pi : p_* i^* \mathcal{B} \longrightarrow \big(X^\vee \otimes Y/Z_1\big)(1)$$

the canonical projection. By 3.6 of [2], the morphism $u : X \to G$ defines a point $\widetilde{b}$ in the fibre of $\mathcal{B}$ over b. We denote again by $\widetilde{b}$ the points of $i^*\mathcal{B}$ and of $p_* i^* \mathcal{B}$ over the point b of B. Let Z be the smallest subgroup of $X^\vee \otimes Y$, containing Z_1 and such that the subtorus $\big(Z/Z_1\big)(1)$ of $(X^\vee \otimes Y/Z_1)(1)$ contains $\pi(\widetilde{b})$. If we put $Z_2 = Z/Z_1$, we have $Z(1) = Z_1(1) \times Z_2(1)$.

With these notations, the unipotent radical $\mathrm{W}_{-1}\big(\mathrm{Lie}\, \mathcal{G}_{\mathrm{mot}}(M)\big)$ of the Lie algebra of $\mathcal{G}_{\mathrm{mot}}(M)$ is the extension of the abelian variety B by the torus $Z(1)$, defined by the adjoint action of $\big(B + Z(1), [-,-]\big)$ on itself.

Remark 4.1. If the morphism of group k-schemes $v : X \to A$ which appears in the definition of

$$M \;=\; [X \stackrel{u}{\longrightarrow} G] \;=\; \big(X, Y^\vee,\, A,\, A^*,\, v,\, v^*,\, \psi\big)$$

is trivial, then M is isogeneous to the 1-motive

$$\big[X \stackrel{u}{\longrightarrow} Y(1)\big] \oplus [0 \longrightarrow G].$$

In this case, the unipotent radical $\mathrm{W}_{-1}\big(\mathcal{G}_{\mathrm{mot}}(M)\big)$ is the direct sum

$$\mathrm{W}_{-2}\big(\mathcal{G}_{\mathrm{mot}}(M)\big) \oplus \mathrm{W}_{-1}\big(\mathcal{G}_{\mathrm{mot}}(\mathrm{W}_{-1} M)\big)$$

and therefore $\mathcal{G}_{\mathrm{mot}}(M)$ is abelian; with the above notations, we have $b_1 = Z_1(1) = 0$ and

$$\mathrm{W}_{-1}\big(\mathrm{Lie}\, \mathcal{G}_{\mathrm{mot}}(M)\big) = B + Z_2(1).$$

If also the morphism $v^* : Y^\vee \longrightarrow A^*$ is trivial, M is isogeneous to the 1-motive

$$[X \xrightarrow{u} Y(1)] \oplus [0 \longrightarrow A].$$

Therefore the unipotent radical $\mathrm{W}_{-1}\big(\mathcal{G}_{\mathrm{mot}}(M)\big)$ is $\mathrm{W}_{-2}\big(\mathcal{G}_{\mathrm{mot}}(M)\big)$ and $\mathcal{G}_{\mathrm{mot}}(M)$ is clearly abelian; with the above notations, we have $b_1 = b_2 = B = Z_1(1) = 0$ and

$$\mathrm{W}_{-1}\big(\mathrm{Lie}\,\mathcal{G}_{\mathrm{mot}}(M)\big) \;=\; \mathrm{W}_{-2}\big(\mathrm{Lie}\,\mathcal{G}_{\mathrm{mot}}(M)\big) \;=\; Z_2(1).$$

Proposition 4.2. *The derived group of the unipotent radical* $\mathrm{W}_{-1}\big(\mathrm{Lie}\,\mathcal{G}_{\mathrm{mot}}(M)\big)$ *of the Lie algebra of* $\mathcal{G}_{\mathrm{mot}}(M)$ *is the torus* $Z_1(1)$.

