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LOCAL EXISTENCE, UNIQUENESS AND REGULARITY OF A SPECIAL SOLUTION FOR A MIXED PROBLEM IN ELASTICITY

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RÉSUMÉ. Le but de cet article est d'établir des résultats d'existence locale, d'unicité et de régularité de solution pour un problème, en élasticité tridimensionnelle, avec conditions aux limites de type Dirichlet et Neuman.

ABSTRACT. The purpose of this paper is to give some results for local existence, uniqueness and regularity of solution of linear and nonlinear problems in three-dimensional elasticity with mixed Dirichlet and Neuman boundary conditions.

Introduction. In nonlinear three-dimensional elasticity, a central problem consists in finding the equilibrium position of an elastic material subject to given internal body forces with surface forces prescribed on part of the boundary and the displacement given on the remainder. More precisely, let Ω be a bounded open subset of $\mathbb{R}^3$ whose boundary Γ consists of disjoint subsets Γ_0, Γ_1 such that $\Gamma_0 \cap \Gamma_1 = \emptyset$, and $\Gamma = \Gamma_0 \cup \Gamma_1$. The closure $\overline{\Omega}$ of Ω represents the reference configuration occupied by an homogeneous elastic material in the abscence of applied forces. The material is subject to given body forces f in Ω, given surface forces g on Γ_1, and zero displacement on Γ_0. When the homogeneous elastic material is isotropic [2], the response function Σ (or second Piola-Kirchhoff stress) is given by:

$$\Sigma(E) = \lambda(\text{trace}(E))I + 2\mu E + o(\|E\|), \tag{1}$$

where $\|\cdot\|$ denotes any norm in the space of 3×3 matrices, $\lambda > 0$, $\mu > 0$ are two constants, known as the Lamé coefficients, and

$$E := E(\nabla u) = 1/2((\nabla u)^t(\nabla u) + (\nabla u)^t + \nabla u)$$

is the nonlinear Green-Saint-Venant strain tensor, ∇u is the displacement gradient.

A special physical case, occuring an important place in nonlinear elasticity, is the Saint-Venant-Kirchhoff material whose the response function is given by the second Piola-Kirchhoff stress tensor,

$$\Sigma(E) = \lambda(\text{trace}(E))I + 2\mu E. \tag{2}$$

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The mathematical problem consists in solving a nonlinear boundary value problem, with mixed Dirichlet and Neuman conditions, for the displacement u such that

$$(S) \quad \begin{cases} -\text{div}((I + \nabla u)(\Sigma(E(\nabla u)))) = f & \text{in } \Omega, \\ ((I + \nabla u)(\Sigma(E(\nabla u)))) \cdot \vec{n} = g & \text{on } \Gamma_1, \\ u = 0 & \text{on } \Gamma_\circ\,. \end{cases}$$

We limit our attention to consider only zero displacement. Note that because the Saint-Venant-Green strain tensor is nonlinear, a problem with nonhomogeneous boundary displacement cannot be reduced to a nonlinear problem with homogeneous boundary displacement and non-zero body force.

The linearized mixed problem of (S) is

$$(S_\ell) \quad \begin{cases} -\text{div}(\Sigma(\epsilon(u))) = f & \text{in } \Omega, \\ \Sigma(\epsilon(u)) \cdot \vec{n} = g & \text{on } \Gamma_1, \\ u = 0 & \text{on } \Gamma_\circ, \end{cases}$$

where $\epsilon(u) = 1/2(\nabla u + (\nabla u)^t)$ is the linear Green-Saint-Venant strain.

In [2], P. G. Ciarlet has proved that the nonlinear problem (S) with Dirichlet boundary condition (i.e. $\Gamma_1 = \emptyset$) has a unique solution. The same problem with mixed Dirichlet and Neuman boundary conditions is, for the moment, unsolved.

