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SIMPLE FAMILIES OF THUE INEQUALITIES

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RÉSUMÉ. Nous résolvons des familles simples d'inégalités de Thue de degré 3, 4 et 6, dont chacune a deux paramètres entiers. Nous obtenons des bornes supérieures pour les solutions suivant la méthode de G. Lettl, A. Pethő, et P. Voutier pour des familles avec un paramètre. Ensuite nous spécifions la partie de droite de chaque inégalité, et nous déterminons toutes les solutions sous certaines conditions pour les paramètres. Pour cette seconde partie, nous avons besoin d'obtenir des bornes inférieures pour les solutions, et pour cela nous utilisons des développements en fractions continues avec quotients partiels rationnels et une généralisation d'un théorème de Legendre.

ABSTRACT. We solve some simple families of Thue inequalities of degree 3, 4 and 6, each of which is parameterized by two integral parameters. We obtain upper bounds for the solutions by following the method given by G. Lettl, A. Pethő, and P. Voutier for families with one parameter. Next we specify the right-hand side of each inequality, and determine all solutions under certain assumptions on the parameters. For this part, we need to obtain lower bounds for the solutions, and for this we use continued fraction expansions with rational partial quotients and a generalization of Legendre's theorem.

1. Introduction

In the present paper we solve the following three "simple" families of Thue inequalities, each of which is parameterized by two integral parameters s and t satisfying certain assumptions:

$$\left|F_{s,t}^{(j)}(x,y)\right| \leq k, \tag{1.1}$$

where $j = 3, 4, 6,\ \ k \in \mathbb{N}$, and

$$\begin{cases} F_{s,t}^{(3)}(X,Y) &= sX^3 - tX^2Y - (t+3s)XY^2 - sY^3, \\ F_{s,t}^{(4)}(X,Y) &= sX^4 - tX^3Y - 6sX^2Y^2 + tXY^3 + sY^4, \\ F_{s,t}^{(6)}(X,Y) &= sX^6 - 2tX^5Y - (5t+15s)X^4Y^2 \\ & \quad -20sX^3Y^3 + 5tX^2Y^4 + (2t+6s)XY^5 + sY^6. \end{cases}$$

The case $s = 1$, that is the case where each of these families is parameterized by one parameter t was treated by many authors. For the cubic case, see Thomas

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[8], Mignotte [6], Mignotte–Pethő–Lemmermeyer [7], Chen [1], and Xia–Chen–Zhang [10]. For the quartic simple family, see Lettl–Pethő [4] and Chen–Voutier [2]. In 1999, Lettl–Pethő–Voutier [5] treated all the three simple families in the case $s = 1$. They used the Padé approximation method, realizing the idea of Chudnovsky [3], that is, they deleted as many as possible common divisors of the coefficients of the hypergeometric polynomials used in the Padé approximation method, and they obtained good upper bounds for the solutions.

In the following, a *primitive solution* means a solution (x, y) of Thue inequality such that $\gcd(x, y) = 1$. We exclude (0,0) from the primitive solutions.

The aim of the present paper is to determine all the solutions of the above simple families of Thue inequalities with two parameters for some specified k; see Theorems 2.4, 2.8 and 2.11. For this aim, we shall proceed as follows.

First, we obtain upper bounds for the solutions. For this, we follow the proof given in [5], since the upper bounds obtained are the known best ones. Of course, we might use the usual Padé approximation method. The estimates obtained by this method would be slightly weaker, but simpler than those obtained with [5]. Actually, the estimates obtained by the Padé approximation method would also work for our purpose.

Secondly, we prove that there are no solutions between the upper bounds thus obtained and the maximum of the small solutions, which can be found easily. The usual method for this second part is to use normal continued fraction expansions of related algebraic numbers. When the families of the Thue inequalities are parameterized by one parameter, this method works well. However, when the families are parameterized by two parameters, it is difficult to calculate normal continued fraction expansions. Therefore, in this second step, we use "generalized continued fraction expansions", that is, continued fraction expansions with rational partial quotients, and also a generalization of a classical theorem of Legendre. This method was developed in our former paper [9] in 2002. This method is useful, and we think that it is even indispensable at present when the families of Thue inequalities are parameterized by two parameters.

2. Results

Our results are the following.

Theorem 2.1 (Sextic case). *Let $s \geq 1$ and $t \geq 90.5s^{5/2}$, and let θ be the zero of $F^{(6)}_{s,t}(X, 1)$ such that $1/2 < \theta < 1$. Then, for any integers p and q with $q \geq \dfrac{t + \frac{3}{2}s}{2.88s}$, we have*

$$\left|\theta - \frac{p}{q}\right| > \frac{1}{c(s,t)q^{\kappa+1}},$$

where

$$\kappa = \frac{\log\left(\sqrt{t^2 + 3st + 9s^2}\right) + 2.56}{\log\left(t + \frac{3}{2}s\right) - 2\log s - 3.09} < 5$$

and

$$c(s,t) = \frac{62.1\sqrt{t^2+3st+9s^2}}{(7.63s)^{\kappa}}.$$

This implies immediately the following.

Theorem 2.2 (Sextic case). *Let $s \geq 1$ and $t \geq 90.5s^{5/2}$. Then, for any primitive solution (x,y) of (1.1), with $1/2 < x/y \leq 4$ and $y \geq \max\left\{\frac{t+\frac{3}{2}s}{2.88s}, \sqrt[6]{\frac{tk}{32.6s^2}}\right\}$, we have*

$$y^{5-\kappa} < \frac{3.6k}{(7.63s)^{\kappa}},$$

with the same κ as in Theorem 2.1.

We specify k in (1.1), and we suppose t is slightly bigger than what is assumed in Theorem 2.1 so that we have $\kappa < 24/5$. Under this hypothesis, we shall determine all primitive solutions.

Theorem 2.3 (Sextic case). *Let $s \geq 1$ and $t \geq 97.3s^{48/19}$. Then the constant κ of Theorem 2.1 is smaller than $24/5$, and the only primitive solutions $(x,y) \in \mathbb{Z}^2$ of*

$$\left|F^{(6)}_{s,t}(x,y)\right| \leq 27(120t+323s), \tag{2.1}$$

with $1/2 < x/y \leq 4$ and $y > 0$, are $(1,1),\ (2,1),\ (3,1),\ (4,1),\ (3,2)$ and $(2,3)$.

Theorem 2.4 (Sextic case). *Let $s \geq 1$ and $t \geq 97.3s^{48/19}$. Then the only primitive solutions $(x,y) \in \mathbb{Z}^2$ of*

$$\left|F^{(6)}_{s,t}(x,y)\right| \leq 120t+323s, \tag{2.2}$$

with $y \geq 0$, are $(\pm1,0),\ (0,1),\ (\pm1,1),\ (\pm2,1),\ (-3,1),\ (\pm1,2),\ (-3,2),\ (-1,3)$ and $(-2,3)$.

Theorem 2.5 (Quartic case). *Let $s \geq 1$ and $t \geq 57.5s^3$, and let θ be the zero of $F^{(4)}_{s,t}(X,1)$ such that $0 < \theta < 1$. Then, for any integers p and q with $q \geq \frac{t}{3.2s}$, we have*

$$\left|\theta - \frac{p}{q}\right| > \frac{1}{c(s,t)q^{\kappa+1}},$$

where

$$\kappa = \frac{\log\left(\sqrt{t^2+16s^2}\right)+0.99}{\log t - 2\log s - 2.37} < 3$$

and

$$c(s,t) = \frac{6.32\sqrt{t^2+16s^2}}{(3.34s)^{\kappa}}.$$

Theorem 2.6 (Quartic case). *Let $s \geq 1$ and $t \geq 57.5s^3$. Then, for any primitive solution (x,y) of (1.1), with $1/2 < x/y \leq 3$ and $y \geq \max\left\{\frac{t}{3.2s}, \sqrt[4]{\frac{tk}{4s^2}}\right\}$, we have*

$$y^{3-\kappa} < \frac{3.29k}{(3.34s)^{\kappa}}$$

with the same κ as in Theorem 2.5.