Proof. The only non trivial component of the bracket $[-,-] : E \otimes E \longrightarrow E$ is

$$[-,-] : E_{-1} \otimes E_{-1} \longrightarrow E_{-2}.$$

Therefore we only have to consider the commutators $[-,-]$ of the elements of $(B + Z(1)) \cap E_{-1} = B$, and such commutators live by definition in the torus $Z_1(1)$. $\square$

Let $\{e_i\}_i$ and $\{f_j^*\}_j$ be basis of $X(\overline{k})$ and $Y^\vee(\overline{k})$ respectively. Choose a point P of $\big[B \cap \big(X^\vee \otimes A + \{0\}\big)\big](k)$ and a point Q of $\big[B \cap \big(\{0\} + A^* \otimes Y\big)\big](k)$ such that the abelian subvariety they generate in $X^\vee \otimes A + A^* \otimes Y$ is isogeneous to B. Denote by

$$\overline{v} : X(\overline{k}) \to A(\overline{k}) \quad \text{and} \quad \overline{v}^* : Y^\vee(\overline{k}) \to A^*(\overline{k})$$

the $\mathrm{Gal}(\overline{k}/k)$-equivariant homomorphisms defined by

$$\overline{v}(e_i) \;=\; \alpha(P, e_i) \quad \text{and} \quad \overline{v}^*(f_j^*) \;=\; \beta(Q, f_j^*),$$

respectively, where α and β are the projections introduced in (4.1). Moreover choose a point $\vec{q} = (q_1, \dots, q_{\mathrm{rk}\,Z_2})$ of $Z_2(1)(k)$ such that the points $q_1, \dots, q_{\mathrm{rk}\,Z_2}$ are multiplicative independent. It is possible to find such a $\vec{q}$ defined over k because of the construction of the torus $Z_2(1)$: in fact, since M is defined over k, the point $\widetilde{b}$ corresponding to the morphism $u : X \to G$, is k-rational and therefore also $\pi(\widetilde{b})$ is k-rational. So the torus $Z_2(1)$ contains at least a k-rational point.

Let

$$\Gamma : Z(1)(\overline{k}) \otimes (X \otimes Y^\vee)(\overline{k}) \longrightarrow \mathbb{Z}(1)(\overline{k})$$

be the $\mathrm{Gal}(\overline{k}/k)$-equivariant homomorphism obtained from the map

$$\gamma : \big(X^\vee \otimes Y(1)\big) \otimes X \longrightarrow Y(1),$$

and denote by $\overline{\psi} : X \otimes Y^\vee(\overline{k}) \longrightarrow \mathbb{Z}(1)(\overline{k})$ the $\mathrm{Gal}(\overline{k}/k)$-equivariant homomorphism defined by

$$\overline{\psi}(e_i, f_j^*) \;=\; \Gamma\left([P,Q],\, \vec{q},\, e_i,\, f_j^*\right). \tag{4.2}$$

The homomorphisms $\overline{v}$, $\overline{v}^*$ and $\overline{\psi}$ define a 1-motive $\big(X, Y^\vee, A, A^*, \overline{v}, \overline{v}^*, \overline{\psi}\big)$ which is a $\otimes$-generator of the Tannakian category generated by M, *i.e.*,

$$\langle M \rangle^{\otimes} \;\cong\; \big\langle \big(X, Y^\vee, A, A^*, \overline{v}, \overline{v}^*, \overline{\psi}\big) \big\rangle^{\otimes} \tag{4.3}$$

(see the proof of Theorem 3.8 of [2]). If we denote by $\overline{v}_i, \overline{v}_j^*$ and $\overline{\psi}_{i,j}$ the $\mathrm{Gal}(\overline{k}/k)$-equivariant homomorphisms obtained by restricting respectively $\overline{v}, \overline{v}^*$ and $\overline{\psi}$ to $\mathbb{Z}e_i$ and

$\mathbb{Z}f_j^*$, by Theorem 1.7 of [1], we have the following equivalence of Tannakian categories:

$$\langle \big(X, Y^\vee, A, A^*, \overline{v}, \overline{v}^*, \overline{\psi}\big) \rangle^\otimes \cong \langle \oplus_{i,j} \big(\mathbb{Z}e_i, \mathbb{Z}f_j^*, A, A^*, \overline{v}_i, \overline{v}_j^*, \overline{\psi}_{i,j}\big) \rangle^\otimes. \tag{4.4}$$