In [1], J. M. Ball introduces the notion of polyconvexity and minimises the stored energy to study the existence of solutions to the mixed boundary value problem of nonlinear elasticity for a wide class of hyperelastic materials, which does not include the Saint-Venant-Kirchhoff material, because its stored energy function, as shown by A. Raoult, is not polyconvex (then neither convex) in the space of 3×3 matrices (see [2] and the reference therein). However, this is not a restriction for the Ball's approach, since P. G. Ciarlet and G. Geymonat (cf. [2] or [3]) have proved that, given two constants $\lambda > 0$ and $\mu > 0$, it is possible to construct a constitutive equation satisfying (2) and to which Ball's theory can be applied. The main difficulty is then to prove if the previous minimum of stored energy function is a solution of the Euler's equilibrium equations of the system (S).

The assumptions $\lambda > 0$ and $\mu > 0$ satisfied by the Lamé coefficients are two physical conditions as shown by an experimental evidence. But many authors (see [3] for example) included the case $\lambda = 0$ corresponding to "limit" of Saint-Venant-Kirchhoff material in the sense that λ is physically very small ($\lambda \approx 0$). Mathematically, this "limit" case plays an important part for the two next reasons: Firstly, the Ciarlet's existence theory for the pure displacement problem (i.e. $\Gamma_1 = \emptyset$) still hold under the weaker assumptions $\mu > 0$ and $3\lambda + 2\mu > 0$, see [2], Chapter 6; and [5], Section 6.1. The second reason is explained by the fact that the study of the nonlinear problem (S) for $\lambda = 0$ is more difficult than other cases, since M. Atteia and M. Raïssouli showed, see [2] or [7], that the associated stored energy function, for $\lambda = 0$, can't be convex, even locally, in a neighbourhood of the point 0 which corresponds to the trifling null solution for the simple case $f = 0$ and $g = 0$. For the previous arguments, we limit our attention throughout the following, not to lengthen the paper, to the case $\lambda = 0$ and $\mu = 1/2$ that corresponds to $\Sigma(E) = E$.

The fundamental goal of this work is to prove, first, that the linearized mixed problem has one and only one regular solution and so to deduce the local existence, uniqueness and regularity of solution of the nonlinear problem with mixed conditions.

The paper is organized as follows. In Section 1 we draw a background material that will be needed throughout the following. Section 2 describes the formulation of the problem which we will study later. Section 3 is devoted to introduce some preliminary results for the linearized mixed problem. In the final Section, we state our fundamental theorem concerning the local existence, uniqueness and regularity of solution for the nonlinear mixed problem.

1. Preliminary. In this short section, we recall some standard notation and results. For some details, one can consult [6] for example.

Let Ω be an open bounded domain in $\mathbb{R}^3$ with its boundary $\Gamma = \partial\Omega$. For a given real number $s > 0$, we recall that the space $H^s(\Omega)$ is equipped with the norm:

$$\|u\|_{H^s(\Omega)} = \left(\|u\|^2_{H^k(\Omega)} + \sum_{|\alpha|=k} \Big\| \frac{\partial^\alpha u(x) - \partial^\alpha u(y)}{|x-y|^{\sigma+3/2}} \Big\|^2_{L^2(\Omega\times\Omega)} \right)^{1/2}, \tag{3}$$

where $s = k+\sigma$, $0 < \sigma < 1$, k is the integer part of s and $|\cdot|$ is the classical Euclidian norm of $\mathbb{R}^3$.

We say that Γ is of class C^∞, [6], if the following conditions are simultaneously satisfied: there exist two real numbers $\alpha > 0$, $\beta > 0$, the local cartesian coordinate systems $(x_{r_1}, x_{r_2}, x_{r_3}) = (x'_r, x_{r_3})$, and the functions $a_r \in C^\infty$, $r = 1, \dots, m$, in the bidimensional closed cube $|x'_r| \le \alpha$ such that every point $x \in \Gamma$ can be represented in the form $x = (x'_r, a_r(x'_r))$. We suppose that the points (x'_r, x_{r_3}) such that $|x'{}_r| \le \alpha$, $a_r(x'_r) < x_{r_3} < a_r(x'_r) + \beta$ are in Ω, and the points (x'_r, x_{r_3}) such that $|x'_r| \le \alpha$, $a_r(x'_r) - \beta < x_{r_3} < a_r(x'_r)$ are out of Ω.