Theorem 2.7 (Quartic case). *Let $s \geq 1$ and $t \geq 70s^{28/9}$. Then the constant κ of Theorem 2.5 is smaller than $14/5$, and the only primitive solutions $(x,y) \in \mathbb{Z}^2$ of*

$$\left|F_{s,t}^{(4)}(x,y)\right| \leq 4(6t+7s), \tag{2.3}$$

with $1/2 < x/y \leq 3$ and $y > 0$, are $(1,1),\ (2,1)$ and $(3,1)$.

Theorem 2.8 (Quartic case). *Let $s \geq 1$ and $t \geq 70s^{28/9}$. Then the only primitive solutions $(x,y) \in \mathbb{Z}^2$ of*

$$\left|F_{s,t}^{(4)}(x,y)\right| \leq 6t+7s,$$

with $y \geq 0$, are $(\pm1,0),\ (0,1),\ (\pm1,1),\ (\pm2,1)$ and $(\pm1,2)$.

Theorem 2.9 (Cubic case). *Let $s \geq 1$ and $t \geq 31s^4$, and let θ be the zero of $F_{s,t}^{(3)}(X,1)$ such that $-1 < \theta < 0$. Then, for any integers p and q with $q \geq \dfrac{t+\frac{3}{2}s}{9.04}$, we have*

$$\left|\theta - \frac{p}{q}\right| > \frac{1}{c(s,t)q^{\kappa+1}},$$

where

$$\kappa = \frac{\log\left(\sqrt{t^2+3ts+9s^2}\right)+0.83}{\log\left(t+\frac{3}{2}s\right) - 2\log s - 1.3} < 2$$

and

$$c(s,t) = 17.43\sqrt{t^2+3ts+9s^2}\left(\frac{2.47}{s^2}\right)^{\kappa}.$$

Theorem 2.10 (Cubic case). *Let $s \geq 1$ and $t \geq 31s^4$. Then, for any primitive solution (x,y) of (1.1), with $-1/2 < x/y \leq 1$ and $y \geq \max\left\{\dfrac{t+\frac{3}{2}s}{9.04},\ \sqrt[3]{\dfrac{tk}{1.99s^2}}\right\}$, we have*

$$y^{2-\kappa} < 18.34\left(\frac{2.47}{s^2}\right)^{\kappa} k$$

with the same κ as in Theorem 2.9.

Theorem 2.11 (Cubic case). *Let $s \geq 1$ and $t \geq 64s^{9/2}$. Then the constant κ of Theorem 2.9 is smaller than $9/5$, and the only primitive solutions $(x,y) \in \mathbb{Z}^2$ of*

$$\left|F_{s,t}^{(3)}(x,y)\right| \leq 2t+3s, \tag{2.4}$$

with $-1/2 < x/y \leq 1$ and $y > 0$, are

$$\begin{cases} (0,1),\ (1,1)\ (-1,t+2) & \text{if } s=1, \\ (0,1),\ (1,1) & \text{if } s \geq 2. \end{cases}$$

Further, the only primitive solutions $(x,y) \in \mathbb{Z}^2$, with $y \geq 0$, are

$$\begin{cases} (1,0),\ (0,1),\ (\pm1,1),\ (-2,1),\ (-1,2), & \\ (-1,t+2),\ (-t-2,t+1),\ (t+1,1) & \text{if } s=1, \\ (1,0),\ (0,1),\ (\pm1,1),\ (-2,1),\ (-1,2) & \text{if } s \geq 2. \end{cases}$$

For comparison, we recall in the rest of this section the results of Lettl–Pethő–Voutier [5].

Theorem 2.12 (Sextic case). *Let $s = 1$ and $t \geq 89$. Then, for any primitive solution (x, y) of (1.1), with $0 < x/y < 1$ and $y \geq 1.572\sqrt[4]{k/t}$, we have*

$$y^{5-\kappa} < \frac{3.57k}{7.5^{\kappa}},$$

where

$$\kappa = \frac{\log\left(\sqrt{t^2+3t+9}\right) + 2.56}{\log\left(t+\frac{3}{2}\right) - 3.09} < 5.$$

Corollary 2.1 (Sextic case). *For $t \geq 89$, the only primitive solutions $(x, y) \in \mathbb{Z}^2$ of*

$$\left|F_{1,t}^{(6)}(x, y)\right| \leq 120t + 323,$$

with $-y/2 < x \leq y$, are $(0, 1),\ (1, 1),\ (1, 2)$ and $(-1, 3)$.

Theorem 2.13 (Quartic case). *Let $s = 1$ and $t \geq 58$. Then, for any primitive solution (x, y) of (1.1), with $0 < x/y < 1$ and $y \geq 4\sqrt{\frac{k}{2t-1}}$, we have*

$$y^{3-\kappa} < \frac{3.28k}{2.97^{\kappa}},$$

where

$$\kappa = \frac{\log\left(\sqrt{t^2+16}\right) + 0.99}{\log t - 2.37} < 3.$$

Corollary 2.2 (Quartic case). *For $t \geq 58$, the only primitive solutions $(x, y) \in \mathbb{Z}^2$ of*

$$\left|F_{1,t}^{(4)}(x, y)\right| \leq 6t + 7,$$

with $|x| \leq y$, are $(0, 1),\ (\pm 1, 1)$ and $(\pm 1, 2)$.

Theorem 2.14 (Cubic case). *Let $s = 1$ and $t \geq 30$. Then, for any primitive solution (x, y) of (1.1), with $-1/2 < x/y \leq 1$ and $y \geq \frac{8k}{2t+3}$, we have*

$$y^{2-\kappa} < 17.78 \cdot 2.59^{\kappa} k,$$

where

$$\kappa = \frac{\log\left(\sqrt{t^2+3t+9}\right) + 0.83}{\log\left(t+\frac{3}{2}\right) - 1.3} < 2.$$

Corollary 2.3 (Cubic case). *For $t \geq 30$, the only primitive solutions $(x, y) \in \mathbb{Z}^2$ of*

$$\left|F_{1,t}^{(3)}(x, y)\right| \leq 2t + 3,$$

with $-y/2 < x \leq y$, are $(0, 1),\ (1, 1)$ and $(-1, t+2)$.

Note that the constants κ in Theorems 2.12, 2.13 and 2.14 are the same as for the case $s = 1$ in Theorems 2.1, 2.5 and 2.9 respectively.

3. Simple forms

Here we recall the definition of simple forms, and a result of [5] about their representatives.

For $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Q})$ and $F \in \mathbb{Q}[X,Y]$ we let

$$F^A := F(aX + bY, cX + dY) \in \mathbb{Q}[X,Y].$$

We call the forms $F, G \in \mathbb{Q}[X,Y]$ *equivalent* if there exist some $A \in GL_2(\mathbb{Q})$ and $r \in \mathbb{Q}^\times$ (that is $r \neq 0$) such that $rG = F^A$.

Definition 3.1. A form $F \in \mathbb{Q}[X,Y]$ is called *simple* if F is irreducible over $\mathbb{Q}$ with $\deg F \geq 3$ and there exists some non-trivial A in $PGL_2(\mathbb{Q}) = GL_2(\mathbb{Q})/\mathbb{Q}^\times E$ such that $\psi_A : z \to Az := \dfrac{az+b}{cz+d}$ permutes the zeros of the underlying polynomial $F(X,1)$ transitively; here E is the identity matrix of order 2, and $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Q})$ represents A.

Remark 3.1. Definition 3.1 implies that the roots of $F(X,1)$ generate a cyclic number field of degree $\deg F$, and its Galois group is generated by ψ_A.

Lettl-Pethő-Voutier [5, Lemma 1] determined the representatives of the simple forms.

Proposition 3.1 ([5])**.** *Up to equivalence, the only simple forms in $\mathbb{Q}[X,Y]$ are such forms among $F^{(3)}_{s,t}$, $F^{(4)}_{s,t}$ and $F^{(6)}_{s,t}$, with $s,t \in \mathbb{Z}$, $s \geq 1$ and $\gcd(s,t) = 1$, that are irreducible over $\mathbb{Q}$.*

Remark 3.2. The proof of Proposition 3.1 is given in [5]. At the final part of the proof, they only consider the case $s = 1$; however s can take positive integers values.