Consider the 1-motives

$$\begin{cases} M^t & = \oplus_{i,j} \Big(\mathbb{Z}e_i, \mathbb{Z}f_j^*, 0, 0, 0, 0, \overline{\psi}_{i,j}^{ab}\Big), \\ M^a & = \oplus_{i,j} \Big(\mathbb{Z}e_i, \mathbb{Z}f_j^*, A, A^*, \overline{v}_i, \overline{v}_j^*, 0\Big), \\ M^{nab} & = \oplus_{i,j} \Big(\mathbb{Z}e_i, \mathbb{Z}f_j^*, A, A^*, \overline{v}_i, \overline{v}_j^*, \overline{\psi}_{i,j}^{nab}\Big), \\ M^{ab} & = \oplus_{i,j} \Big(\mathbb{Z}e_i, \mathbb{Z}f_j^*, A, A^*, \overline{v}_i, \overline{v}_j^*, \overline{\psi}_{i,j}^{ab}\Big), \end{cases}$$

where

$$\begin{cases} \overline{\psi}_{i,j}^{nab}(e_i, f_j^*) & = \Gamma\big([P,Q], \vec{1}, e_i, f_j^*\big), \\ \overline{\psi}_{i,j}^{ab}(e_i, f_j^*) & = \Gamma\big(\vec{1}, \vec{q}, e_i, f_j^*\big). \end{cases} \tag{4.5}$$

The 1-motive M^a (resp. M^t) is without toric part (resp. abelian part) and its motivic Galois group is clearly commutative. We prove now that the Tannakian category $\langle M^{ab}\rangle^\otimes$ $\big($resp. $\langle M^{nab}\rangle^\otimes$) is the biggest subcategory of $\langle M\rangle^\otimes$ whose motivic Galois group is commutative (resp. non commutative$\big)$.

Lemma 4.3. *The Tannakian category generated by M is equivalent to the Tannakian category generated by the 1-motive $M^t \oplus M^{nab}$. Moreover, the 1-motives M^{ab} and $M^a \oplus M^t$ generate the same Tannakian category.*

Proof. Through the projection

$$p : X^\vee \otimes Y(1) \longrightarrow \big(X^\vee \otimes Y/Z_1\big)(1)$$

the $\Sigma - (X^\vee \otimes Y)(1)$-torsor $i^*\mathcal{B}$ becomes trivial. Confronting (4.2) and (4.5), we observe that having the trivializations $\overline{\psi}_{i,j}^{ab}$ and $\overline{\psi}_{i,j}^{nab}$ is the same thing as having the trivialization $\overline{\psi}_{i,j}$, and so the Tannakian category

$$\big\langle \oplus_{i,j} \big(\mathbb{Z}e_i, \mathbb{Z}f_j^*, A, A^*, \overline{v}_i, \overline{v}_j^*, \overline{\psi}_{i,j}\big)\big\rangle^\otimes$$

is equivalent to $\langle M^t \oplus M^{nab}\rangle^\otimes$. Via (4.3) and (4.4), we can then conclude that $\langle M\rangle^\otimes$ is equivalent to $\langle M^t \oplus M^{nab}\rangle^\otimes$.

Again because the $\Sigma - (X^\vee \otimes Y/Z_1)(1)$–torsor $p_* i^*\mathcal{B}$ is trivial, the trivialization $\overline{\psi}_{i,j}^{ab}$ is independent of the abelian part of the 1-motive M, *i.e.*, it is independent of $\overline{v}_i, \overline{v}_j^*$. Therefore $\langle M^{ab}\rangle^\otimes$ is equivalent to $\langle M^a \oplus M^t\rangle^\otimes$. $\square$