With the above notation, let us put $\triangle_r = \{x'_r\, ,\ |x'_r| < \alpha\}$ and consider a function f defined on Γ satisfying:

$$f(x) := f(x'_r, a_r(x'_r)) = f_r(x'_r), \tag{4}$$

where f_r is defined in $\triangle_r$ for $r = 1, 2, \dots, m$.

In order to define the surface integral of a function f, we need a partition of unity associated with the covering of the boundary Γ by the open sets:

$$U_r = \{(x'_r, x_{r_3}),\ |x'_r| < \alpha,\ a_r(x'_r) - \beta < x_{r_3} < a_r(x'_r) + \beta\},$$

that is, a family of function $\psi_r : \mathbb{R}^3 \longrightarrow \mathbb{R}$, that satisfy:

$$\text{supp}\, \psi_r \subset U_r,\ 0 \le \psi_r \le 1 \text{ for } r = 1, \dots, m, \text{ and } \sum_{r=1}^{m} \psi_r(x) = 1 \text{ for all } x \in \Gamma,$$

with $\psi_r \in D(\Gamma)$ space of functions infinitely differentiable of compact support.

Then, we define

$$\int_\Gamma f d\sigma = \sum_{r=1}^{m} \int_{\triangle_r} f(x'_r, a_r(x'_r))\psi_r(x'_r, a_r(x'_r)) \left(1 + \sum_{i=1}^{2} (\frac{\partial a_r}{\partial x_{r_i}})^2 \right)^{1/2} dx'_r.$$

With the above notations, we define the space $L^2(\Gamma)$ by

$$f \in L^2(\Gamma) \iff \int_\Gamma |f|^2 d\sigma < +\infty,$$

and $L^2(\Gamma)$ can be equipped with the norm:

$$|f|_{L^2(\Gamma)} = \left(\int_\Gamma |f|^2 d\sigma\right)^{1/2},$$

which according to [6, page 120], is equivalent to the next one

$$\|f\|_{L^2(\Gamma)} = \left(\sum_{r=1}^{m} \int_{\triangle_r} |f(x'_r, a_r(x'_r))|^2 dx'_r\right)^{1/2}.$$

As is pointed out in [6], we notice that the previous norms are independent of the local coordinate systems considered and independent of the associated partition of unity.

Endowed with $\|\cdot\|_{L^2(\Gamma)}$ (or $|\cdot|_{L^2(\Gamma)}$), $L^2(\Gamma)$ is a reflexive and separable Banach space

In order to simplify the statement below, we use the $\|\cdot\|_{L^2(\Gamma)}$ norm.

Similarly to the above definitions, the space $H^s(\Gamma)$ is defined by:

$$f \in H^s(\Gamma) \iff f_r \in H^s(\triangle_r),$$

for every $r = 1, 2, \ldots, m$ and we can define on $H^s(\Gamma)$ the following norm:

$$\|f\|_{H^s(\Gamma)} = \left(\sum_{r=1}^{m} \|f_r\|^2_{H^s(\triangle_r)}\right)^{1/2},$$

for which $H^s(\Gamma)$ is a reflexive and separable Banach space.

2. Formulation of the problem. In what follows, we describe brievly the mathematical model of the problem which we shall study later. For further details, we refer the reader to [2] and the rich reference therein.