Remark 3.3. There are infinitely many forms among $F^{(j)}_{s,t}$ which are not irreducible over $\mathbb{Q}$. In the present paper, we consider all forms $F^{(j)}_{s,t}$, for $j = 3,4,6$, with $\gcd(s,t) = 1$ and $s \geq 1$, including reducible ones. Also, it may happen that two forms having different values for the parameters are equivalent.

Definition 3.2. Let $F \in \mathbb{Q}[X,Y]$ be a form. We call $A \in GL_2(\mathbb{Z})$ an *automorphism* of F if $F^A = F$.

Cubic simple forms have non-trivial automorphisms. Quartic and sextic simple forms have a slightly different but almost similar property.

Proposition 3.2. **(A)** (Sextic case) *Let $g = \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix}$. Then*

$$\left(F^{(6)}_{s,t}\right)^g = -27F^{(6)}_{s,t},$$

and $g^6 = E$ in $PGL_2(\mathbb{Q})$. Also, ψ_g permutes the zeros of $F^{(6)}_{s,t}(X,1)$ transitively, and permutes the intervals $(1/2,4]$, $(-1/5,1/2]$, $(-2/3,-1/5]$, $(-5/4,-2/3]$, $(-3,-5/4]$ and $(-\infty,-3] \cup (4,\infty)$ transitively.

In addition, let x, y *be integers with* $\gcd(x, y) = 1$.

(i) *If* $x \not\equiv y \pmod 3$, *then* $\gcd(x - y, x + 2y) = 1$ *and*

$$F_{s,t}^{(6)}(x - y, x + 2y) = -27F_{s,t}^{(6)}(x, y).$$

(ii) *If* $x \equiv y \pmod 3$, *then* $\dfrac{x - y}{3}$ *and* $\dfrac{x + 2y}{3}$ *are integers and*

$$\gcd\left(\frac{x - y}{3}, \frac{x + 2y}{3}\right) = 1.$$

Moreover,

$$F_{s,t}^{(6)}\left(\frac{x - y}{3}, \frac{x + 2y}{3}\right) = -\frac{1}{27}F_{s,t}^{(6)}(x, y).$$

(B) (Quartic case) *Let* $g = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$. *Then*

$$\left(F_{s,t}^{(4)}\right)^g = -4F_{s,t}^{(4)},$$

and $g^4 = E$ *in* $PGL_2(\mathbb{Q})$. *Also,* ψ_g *permutes the zeros of* $F_{s,t}^{(4)}(X, 1)$ *transitively, and permutes the intervals* $(1/2, 3],\ (-1/3, 1/2],\ (-2, -1/3]$ *and* $(-\infty, -2] \cup (3, \infty)$ *transitively.*

In addition, let x, y *be integers with* $\gcd(x, y) = 1$.

(i) *If* $x \not\equiv y \pmod 2$, *then* $\gcd(x - y, x + y) = 1$ *and*

$$F_{s,t}^{(4)}(x - y, x + y) = -4F_{s,t}^{(4)}(x, y).$$

(ii) *If* $x \equiv y \pmod 2$, *then* $\dfrac{x - y}{2}$ *and* $\dfrac{x + y}{2}$ *are integers and*

$$\gcd\left(\frac{x - y}{2}, \frac{x + y}{2}\right) = 1.$$

Moreover,

$$F_{s,t}^{(4)}\left(\frac{x - y}{2}, \frac{x + y}{2}\right) = -\frac{1}{4}F_{s,t}^{(4)}(x, y).$$

(C) (Cubic case) *Let* $g = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$. *Then* $\left(F_{s,t}^{(3)}\right)^g = F_{s,t}^{(3)}$, *that is* g *is an automorphism of* $F_{s,t}^{(3)}$, *and* $g^3 = E$. *Also,* ψ_g *permutes the zeros of* $F_{s,t}^{(3)}(X, 1)$ *transitively, and permutes the intervals* $(-1/2, 1]$, $(-2, -1/2]$ *and* $(-\infty, -2] \cup (1, \infty)$ *transitively.*

In addition, let x, y *be integers with* $\gcd(x, y) = 1$. *Then*

$$\gcd(y, -x - y) = 1$$

and

$$F_{s,t}^{(3)}(y, -x - y) = F_{s,t}^{(3)}(x, y).$$

In order to find all primitive solutions of the Thue inequalities (1.1), we consider the Thue inequalities

$$\left|F_{s,t}^{(6)}(x, y)\right| \le 27k, \tag{3.1}$$

$$\left|F_{s,t}^{(4)}(x,y)\right| \le 4k$$

and

$$\left|F_{s,t}^{(3)}(x,y)\right| \le k.$$

From Proposition 3.2, it is enough to find all primitive solutions (x, y) of each of these inequalities such that x/y belongs to one fixed interval, say I_1, among the intervals listed in the proposition. Indeed, for the sextic case for example, the images of the primitive solutions of (3.1) in I_1 by the linear transformation g^i, for $i = 1, 2, \ldots, 5$, satisfy $\left|F_{s,t}^{(6)}(x,y)\right| \le 27^2 k$, and all primitive solutions of $\left|F_{s,t}^{(6)}(x,y)\right| \le k$ can be obtained in this way; namely, all primitive solutions of $\left|F_{s,t}^{(6)}(x,y)\right| \le k$ can be found among these images by g^i or the images divided by the common factor of their coordinates. For the quartic and cubic cases, the situation is similar. In addition, we may assume that $y > 0$. Therefore, below we assume the following.

Assumption 3.1. For the solutions (x, y) of our Thue inequalities, we suppose that $\gcd(x, y) = 1,\ y > 0$, and

$$\begin{cases} 1/2 < x/y \le 4 & \text{for the sextic case,} \\ 1/2 < x/y \le 3 & \text{for the quartic case,} \\ -1/2 < x/y \le 1 & \text{for the cubic case.} \end{cases}$$

In order to solve our Thue inequalities under Assumption 3.1, we need to obtain an irrationality measure for the corresponding zero of $F_{s,t}^{(j)}(X, 1)$, for $j = 3, 4, 6$. Hence we assume that $t \ge 6s$, and we denote by θ the zero of $F_{s,t}^{(j)}(X, 1)$ with

$$\begin{cases} 1/2 < \theta < 1 & \text{for } j = 6, \\ 0 < \theta < 1 & \text{for } j = 4, \\ -1 < \theta < 0 & \text{for } j = 3. \end{cases} \tag{3.2}$$

Note that under the assumption $t \ge 6s$, the zeros of $F_{s,t}^{(j)}(X, 1)$ lie one by one in each interval listed in Proposition 3.2; see [5, Lemma 3, Lemma 9, Lemma 10].

4. Irrationality measure

In order to prove Theorems 2.1, 2.2, 2.5, 2.6, 2.9 and 2.10, we use the method developed in [5]. In [5], Lettl, Pethő and Voutier treated the case where $s = 1$ and t is an integer greater than a certain quantity. In our case, s is an arbitrary positive integer and t is a large integer in comparison with s. We replace t in the proof of [5] by t/s, and follow their proof. Asking the readers to consult [5], we will just give some remarks about the proofs of our results.

Proof of Theorem 2.1 (Sextic case).

(1) We assume

$$q \ge \frac{1}{2\ell_0} = \frac{t + \frac{3}{2}s}{2.88s},$$

with ℓ_0 so that the folklore lemma in the Padé approximation method (see, for instance, [5, Lemma 7]) works.

(2) Since θ is the β_2 of [5], we choose $X = 1$ ([5, p.1884]) to get a good rational approximation for θ.

(3) We use the estimate $\arg \boldsymbol{w}(1) < \dfrac{3\sqrt{3}}{t+\frac{3}{2}s}$ without relaxing the right-hand side to $\dfrac{3\sqrt{3}}{t}$ as done in [5, (21)].

(4) For $r \in \mathbb{N}$, we obtain integers p_r and q_r satisfying

$$(4.1) \qquad |q_r| < k_0 Q^r, \quad |\theta q_r - p_r| \le \ell_0 E^{-r} \quad \text{and} \quad p_r q_{r+1} \neq p_{r+1} q_r$$

with

$$k_0 = 2.4, \quad Q = e^{2.56}\sqrt{t^2+3ts+9s^2}, \quad \ell_0 = \frac{1.44s}{t+\frac{3}{2}s} \quad \text{and} \quad E = \frac{t+\frac{3}{2}s}{e^{3.09}s^2},$$

where we use the notation in [5, Lemma 7]. From this lemma, the value of κ is given by $\kappa = \dfrac{\log Q}{\log E}$ and, if $t \ge 90.5s^{5/2}$, then $\kappa < 5$. We also have $c(s,t) = 2k_0 Q(2\ell_0 E)^\kappa$, and we obtain Theorem 2.1. □

Proof of Theorem 2.2 (Sextic case).