Theorem 4.4. *The Tannakian category generated by M^{ab} is the biggest Tannakian subcategory of $\langle M\rangle^{\otimes}$ whose motivic Galois group is commutative. We have the following diagram of affine group $\langle M\rangle^{\otimes}$-schemes*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Z_2(1) & \longrightarrow & \mathrm{Lie}\,\mathcal{G}_{\mathrm{mot}}(M) & \longrightarrow & \mathrm{Lie}\,\mathcal{G}_{\mathrm{mot}}(M^{nab}) & \longrightarrow & 0 \\
 & & \downarrow & & \| & & \downarrow & & \\
0 & \longrightarrow & Z(1) & \longrightarrow & \mathrm{Lie}\,\mathcal{G}_{\mathrm{mot}}(M) & \longrightarrow & \mathrm{Lie}\,\mathcal{G}_{\mathrm{mot}}(M^{a}) & \longrightarrow & 0 \\
 & & \uparrow & & \| & & \uparrow & & \\
0 & \longrightarrow & Z_1(1) & \longrightarrow & \mathrm{Lie}\,\mathcal{G}_{\mathrm{mot}}(M) & \longrightarrow & \mathrm{Lie}\,\mathcal{G}_{\mathrm{mot}}(M^{ab}) & \longrightarrow & 0 \\
 & & \downarrow & & \| & & \downarrow & & \\
0 & \longrightarrow & B+Z_1(1) & \longrightarrow & \mathrm{Lie}\,\mathcal{G}_{\mathrm{mot}}(M) & \longrightarrow & \mathrm{Lie}\,\mathcal{G}_{\mathrm{mot}}(M^{t}) & \longrightarrow & 0
\end{array}$$

where all horizontal short sequences are exact and where the vertical arrows on the left are inclusions and those on the right are surjections.

Proof. By 3.10 of [2], the Lie algebra $\mathrm{W}_{-2}\mathrm{Lie}\,\mathcal{G}_{\mathrm{mot}}(M)$ is the torus $Z(1)$. Moreover by construction, the 1-motive without toric part M^a generates the same Tannakian category as the 1-motive $\mathrm{W}_0/\mathrm{W}_{-2}M + \mathrm{W}_{-1}M$. Hence, thanks to Remark 3.5, we obtain the second horizontal short exact sequence.

From the definition of the 1-motives M^{nab}, M^{ab} and M^t, and from (4.5) we observe that the torus $Z_2(1)$ acts trivially on M^{nab}, the torus $Z_1(1)$ acts trivially on M^{ab} and that the motive $B + Z_1(1)$ acts trivially on M^t. In other words we have the inclusions

$$\begin{cases} Z_2(1) & \subseteq & \mathrm{Lie}\,H_{\langle M\rangle^{\otimes}}\big(\langle M^{nab}\rangle^{\otimes}\big), \\ Z_1(1) & \subseteq & \mathrm{Lie}\,H_{\langle M\rangle^{\otimes}}\big(\langle M^{ab}\rangle^{\otimes}\big), \\ B+Z_1(1) & \subseteq & \mathrm{Lie}\,H_{\langle M\rangle^{\otimes}}\big(\langle M^{t}\rangle^{\otimes}\big). \end{cases}$$

Now we prove that these inclusions are in fact identities. Since the 1-motive M^t has no abelian part, $\mathrm{Lie}\,H_{\langle M\rangle^{\otimes}}\big(\langle M^t\rangle^{\otimes}\big)$ must be contained in the Lie subalgebra of $\big(B + Z(1), [\,,\,]\big)$ defined by the morphisms $v : X \to A$ and $v^* : X^\vee \to A^*$, *i.e.*,

$$\mathrm{Lie}\,H_{\langle M\rangle^{\otimes}}\big(\langle M^t\rangle^{\otimes}\big) \subseteq B + Z_1(1).$$

According to Lemma 4.3, the 1-motives M and $M^t \oplus M^{nab}$ generate the same Tannakian category, and therefore

$$\begin{aligned} \mathrm{Lie}\,H_{\langle M\rangle^{\otimes}}\big(\langle M^{nab}\rangle^{\otimes}\big) &\subseteq B+Z(1)/\mathrm{Lie}\,H_{\langle M\rangle^{\otimes}}\big(\langle M^{t}\rangle^{\otimes}\big) \\ &= Z_2(1). \end{aligned}$$

Again, by Lemma 4.3, the 1-motives M^a and M^t generate Tannakian subcategories of $\langle M^{ab}\rangle^{\otimes}$, and so Lemma 2.1 implies that

$$\begin{aligned} \mathrm{Lie}\,H_{\langle M\rangle^{\otimes}}\big(\langle M^{ab}\rangle^{\otimes}\big) &\subseteq \mathrm{Lie}\,H_{\langle M\rangle^{\otimes}}\big(\langle M^{a}\rangle^{\otimes}\big) \cap \mathrm{Lie}\,H_{\langle M\rangle^{\otimes}}\big(\langle M^{t}\rangle^{\otimes}\big) \\ &= Z(1) \cap \big(B+Z(1)\big) = Z(1). \end{aligned}$$

This finishes the proof of the first, the third and the fourth short exact sequence.