Throughout this paper, $\Omega \subset \mathbb{R}^3$ is a nonempty bounded open domain with its boundary $\Gamma = \partial\Omega$ of class C^∞, we assume that $\Gamma = \Gamma_0 \cup \Gamma_1$ where $\Gamma_\circ$ and Γ_1 are two measurable portions of Γ with $\Gamma_0 \cap \Gamma_1 = \emptyset$.

Let $f \in (H^1(\Omega))^3$ and $g \in (H^{3/2}(\Gamma_1))^3$. In order to fix ideas, and for the reasons explained in the above first section, we will study the problem (S) for the Saint-Venant-Kirchhoff material with limiting values $\lambda = 0$ and $\mu = 1/2$. This restriction, adopted only for simplicity, can be relaxed and the general case treated similarly. Our problem is then formulated as follows:

Find $u \in (H^3(\Omega))^3$ such that

$$(P) \quad \begin{cases} -\text{div}((I + \nabla u)(E(\nabla u))) = f & \text{in } \Omega, \\ ((I + \nabla u)(E(\nabla u))) \cdot \vec{n} = g & \text{on } \Gamma_1, \\ u = 0 & \text{on } \Gamma_0, \end{cases}$$

where, as already noted,

$$E(\nabla u) = 1/2((\nabla u)^t(\nabla u) + (\nabla u)^t + \nabla u),$$

is the nonlinear Green-Saint-Venant strain and $\vec{n}$ the exterior normal vector to Γ.

The linearised problem of (P) is the following:

Find $u \in (H^3(\Omega))^3$ such that

$$(P_\ell) \quad \begin{cases} -\text{div}(\epsilon(u)) = f & \text{in } \Omega, \\ (\epsilon(u)) \cdot \vec{n} = g & \text{on } \Gamma_1, \\ u = 0 & \text{on } \Gamma_0, \end{cases}$$

where

$$\epsilon(u) = (\epsilon_{ij}(u))_{1 \le i,j \le 3} = 1/2(\nabla u + (\nabla u)^t),$$

is the linear Green-Saint-Venant strain.

3. Existence, uniqueness and regularity of solution for (P_ℓ). We start by stating some results for the linearized problem (P_ℓ) which will allow us to study the nonlinear mixed problem (P). For this, we need additional notation. Let us put:

$$\mathbf{V} = \{v \in (H^1(\Omega))^3 \mid v = 0 \text{ on } \Gamma_0\}$$

$$\mathbf{V}_1 = \{v = (v_1, v_1, v_1) \in (H^1(\Omega))^3 \mid v = 0 \text{ on } \Gamma_0\}$$

Clearly, $\mathbf{V}_1 \subset \mathbf{V}$ and $\mathbf{V}_1$, $\mathbf{V}$ are two closed vector subspaces of $(H^1(\Omega))^3$.

For every $v \in (H^1(\Omega))^3$ we define

$$|\epsilon(v)|^2_{\circ,\Omega} = \sum_{i,j=1}^{3} \int_\Omega (\epsilon_{ij}(v))^2.$$

With the above definitions, we have the next proposition.

Proposition 3.1. *The semi-norm $|\epsilon(\cdot)|_{\circ,\Omega}$ is a norm in $\mathbf{V}_1$ equivalent to the norm $\|\cdot\|_{(H^1(\Omega))^3}$ of $(H^1(\Omega))^3$.*

Proof. Since $\mathbf{V}_1 \subset \mathbf{V}$ and $|\epsilon(\cdot)|_{\circ,\Omega}$ is a norm in $\mathbf{V}$ (see [2, Chapter 6]), equivalent to the norm of $(H^1(\Omega))^3$, the desired result follows. □