We assume $t \ge 90.5s^{5/2}$. Using the estimate given in [5, Lemma 3 (a)] for the intervals containing the zeros of $F^{(6)}_{1,t}(X,1)$, we obtain the same estimates for the intervals containing the zeros of $F^{(6)}_{s,t}(X,1)$, just by replacing t by t/s. The assumption

$$y \ge \sqrt[6]{\frac{tk}{32.6s^2}}$$

implies that the primitive solutions (x,y) of (1.1), with $1/2 < x/y \le 4$, in fact satisfy $1 - \dfrac{3s}{2t} < x/y < 1$. Using this estimate, we obtain

$$\left|\theta - \frac{x}{y}\right| < \frac{k}{(18t-39s)y^6}.$$

From this inequality and Theorem 2.1, we easily obtain Theorem 2.2. □

Proof of Theorem 2.5 (Quartic case).

(1) We assume $q \ge \dfrac{t}{3.2s}$ so that the folklore lemma works.

(2) Since θ is the β_2 of [5], we choose $X = 1$ ([5, p.1889]) to get a good rational approximation for θ.

(3) For $r \in \mathbb{N}$, we obtain integers p_r and q_r satisfying (4.1) with

$$k_0 = 0.83\sqrt{2}, \quad Q = e^{0.99}\sqrt{t^2+16s^2}, \quad \ell_0 = \frac{1.6s}{t} \quad \text{and} \quad E = \frac{t}{e^{2.37}s^2}.$$

Here we obtain the more precise value 1.6 instead of 1.8 in [5, p.1890, line 11]. We obtain the values κ and $c(s,t)$ as before, and if $t \ge 57.5s^3$, then $\kappa < 3$. Thus we obtain Theorem 2.5. □

Proof of Theorem 2.6 (Quartic case).

We assume that $t \geq 57.5s^3$. We use the estimate for the zeros of $F_{s,t}^{(4)}(X,1)$ given in [5, Lemma 9 (c)] just replacing t by t/s. The assumption

$$y \geq \sqrt[4]{\frac{tk}{4s^2}}$$

implies that the primitive solutions (x, y) of (1.1), with $1/2 < x/y \leq 3$, in fact satisfy $1 - \dfrac{2s}{t} < x/y < 1$. Using this estimate we obtain

$$\left|\theta - \frac{x}{y}\right| < \frac{k}{2(t-2s)y^4}.$$

From this inequality and Theorem 2.5, we obtain Theorem 2.6. □

Proof of Theorem 2.9 (Cubic case).

(1) We assume that $q \geq \dfrac{t + \frac{3}{2}s}{9.04}$ so that the folklore lemma works.

(2) Since θ is the β_2 of [5], we choose $X = 0$ ([5, p.1892]) to get a good rational approximation for θ.

(3) We use the estimate $\arg \boldsymbol{w}(0) = 2\arg(2t + 3s + 3\sqrt{-3}) < \dfrac{3\sqrt{3}}{t + \frac{3}{2}s}$ without relaxing the right-hand side to $\dfrac{3\sqrt{3}}{t}$ as done in [5, p.1893, line 3].

(4) For $r \in \mathbb{N}$, we obtain integers p_r and q_r satisfying (4.1) with

$$k_0 = 3.8, \quad Q = e^{0.83}\sqrt{t^2 + 3ts + 9s^2}, \quad \ell_0 = \frac{2.61\sqrt{3}}{t + \frac{3}{2}s} \quad \text{and} \quad E = \frac{t + \frac{3}{2}s}{e^{1.3}s^2}.$$

We obtain the values κ and $c(s,t)$ as before, and if $t \geq 31s^4$, then $\kappa < 2$. Thus we obtain Theorem 2.9. □

Proof of Theorem 2.10 (Cubic case).

We assume that $t \geq 31s^4$. We use the estimate for the zeros of $F_{s,t}^{(3)}(X,1)$ given in [5, Lemma 10 (c)] just replacing t by t/s. The assumption

$$y \geq \sqrt[3]{\frac{tk}{1.99s^2}}$$

implies that the primitive solutions (x, y) of (1.1), with $-1/2 < x/y \leq 1$, in fact satisfy $-s/t < x/y < 0$. Using this estimate, we obtain

$$\left|\theta - \frac{x}{y}\right| < \frac{k}{ty^3}.$$

From this inequality and Theorem 2.9, we obtain Theorem 2.10. □

5. Continued fractions and Legendre's theorem

In order to prove Theorems 2.3, 2.4, 2.7, 2.8 and 2.11, we need to obtain lower bounds for the solutions of the Thue inequalities. One of the methods is to calculate the normal continued fractions for the zero θ of the corresponding polynomial and to use classical Legendre's theorem. However, in our case, it is difficult to calculate the normal continued fractions, since each of our polynomials is parameterized by two parameters. For example, for the sextic case, the beginning of the normal continued fraction expansion of θ is

$$\theta = [0; 1, \lfloor (2t/s+1)/3 \rfloor, \dots],$$

and for the next term we must consider a lot of cases; here $\lfloor \cdot \rfloor$ is the Gauss symbol, that is the greatest integer function. The same situation occurred in our former paper [9], and we developed a method applicable to such a situation; namely, we considered continued fractions with rational partial quotients, and we generalized a classical theorem of Legendre. For the convenience of readers, we recall this method.

For a real number ξ we consider a *continued fraction expansion with rational partial quotients*

$$\xi = [k_0; k_1, k_2, k_3, \dots] = k_0 + \cfrac{1}{k_1 + \frac{1}{k_2 + \frac{1}{k_3 + \cdots}}}$$

where the k_i's are rational numbers satisfying

$$k_0 < \xi < k_0 + 1 \quad \text{and} \quad k_i \ge 1 \quad \text{for } i = 1, 2, \dots.$$

Let us call this expansion a *generalized continued fraction expansion*. Note that this expansion is not unique. We recursively define p_n and q_n by

$$\begin{cases} p_n = 1 \quad \text{and} \quad q_n = 0 & \text{for } n = 0, \\ p_n = k_0 \quad \text{and} \quad q_n = 1 & \text{for } n = 1, \\ p_{n+1} = k_n p_n + p_{n-1} \quad \text{and} \quad q_{n+1} = k_n q_n + q_{n-1} & \text{for } n \ge 1. \end{cases}$$

The quotients p_n/q_n, for $n = 1, 2, \dots$, are called *convergents* to ξ. For $n \ge 0$, we take a positive integer d_n such that both $d_n p_n$ and $d_n q_n$ belong to $\mathbb{Z}$. We call d_n a *common denominator* of p_n and q_n. Below, smaller d_n is better.

Lemma 5.1 (Generalization of Legendre's theorem, [9, Theorem 5])**.** *Let ξ be a real number, and let p_n/q_n, for $n = 1, 2, \dots$, be the convergents defined by a generalized continued fraction expansion for ξ. For $n \ge 0$, let d_n be a common denominator of p_n and q_n. For a fixed $n \ge 1$, assume that $q_n/d_{n-1} < q_{n+1}/d_n$. Under this assumption, if some integers p and q satisfy*

$$\frac{q_n}{d_{n-1}} \le q < \frac{q_{n+1}}{d_n} \tag{5.1}$$

and

$$\left| \xi - \frac{p}{q} \right| < \frac{1}{d_n(d_{n-1} + d_{n+1})q^2}, \tag{5.2}$$

then

$$\frac{p}{q} = \frac{p_n}{q_n}.$$

In order to obtain a lower bound for the solutions of a given Thue inequality, we must check that the convergents (except some small ones) do not give solutions of the Thue inequality. However, it is sometimes not easy to check. We have the following alternative way. (In [9, Lemma 12], the leading coefficient of $F(X,1)$ is supposed to be 1 without mentioning it.)