According to Lemma 4.3, the 1-motives M^a and M^t generate Tannakian subcategories of $\langle M^{ab}\rangle^{\otimes}$. By construction the 1-motive M^a generates a Tannakian subcategory of $\langle M^{nab}\rangle^{\otimes}$. Hence in order to prove that the left vertical arrows are inclusions and that the right vertical arrows are surjections, it is enough to apply Lemma 2.1.

The third exact sequence of the above diagram implies that the motivic Galois group of M^{ab} is isomorphic to the quotient $\mathrm{Lie}\,\mathcal{G}_{\mathrm{mot}}(M)/Z_1(1)$. But, according to Proposition 4.2, $Z_1(1)$ is the derived group of $\mathrm{Lie}\,\mathcal{G}_{\mathrm{mot}}(M)$ and hence we can conclude that $\langle M^{ab}\rangle^{\otimes}$ is the biggest Tannakian subcategory of $\langle M\rangle^{\otimes}$ whose motivic Galois group is commutative. $\square$

Remark 4.5. Among the non degenerate 1-motives, the 1-motive M^{nab} is the one which generates the biggest Tannakian subcategory of $\langle M\rangle^{\otimes}$, whose motivic Galois group is non commutative. Recall that a 1-motive is said to be non degenerate if the dimension of the Lie algebra $\mathrm{W}_{-1}\mathrm{Lie}\,\mathcal{G}_{\mathrm{mot}}(M)$ is maximal (*cf.* 2.3 of [1]).

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REFERENCES

[1] C. Bertolin, *The Mumford-Tate group of 1-motives,* Ann. Inst. Fourier (Grenoble) **52** (2002), no. 4, 1041–1059.

[2] C. Bertolin, *Le radical unipotent du groupe de Galois motivique d'un 1-motif,* Math. Ann. **327** (2003), no. 3, 585–607.

[3] S. Bosch, W. Lütkebohmert, M. Raynaud, *Néron models*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], 21, Springer-Verlag, Berlin, 1990, x+325 pp.

[4] P. Deligne, *Théorie de Hodge III*, Inst. Hautes Études Sci. Publ. Math. **44** (1974), 5–77.

[5] P. Deligne and J.S. Milne, *Tannakian categories*, in “Hodge cycles, motives, and Shimura varieties”, Lecture Notes in Mathematics **900**, Springer-Verlag, Berlin-New York, 1982.

[6] P. Deligne, *Le groupe fondamental de la droite projective moins trois points*, Galois groups over $\mathbb{Q}$ (Berkeley, CA, 1987), 79–297, Math. Sci. Res. Inst. Publ. **16** Springer, New York, 1989.

[7] P. Deligne, *Catégories tannakiennes*, The Grothendieck Festschrift, Vol. II, 111–195, Progr. Math. **87** Birkhäuser Boston, Boston, MA, 1990.

[8] A. Grothendieck *et al.*, *Groupes de monodromie en géométrie algébrique*, Séminaire de géométrie algébrique du Bois-Marie 1967–1969 (SGA 7 I), Lecture Notes in Mathematics, **288** Springer-Verlag, Berlin-New York, 1972, viii+523 pp.

[9] U. Jannsen, *Mixed motives and algebraic K-theory*, Lecture Notes in Mathematics **1400** Springer-Verlag, Berlin, 1990.

[10] J.S. Milne, *Motives over finite fields*, in “Motives” (Seattle, WA, 1991), 401–459, Proc. Sympos. Pure Math. **55** Part 1, Amer. Math. Soc., Providence, RI, 1994.

[11] N. Saavedra Rivano, *Catégories Tannakiennes*, Lecture Notes in Mathematics **265** Springer-Verlag, Berlin-New York, 1972.

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