Proposition 3.2. *Let $f \in (L^2(\Omega))^3$ and $g \in (L^2(\Gamma_1))^3$, then the problem (P_ℓ) has one and only one solution $u \in \mathbf{V}_1$ satisfying:*

$$\forall\, v \in \mathbf{V}_1\,,\ a(u,v) = L(v),$$

where

$$a(u,v) = \sum_{i,j=1}^{3} \int_\Omega \epsilon_{ij}(u)\epsilon_{ij}(v),$$

and

$$L(v) = \int_\Omega f \cdot v + \int_{\Gamma_1} g \cdot v.$$

Proof. Let $u, v \in (H^1(\Omega))^3$, by using the Green's formula and the symmetry of $\epsilon(\,\cdot\,)$, we have:

$$a(u,v) = \sum_{i,j=1}^{3} \int_\Omega \epsilon_{ij}(u)\epsilon_{ij}(v) = \int_\Omega f \cdot v + \int_{\Gamma_1} g \cdot v = L(v).$$

Let us consider the mapping:

$$j : \mathbf{V}_1 \subset (H^1(\Omega))^3 \longrightarrow \mathbb{R}$$
$$u \longmapsto j(u) = (1/2)a(u,u) - L(u)$$

It is easy to see that j is a continuous convex and coercive functional. The space $(H^1(\Omega))^3$ is reflexif, we conclude that j attains its minimum in the closed space $\mathbf{V}_1$.

Since $a(\cdot,\cdot)$ is coercive then the minimum of j is unique and satisfies $a(u,v) = L(v)$, for all $v \in \mathbf{V}_1$. This concludes the proof. □

Now, we are in a position to state our first lemma which will be needed in the sequel.

Lemma 3.1. *Let $g \in (H^{1/2}(\Gamma_1))^3$ and $u = (u_1, u_1, u_1) \in \mathbf{V}_1$ that satisfy $\epsilon(u)\cdot\vec{n} = g$. If we assume that the exterior normal vector $\vec{n} = (n_1, n_2, n_3)$ is such that $n_1+n_2+n_3 \neq 0$ in Γ_1, then $\nabla u_1 \in (H^{1/2}(\Gamma_1))^3$.*

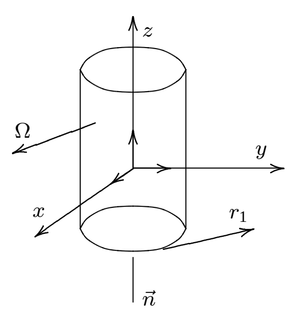
Proof. Let $u = (u_1, u_1, u_1) \in \mathbf{V}_1$, we have successively

$$\epsilon(u)\cdot\vec{n} = g \iff \forall\, i = 1,2,3 \ \sum_{j=1}^{3} \epsilon_{ij}(u)n_j = g_i \iff AU = g,$$

where $U = (\partial_1 u_1, \partial_2 u_1, \partial_3 u_1)$ and $A = (a_{ij})_{1\le i,j\le 3}$ with $a_{11} = 2n_1 + n_2 + n_3$, $a_{22} = 2n_2 + n_1 + n_3$, $a_{33} = n_1 + n_2 + 2n_3$, $a_{21} = a_{31} = n_1$, $a_{32} = a_{12} = n_2$, $a_{13} = a_{23} = n_3$.

The linear system $AU = g$ has one and only one solution U if and only if $\det A = (n_1 + n_2 + n_3)^3 \neq 0$, and in this case $U = A^{-1}g \in (H^{1/2}(\Gamma_1))^3$. This concludes the proof. □

Example 3.1. We state an example illustrating the hypothesis of the previous lemma. Let Ω be the interior of the following cylinder:



Here, we have $\vec{n} = (0,0,-1)$ and $\partial_1 u_1 = -2g_1+g_3$, $\partial_2 u_1 = -2g_2+g_3$, $\partial_3 u_1 = -g_3$.

Now, using the above results we shall prove the following theorem.