Lemma 5.2 ([9, Lemma 12]). *Let $F(X,Y) = a_0X^d + \cdots$, with $a_0 \neq 0$, be a homogeneous polynomial of degree $d \geq 3$ with integer coefficients, and let k be a positive integer. Let $\xi = \xi^{(1)}, \xi^{(2)}, \ldots, \xi^{(d)}$ be the zeros of $F(X,1)$, and suppose that ξ is a simple zero. Let (x,y) be a solution of the Thue inequality*

$$|F(x,y)| \leq k. \tag{5.3}$$

Suppose that, for a positive number A, we have

$$\left| a_0 \prod_{i=2}^{d} \left(\xi^{(i)} - \frac{x}{y} \right) \right| \geq A. \tag{5.4}$$

Take a generalized continued fraction expansion $\xi = [k_0; k_1, k_2, \ldots, k_n, \ldots]$ for ξ. Let the p_n/q_n's be its convergents and let d_n be a common denominator of p_n and q_n. Suppose further that for a fixed $n \geq 1$ the following three conditions (5.5), (5.6) *and* (5.7) *are satisfied*:

$$\frac{q_n}{d_{n-1}} < \frac{q_{n+1}}{d_n}, \tag{5.5}$$

$$\left(\frac{k d_n (d_{n-1} + d_{n+1})}{A} \right)^{1/(d-2)} d_{n-1} < q_n, \tag{5.6}$$

and

$$\begin{cases} \left(\dfrac{k(k_n + 1 + 1/k_{n-1})}{A} \right)^{1/(d-2)} d_{n-1}^{d/(d-2)} \leq q_n & \text{if } n \geq 2, \\ \left(\dfrac{k(k_1+1)}{A} \right)^{1/(d-2)} d_0^{d/(d-2)} \leq q_1 & \text{if } n = 1. \end{cases} \tag{5.7}$$

Then

$$y \notin [q_n/d_{n-1}, q_{n+1}/d_n).$$

6. Proofs of Theorems 2.3 and 2.4 (Sextic case)

Proof of Theorem 2.3

We assume that

$$t \geq 97.3 s^{48/19}. \tag{6.1}$$

Under this assumption, we have $\kappa < 24/5$ by the formula for κ in Theorem 2.1. Under this assumption and Assumption 3.1, we shall find all solutions of (2.1). For the solutions of (2.1), we have an upper bound by Theorem 2.2. In order to obtain a lower bound for the solutions, except for some small ones, we calculate a generalized continued fraction expansion for the zero $\theta \in (1/2, 1)$ by using the software PARI. We calculate

$$\theta = [k_0; k_1, k_2, \ldots, k_{11}, \ldots]$$

up to k_{11}. They are given by

$$\begin{cases} k_0 = 0, \quad k_1 = 1, \quad k_2 = \dfrac{2t}{3s} + \dfrac{1}{2}, \quad k_3 = \dfrac{24t}{35s} + \dfrac{36}{35}, \quad k_4 = \dfrac{350t}{429s} + \dfrac{175}{143}, \\ k_5 = \dfrac{1144t}{1615s} + \dfrac{1716}{1615}, \quad k_6 = \dfrac{13566t}{16445s} + \dfrac{20349}{16445}, \quad k_7 = \dfrac{1447160t}{2032639s} + \dfrac{2170740}{2032639}, \\ k_8 = \dfrac{580754t}{702075s} + \dfrac{290377}{234025}, \quad k_9 = \dfrac{365079000t}{511934651s} + \dfrac{547618500}{511934651}, \\ k_{10} = \dfrac{17405778134t}{21019423425s} + \dfrac{8702889067}{7006474475}, \quad k_{11} = \dfrac{1019123560t}{1428028237s} + \dfrac{1582685340}{1428028237}. \end{cases}$$

The first convergents are given by

$$\begin{cases} p_0 = 1, \quad p_1 = 0, \quad p_2 = 1, \quad p_3 = \dfrac{2t}{3s} + \dfrac{1}{2}, \quad p_4 = \dfrac{16t^2}{35s^2} + \dfrac{36t}{35s} + \dfrac{53}{35}, \quad \ldots \\ q_0 = 0, \quad q_1 = 1, \quad q_2 = 1, \quad q_3 = \dfrac{2t}{3s} + \dfrac{3}{2}, \quad q_4 = \dfrac{16t^2}{35s^2} + \dfrac{12t}{7s} + \dfrac{89}{35}, \quad \ldots, \\ q_{12} = \dfrac{13631488t^{10}}{252004983s^{10}} + \dfrac{71565312t^9}{84001661s^9} + \cdots \end{cases}$$

and the corresponding common denominators d_n of p_n and q_n are given by

$$\begin{cases} d_0 = 1, \quad d_1 = 1, \quad d_2 = 1, \quad d_3 = 6s, \quad d_4 = 35s^2, \quad d_5 = 858s^3, \\ d_6 = 6783s^4, \quad d_7 = 98670s^5, \quad d_8 = 290377s^6, \quad d_9 = 18253950s^7, \\ d_{10} = 511934651s^8, \quad d_{11} = 764342670s^9, \quad d_{12} = 81397609509s^{10}. \end{cases}$$

A generalized continued fraction expansion is not unique. Here we choose k_n in each step so that the denominator of k_n is as small as possible and p_{n+1}/q_{n+1} approaches well θ.

We first determine small solutions of (2.1).

CLAIM 1. The only primitive solutions of (2.1), with $1/2 < x/y \leq 4$ and $0 < y \leq 3$, are $(1,1), (2,1), (3,1), (4,1), (3,2)$ and $(2,3)$.

PROOF. This can be checked easily. □

CLAIM 2. The inequality (2.1) has no primitive solutions satisfying $1/2 < x/y \leq 4$ and $4 \leq y \leq 15s^{29/19}$.

PROOF. The proof is elementary. Let

$$f_t(X) = X^6 - 2tX^5 - (5t+15)X^4 - 20X^3 + 5tX^2 + (2t+6)X + 1.$$

For $y \geq 4$ and $t \geq 1$, we have by elementary estimations

$$f_t\left(1 - \frac{1}{y}\right) > \frac{7.5t}{y} - 27, \quad f_t(1/2) > \frac{7.5t}{y} - 27 \quad \text{and} \quad f_t\left(1 + \frac{1}{y}\right) < -\frac{18t}{y} - 27,$$

and, for $1/2 < X \le 1$ and $t \ge 1$, we have $f_t''(X) < 0$. Hence, for $y \ge 4$ and $t \ge 1$,

$$f_t(X) > \frac{7.5t}{y} - 27 \qquad \text{for } \frac{1}{2} < X \le 1 - \frac{1}{y}\cdot$$

For $1 \le X \le 4$, we have $f_t'(X) < 0$. Hence, for $y \ge 4$ and $t \ge 1$, we have

$$f_t(X) < -\frac{18t}{y} - 27 \qquad \text{for } 1 + \frac{1}{y} \le X \le 4.$$

Suppose that there exists a primitive solution of (2.1) satisfying $1/2 < x/y < 1$ and $4 \le y \le 15s^{29/19}$. Then we would have

$$\begin{aligned} k = 27(120t + 323s) &\ge \left|F_{s,t}^{(6)}(x,y)\right| = sy^6\left|f_{t/s}(x/y)\right| \\ &> sy^6\left(\frac{7.5t}{ys} - 27\right) = y^5(7.5t - 27sy). \end{aligned}$$

However, by (6.1), we have the inequalities

$$3330t > 27(120t + 323s) \qquad \text{and} \qquad y^5(7.5t - 27sy) > 3410t,$$

which clearly lead to a contradiction.

Suppose that there exists a primitive solution of (2.1) satisfying $1 < x/y \le 4$ and $4 \le y \le 15s^{29/19}$. Then we would have

$$\begin{aligned} k = 27(120t + 323s) &\ge \left|F_{s,t}^{(6)}(x,y)\right| = sy^6\left|f_{t/s}(x/y)\right| \\ &> sy^6\left(\frac{18t}{ys} + 27\right) = y^5(18t + 27sy). \end{aligned}$$

However, by (6.1), we have the inequalities

$$3330t > 27(120t + 323s) \qquad \text{and} \qquad y^5(18t + 27sy) > 18400t,$$

leading to another contradiction. Therefore, (2.1) has no primitive solutions satisfying $1/2 < x/y \le 4$ and $4 \le y \le 15s^{29/19}$. □

CLAIM 3. The inequality (2.1) has no primitive solutions satisfying $1/2 < x/y \le 4$ and $15s^{29/19} < y < \dfrac{q_3}{d_2} = \dfrac{2t}{3s} + \dfrac{3}{2}\cdot$

PROOF. We use the generalization of Legendre's theorem. Suppose that there exists a primitive solution satisfying the condition. Then, using the estimate given in [5, Lemma 3 (a)] for the zeros of $F_{1,t}^{(6)}(X,1)$, we obtain, for $1/2 < x/y \le 4$,

$$\left|\theta - \frac{x}{y}\right| < \frac{16k}{15y^6(4t - 3s)}.$$

In addition, from $k = 27(120t + 323s)$, (6.1) and $y > 15s^{29/19}$, we get

$$\frac{16k}{15y^6(4t - 3s)} < \frac{895}{y^6} < \frac{1}{d_2(d_1 + d_3)y^2}.$$

Therefore, from Lemma 5.1, we have $x/y = p_2/q_2 = 1$, hence $x = y = 1$, which is a contradiction. □

Next we treat large solutions.

CLAIM 4. The inequality (2.1) has no primitive solutions satisfying $1/2 < x/y \leq 4$ and $\dfrac{q_3}{d_2} = \dfrac{2t}{3s} + \dfrac{3}{2} \leq y < \dfrac{q_{12}}{d_{11}}$.

PROOF. We use the generalized continued fraction expansion for θ and Lemma 5.2. Suppose that there exists a primitive solution satisfying the condition. From (6.1), we have

$$\frac{2t}{3s} + \frac{3}{2} > \sqrt[6]{\frac{tk}{32.6s^2}}.$$

Hence we have $y > \sqrt[6]{\dfrac{tk}{32.6s^2}}$, and from this we have $1 - \dfrac{3s}{2t} < x/y < 1$ as in the proof of Theorem 2.2. Then, using the estimate given in [5, Lemma 3 (a)] for the zeros of $F^{(6)}_{1,t}(X,1)$, we obtain

$$s\left|\prod_{i=2}^{6}\left(\theta^{(i)} - \frac{x}{y}\right)\right| > 18t - 39s =: A,$$

where $\theta^{(i)}$, for $i = 2, \ldots, 6$, are the zeros of $F^{(6)}_{s,t}(X,1)$ other than θ. We may suppose that y belongs to the interval $[q_n/d_{n-1}, q_{n+1}/d_n]$ for some $n = 3, \ldots, 11$. Using the data of the generalized continued fraction expansion for θ, we verify that (5.6) and (5.7) are satisfied for every $n = 3, \ldots, 11$. Thus, from Lemma 5.2, we obtain

$$y \notin [q_n/d_{n-1}, q_{n+1}/d_n).$$

This is a contradiction, and the assertion follows. □

CLAIM 5. The inequality (2.1) has no primitive solutions satisfying $1/2 < x/y \leq 4$ and $y \geq q_{12}/d_{11}$.

PROOF. Indeed, suppose that there exists such a solution. Then the assumptions of Theorem 2.2 are satisfied, and Theorem 2.2 gives an upper bound for y, that is

$$y^{5-\kappa} < \frac{3.6k}{(7.63s)^{\kappa}}.$$

From $\kappa < 24/5$, we have $1/(5-\kappa) < 5$ and $\kappa/(5-\kappa) = 5/(5-\kappa) - 1$. This implies

$$\begin{aligned}
\frac{q_{12}}{d_{11}} \leq \; y \; &< \frac{(3.6 \cdot 27(120t + 323s))^{1/(5-\kappa)}}{(7.63s)^{\kappa/(5-\kappa)}} \\
&= 7.63s\left(\frac{3.6 \cdot 27(120t + 323s)}{(7.63s)^5}\right)^{1/(5-\kappa)} \\
&\leq \begin{cases} 7.63s \quad \text{if } \dfrac{3.6 \cdot 27(120t + 323s)}{(7.63s)^5} \leq 1, \\[2ex] 7.63\left(\dfrac{3.6 \cdot 27 \cdot 120}{7.63^5}\right)^5\left(1 + \dfrac{323}{120 \cdot 97.3}\right)^5 \dfrac{t^5}{s^{24}} = \dfrac{0.164t^5}{s^{24}} \quad \text{otherwise.} \end{cases}
\end{aligned}$$

However, (6.1) implies that the right-hand side of the above sequence of inequalities is strictly smaller than the corresponding left-hand side. This is a contradiction, and the assertion follows. □

Because of Claims 1 to 5, the proof of Theorem 2.3 is accomplished. □

Proof of Theorem 2.4

By Proposition 3.2 (a) and the explanation following it, it is enough to pick up solutions of (2.2) among the images by g^i, for $i = 1, \dots, 5$, of the solutions in Theorem 2.3 or the images divided by the common factor of their coordinates. $\square$

7. Proofs of Theorems 2.7 and 2.8 (Quartic case)

Proof of Theorem 2.7

We assume that

$$t \geq 70s^{28/9}. \tag{7.1}$$

Under this assumption, we have $\kappa < 14/5$. Under this assumption and Assumption 3.1, we shall find all solutions of (2.3). For the solutions of (2.3), we have an upper bound by Theorem 2.6. In order to obtain a lower bound, we calculate a generalized continued fraction expansion for the zero $\theta \in (0,1)$, that is,

$$\theta = [k_0; k_1, k_2, \dots, k_{13}, \dots]$$

up to k_{13}. They are given by

$$\begin{cases} k_0 = 0, \quad k_1 = 1, \quad k_2 = \dfrac{t}{2s} - \dfrac{1}{2}, \quad k_3 = \dfrac{2t}{5s}, \quad k_4 = \dfrac{25t}{42s}, \quad k_5 = \dfrac{294t}{715s}, \\ k_6 = \dfrac{143t}{238s}, \quad k_7 = \dfrac{102t}{247s}, \quad k_8 = \dfrac{35321t}{58650s}, \quad k_9 = \dfrac{97750t}{236379s}, \quad k_{10} = \dfrac{21489t}{35650s}, \\ k_{11} = \dfrac{121210t}{292929s}, \quad k_{12} = \dfrac{2996889t}{4969610s}, \quad k_{13} = \dfrac{22860206t}{55228383s}. \end{cases}$$

The first convergents are given by

$$\begin{cases} p_0 = 1, \quad p_1 = 0, \quad p_2 = 1, \quad p_3 = \dfrac{t}{2s} - \dfrac{1}{2}, \quad p_4 = \dfrac{t^2}{5s^2} - \dfrac{t}{5s} + 1, \quad \dots \\ q_0 = 0, \quad q_1 = 1, \quad q_2 = 1, \quad q_3 = \dfrac{t}{2s} + \dfrac{1}{2}, \quad q_4 = \dfrac{t^2}{5s^2} + \dfrac{t}{5s} + 1, \quad \dots, \\ q_{14} = \dfrac{391t^{12}}{2076255s^{12}} + \dfrac{391t^{11}}{2076255s^{11}} + \cdots \end{cases}$$

and the corresponding common denominators d_n of p_n and q_n are given by

$$\begin{cases} d_0 = 1, \quad d_1 = 1, \quad d_2 = 1, \quad d_3 = 2s, \quad d_4 = 5s^2, \quad d_5 = 42s^3, \quad d_6 = 143s^4, \\ d_7 = 102s^5, \quad d_8 = 2717s^6, \quad d_9 = 19550s^7, \quad d_{10} = 21489s^8, \\ d_{11} = 121210s^9, \quad d_{12} = 428127s^{10}, \quad d_{13} = 4969610s^{11}, \quad d_{14} = 39448845s^{12}. \end{cases}$$

We first determine small solutions of (2.3).

CLAIM 1. The only primitive solutions of (2.3), with $1/2 < x/y \leq 3$ and $0 < y \leq 3$, are $(1,1), (2,1)$ and $(3,1)$.

PROOF. This can be checked easily. $\square$

CLAIM 2. The inequality (2.3) has no primitive solutions satisfying $1/2 < x/y \leq 3$ and $4 \leq y \leq 13s^{19/9}$.

PROOF. The proof is elementary. Let

$$f_t(X) = X^4 - tX^3 - 6X^2 + tX + 1.$$

For $y \geq 4$ and $t \geq 1$, we have by elementary estimations

$$f_t\left(1 - \frac{1}{y}\right) > \frac{21t}{16y} - 5, \quad f_t(1/2) > \frac{21t}{16y} - 5, \quad \text{and} \quad f_t\left(1 + \frac{1}{y}\right) < -\frac{2t}{y},$$

and, for $1/2 < X \leq 1$ and $t \geq 1$, we have $f_t''(X) < 0$. Hence,

$$f_t(X) > \frac{21t}{16y} - 5 \qquad \text{for } \frac{1}{2} < X \leq 1 - \frac{1}{y}.$$

For $1 \leq X \leq 3$, we have $f_t'(X) < 0$, hence we have

$$f_t(X) < -\frac{2t}{y} \qquad \text{for } 1 + \frac{1}{y} \leq X \leq 3.$$

Suppose that there exists a primitive solution of (2.3) satisfying $1/2 < x/y < 1$ and $4 \leq y \leq 13s^{19/9}$. Then we would have

$$\begin{aligned} k = 4(6t + 7s) &\geq \left|F_{s,t}^{(4)}(x, y)\right| = sy^4 \left|f_{t/s}(x/y)\right| \\ &> sy^4 \left(\frac{21t}{16ys} - 5\right) = y^3 \left(\frac{21}{16}t - 5sy\right). \end{aligned}$$

However, by (7.1), we have the inequalities

$$24.4t > 4(6t + 7s) \qquad \text{and} \qquad y^3 \left(\frac{21}{16}t - 5sy\right) > 24.57t,$$

which clearly lead to a contradiction.

Suppose that there exists a primitive solution of (2.3) satisfying $1 < x/y \leq 3$ and $4 \leq y \leq 13s^{19/9}$. Then we would have

$$\begin{aligned} k = 4(6t + 7s) &\geq \left|F_{s,t}^{(4)}(x, y)\right| = sy^4 \left|f_{t/s}(x/y)\right| \\ &> sy^4 \frac{2t}{ys} = 2ty^3. \end{aligned}$$

However, by (7.1), we have the inequalities

$$24.4t > 4(6t + 7s) \qquad \text{and} \qquad 2ty^3 \geq 128t,$$

leading to another contradiction. Therefore, (2.3) has no primitive solutions satisfying $1/2 < x/y \leq 3$ and $4 \leq y \leq 13s^{19/9}$. $\square$

CLAIM 3. The inequality (2.3) has no primitive solutions satisfying $1/2 < x/y \leq 3$ and $13s^{19/9} < y < \dfrac{q_3}{d_2} = \dfrac{t}{2s} + \dfrac{1}{2}$.

PROOF. We use the generalization of Legendre's theorem. Suppose that there exists a primitive solution satisfying the condition. Then, by [5, Lemma 9 (c)], we obtain, for $1/2 < x/y \leq 3$,

$$\left|\theta - \frac{x}{y}\right| < \frac{4k}{3(t - 3s)y^4}.$$

In addition, from $k = 4(6t + 7s)$, (7.1) and $y > 13s^{19/9}$, we get

$$\frac{4k}{3(t-3s)y^4} < \frac{32.1}{y^4} < \frac{1}{d_2(d_1+d_3)y^2}.$$

Therefore, from Lemma 5.1, we have $x/y = p_2/q_2 = 1$, hence $x = y = 1$; a contradiction. □

CLAIM 4. The inequality (2.3) has no primitive solutions satisfying $1/2 < x/y \leq 3$ and $\frac{q_3}{d_2} = \frac{t}{2s} + \frac{1}{2} \leq y < \frac{q_{14}}{d_{13}}$.

PROOF. We use the generalized continued fraction expansion for θ and Lemma 5.2. Suppose that there exists a primitive solution satisfying the condition. From (7.1), we have $\frac{t}{2s} + \frac{1}{2} > \sqrt[4]{\frac{tk}{4s^2}}$. Hence we have $y > \sqrt[4]{\frac{tk}{4s^2}}$, and from this we have

$$1 - \frac{2s}{t} < x/y < 1$$

as in the proof of Theorem 2.6. Then, using the estimate given in [5, Lemma 9 (c)], we obtain

$$s\left|\prod_{i=2}^{4}\left(\theta^{(i)} - \frac{x}{y}\right)\right| > 2(t-2s) =: A,$$

where the $\theta^{(i)}$'s, for $i = 2, 3, 4$, are the zeros of $F_{s,t}^{(4)}(X, 1)$ other than θ. We suppose that $q_n/d_{n-1} \leq y < q_{n+1}/d_n$ for some $n = 3, \ldots, 13$, and we verify that (5.6) and (5.7) are satisfied for every $n = 3, \ldots, 13$. Then Lemma 5.2 implies that $y \notin [q_n/d_{n-1}, q_{n+1}/d_n)$. This is a contradiction, and the assertion follows. □

CLAIM 5. Inequality (2.3) has no primitive solutions satisfying $1/2 < x/y \leq 3$ and $y \geq q_{14}/d_{13}$.

PROOF. Indeed, suppose that there exists such a solution. Then the assumptions of Theorem 2.6 are satisfied, and Theorem 2.6 gives an upper bound for y, that is

$$y^{3-\kappa} < \frac{3.29k}{(3.34s)^{\kappa}}.$$

From $\kappa < 14/5$, we have $1/(3-\kappa) < 5$. This implies

$$\begin{aligned}\frac{q_{14}}{d_{13}} \leq y &< \frac{(3.29 \cdot 4(6t+7s))^{1/(3-\kappa)}}{(3.34s)^{\kappa/(3-\kappa)}} \\ &< 3.34s\left(\frac{80.3t}{3.34^3 s^3}\right)^5 = \frac{156t^5}{s^{14}}.\end{aligned}$$

However, (7.1) implies

$$\frac{156t^5}{s^{14}} < \frac{q_{14}}{d_{13}}.$$

This is a contradiction, and the assertion follows. □

Because of Claims 1 to 5, the proof of Theorem 2.7 is accomplished. □

Proof of Theorem 2.8.

The proof of Theorem 2.8 is similar to that of Theorem 2.4.

8. Proof of Theorem 2.11 (Cubic case)

We assume that

$$t \geq 64s^{9/2}. \tag{8.1}$$

Under this assumption, we have $\kappa < 9/5$. Under this assumption and Assumption 3.1, we shall find all the solutions of (2.4). We calculate a generalized continued fraction expansion for the zero $\theta \in (-1, 0)$, that is,

$$\theta = [k_0; k_1, k_2, \ldots, k_{18}, \ldots]$$

up to k_{18}. They are given by

$$\begin{cases} k_0 = -1, \quad k_1 = 1, \quad k_2 = \dfrac{t}{s} + 1, \quad k_3 = \dfrac{t}{2s} + \dfrac{3}{4}, \quad k_4 = \dfrac{8t}{7s} + \dfrac{12}{7}, \\ k_5 = \dfrac{49t}{96s} + \dfrac{49}{64}, \quad k_6 = \dfrac{1152t}{1001s} + \dfrac{1728}{1001}, \quad k_7 = \dfrac{1573t}{3072s} + \dfrac{1573}{2048}, \\ k_8 = \dfrac{4096t}{3553s} + \dfrac{6144}{3553}, \quad k_9 = \dfrac{4199t}{8192s} + \dfrac{12597}{16384}, \quad k_{10} = \dfrac{32768t}{28405s} + \dfrac{49152}{28405}, \\ k_{11} = \dfrac{705755t}{1376256s} + \dfrac{705755}{917504}, \quad k_{12} = \dfrac{38535168t}{33393355s} + \dfrac{57802752}{33393355}, \\ k_{13} = \dfrac{45179245t}{88080384s} + \dfrac{45179245}{58720256}, \quad k_{14} = \dfrac{16777216t}{14535931s} + \dfrac{25165824}{14535931}, \\ k_{15} = \dfrac{34427205t}{67108864s} + \dfrac{103281615}{134217728}, \quad k_{16} = \dfrac{268435456t}{232548515s} + \dfrac{402653184}{232548515}, \\ k_{17} = \dfrac{9089613695t}{17716740096s} + \dfrac{9089613695}{11811160064}, \\ k_{18} = \dfrac{779536564224t}{675270333535s} + \dfrac{1169304846336}{675270333535}. \end{cases}$$

The first convergents are given by

$$\begin{cases} p_0 = 1, \quad p_1 = -1, \quad p_2 = 0, \quad p_3 = -1, \quad p_4 = -\dfrac{t}{2s} - \dfrac{3}{4}, \quad \ldots \\ q_0 = 0, \quad q_1 = 1, \quad q_2 = 1, \quad q_3 = \dfrac{t}{s} + 2, \quad q_4 = \dfrac{t^2}{2s^2} + \dfrac{7t}{4s} + \dfrac{5}{2}, \quad \ldots, \\ q_{19} = \dfrac{2162688t^{17}}{150226993s^{17}} + \dfrac{56229888t^{16}}{150226993s^{16}} + \cdots \end{cases}$$

and the corresponding common denominators d_n of p_n and q_n are given by

$$\begin{cases} d_0 = 1, \quad d_1 = 1, \quad d_2 = 1, \quad d_3 = s, \quad d_4 = 4s^2, \quad d_5 = 7s^3, \quad d_6 = 48s^4, \\ d_7 = 143s^5, \quad d_8 = 128s^6, \quad d_9 = 4199s^7, \quad d_{10} = 256s^8, \quad d_{11} = 37145s^9, \\ d_{12} = 21504s^{10}, \quad d_{13} = 1964315s^{11}, \quad d_{14} = 24576s^{12}, \quad d_{15} = 11475735s^{13}, \\ d_{16} = 98304s^{14}, \quad d_{17} = 293213345s^{15}, \quad d_{18} = 2162688s^{16}, \\ d_{19} = 13505406707s^{17}. \end{cases}$$

We first determine small solutions of (2.4).

CLAIM 1. The only primitive solutions of (2.4), with $-1/2 < x/y \leq 1$ and $0 < y \leq 2$, are $(0, 1)$ and $(1, 1)$.

PROOF. This can be checked easily. □

CLAIM 2. The inequality (2.4) has no primitive solutions satisfying $0 < x/y \leq 1$ and $y \geq 3$.

PROOF. Indeed, the condition implies

$$F_{s,t}^{(3)}(x, y) < -tx^2y - (t + 3s)xy^2 \leq -12t - 27s < -2t - 3s;$$

a contradiction. □

CLAIM 3. The inequality (2.4) has no primitive solutions satisfying $-1/2 < x/y < 0$ and $3 \leq y < \dfrac{t}{4s}$.

PROOF. Indeed, by elementary estimations, we have $F_{s,t}^{(3)}(x, y) > 2t + 3s$; a contradiction. □

CLAIM 4. The inequality (2.4) has no primitive solutions satisfying $-1/2 < x/y < 0$ and $\dfrac{t}{4s} \leq y < \dfrac{t}{s} + 2$.

PROOF. The proof is elementary. We have $\dfrac{t}{4s} \leq y \leq \dfrac{t}{s} + 2 - \dfrac{1}{s}\cdot$ We first suppose that $x = -1$. Then we see $F_{s,t}^{(3)}(-1, y) > 2t + 3s$ by using for example the fact that $sy + t/y$ is an increasing function of $y \geq \dfrac{t}{4s}$ under (8.1). We next suppose that $-x \geq 2$. For this case also we see that $F_{s,t}^{(3)}(x, y) > 2t + 3s$ by (8.1); a contradiction. □

CLAIM 5. The only primitive solution of (2.4) satisfying

$$-1/2 < x/y < 0 \quad \text{and} \quad \frac{q_3}{d_2} = \frac{t}{s} + 2 \leq y < \frac{q_4}{d_3}$$

is $(-1, t + 2)$ if $s = 1$, and there is no such solution if $s \geq 2$.

PROOF. We use the generalization of Legendre's theorem. Let (x, y) be a solution satisfying the condition. By [5, Lemma 10 (c)], we obtain, for $-1/2 < x/y < 0$,

$$\left|\theta - \frac{x}{y}\right| < \frac{2(2t + 3s)}{(t + s)y^3}\cdot$$

In addition, from (8.1) and $y \geq \dfrac{t}{s} + 2$, we have

$$\frac{2(2t + 3s)}{(t + s)y^3} < \frac{1}{d_3(d_2 + d_4)y^2}\cdot$$

Therefore, from Lemma 5.1, we have

$$\frac{x}{y} = \frac{p_3}{q_3} = \frac{-s}{t + 2s}\cdot$$

Since $\gcd(s,t)=1$, and $\gcd(x,y)=1$, this implies $x=-s$ and $y=t+2s$. We have $F_{s,t}^{(3)}(-s,t+2s)=s^3(2t+3s)$. Hence, if $s=1$, then $(-1,2t+3)$ is a solution of (2.4) and, if $s\geq 2$, then $(-s,t+2s)$ is not a solution. The assertion follows. □

CLAIM 6. The inequality (2.4) has no primitive solutions satisfying $-1/2<x/y<0$ and $q_4/d_3\leq y<q_{19}/d_{18}$.

PROOF. We use the generalized continued fraction expansion for θ and Lemma 5.2. Indeed, suppose that there exists a primitive solution satisfying the condition. From (8.1), we have

$$\frac{q_4}{d_3}=\frac{1}{s}\left(\frac{t^2}{2s^2}+\frac{7t}{4s}+\frac{5}{2}\right)>\sqrt[3]{\frac{tk}{1.99s^2}}.$$

Hence, we have $y>\sqrt[3]{\frac{tk}{1.99s^2}}$, and from this we have $-s/t<x/y<0$ as in the proof of Theorem 2.10. Then, using the estimate given in [5, Lemma 10 (c)], we obtain

$$s\left|\prod_{i=2}^{3}\left(\theta^{(i)}-\frac{x}{y}\right)\right|>t=:A,$$

where the $\theta^{(i)}$'s, for $i=2,3$, are the zeros of $F_{s,t}^{(3)}(X,1)$ other than θ. As before, we suppose that $q_n/d_{n-1}\leq y<q_{n+1}/d_n$ for some $n=4,\ldots,18$, and we verify that (5.6) and (5.7) are satisfied for every $n=4,\ldots,18$. Then Lemma 5.2 implies $y\notin[q_n/d_{n-1},q_{n+1}/d_n)$. This is a contradiction, and the assertion follows. □

CLAIM 7. The inequality (2.4) has no primitive solutions satisfying $-1/2<x/y<0$ and $y\geq q_{19}/d_{18}$.

PROOF. Indeed, suppose that there exists such a solution. Then the assumptions of Theorem 2.10 are satisfied, and Theorem 2.10 gives an upper bound for y, that is

$$y^{2-\kappa}<18.34\left(\frac{2.47}{s^2}\right)^{\kappa}k.$$

From $\kappa<9/5$, we have $1/(2-\kappa)<5$ and $\kappa/(2-\kappa)=2/(2-\kappa)-1$. This implies

$$\begin{aligned}\frac{q_{19}}{d_{18}}\leq y &< 18.34^5\left(\frac{2.47}{s^2}\right)^{\kappa/(2-\kappa)}(2t+3s)^{1/(2-\kappa)}\\ &= 18.34^5\frac{s^2}{2.47}\left(\frac{2.47^2(2t+3s)}{s^4}\right)^{1/(2-\kappa)}\\ &< 18.34^5\frac{s^2}{2.47}\left(\frac{2.47^2(2t+3s)}{s^4}\right)^{5}.\end{aligned}$$

However, (8.1) implies

$$18.34^5\frac{s^2}{2.47}\left(\frac{2.47^2(2t+3s)}{s^4}\right)^{5}<\frac{q_{19}}{d_{18}}.$$

This is a contradiction, and the assertion follows. □

Because of Claims 1 to 7, the proof of the first part of Theorem 2.11 is accomplished. The second part is easily shown from the first part by using automorphisms.

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