Theorem 3.1. *Let $f \in (H^1(\Omega))^3$ and $g \in (H^{1/2}(\Gamma_1))^3$. Assume that Γ_1 satisfies the condition of Lemma 3.1 and $\sup_{\Gamma_1} |g| < +\infty$, then the solution $u \in \mathbf{V}_1$ of the linear mixed problem (P_ℓ) belongs to $(H^2(\Omega))^3$.*

Proof. By Proposition 3.2, let $u = (u_1, u_1, u_1) \in \mathbf{V}_1$ be the solution of (P_l) problem. Since $\epsilon(u) \cdot \vec{n} = g$ in Γ_1, Lemma 3.1 yields $\partial_i u_1 \in H^{1/2}(\Gamma_1)$ for $i = 1, 2, 3$. Because Γ is of class C^∞, let $(x_{r_1}, x_{r_2}, x_{r_3})$ be a cartesian coordinate systems such that:

$$\Gamma_1 \subset \bigcup_{r=1}^{m'} \{(x'_r, a_r(x'_r)) \, , \; x'_r \in \triangle_r\},$$

and

$$\Gamma_0 \subset \bigcup_{r=m'+1}^{m} \{(x'_r \, , \, a_r(x'_r)), \; x'_r \in \triangle_r\}.$$

We shall show that $u_{|\Gamma_1} \in (H^{3/2}(\Gamma_1))^3$. First, we have

$$\|u\|^2_{(H^{3/2}(\Gamma_1))^3} = \sum_{i=1}^{3} \|u_i\|^2_{H^{3/2}(\Gamma_1)} = 3 \, \|u_1\|^2_{H^{3/2}(\Gamma_1)},$$

and, (3) with an elementary transformation yields

$$\|u_1\|^2_{H^{3/2}(\Gamma_1)} = \sum_{r=1}^{m'} \|(u_1)_r\|^2_{H^{3/2}(\triangle_r)} = \sum_{r=1}^{m'} \left(\|(u_1)_r\|^2_{L^2(\triangle_r)} + \|\nabla(u_1)_r\|^2_{(H^{1/2}(\triangle_r))^2} \right).$$

Knowing that $u_{|\Gamma_1} \in (H^{1/2}(\Gamma_1))^3$, we deduce, by definition, that $(u)_r \in (L^2(\triangle_r))^3$ and thus $(u_1)_r \in L^2(\triangle_r)$.

Due to relation (4), we can write

$$\partial_j((u_1)_r) = (\partial_j u_1)_r + (\partial_3 u_1)_r \partial_j a_r, \quad j = 1, 2. \tag{5}$$

Then, there exists $C_1 > 0$ such that

$$\begin{aligned} \|\nabla((u_1)_r)\|^2_{(H^{1/2}(\triangle_r))^2} &= \sum_{k=1}^{2} \|\partial_k((u_1)_r)\|^2_{H^{1/2}(\triangle_r)} \\ &= \sum_{k=1}^{2} \|(\partial_k u_1)_r + (\partial_3 u_1)_r \partial_k a_r\|^2_{H^{1/2}(\triangle_r)} \\ &\leq C_1 \sum_{k=1}^{2} \left(\|(\partial_k u_1)_r\|^2_{H^{1/2}(\triangle_r)} + \|(\partial_3 u_1)_r \partial_k a_r\|^2_{H^{1/2}(\triangle_r)} \right). \end{aligned}$$

Since $\partial_k u_1 \in H^{1/2}(\Gamma_1)$ for $k = 1, 2$ then $(\partial_k u_1)_r \in H^{1/2}(\triangle_r)$.

Moreover, with (3) we can write ($k = 1, 2$):

$$\begin{aligned}
\big\|(\partial_3 u_1)_r \partial_k a_r\big\|^2_{H^{1/2}(\triangle_r)} &= \big\|(\partial_3 u_1)_r \partial_k a_r\big\|^2_{L^2(\triangle_r)} \\
&\quad + \bigg\|\frac{(\partial_3 u_1(x))_r \partial_k a_r(x) - (\partial_3 u_1(y))_r \partial_k a_r(y)}{|x-y|^2}\bigg\|^2_{L^2(\triangle_r\times\triangle_r)} \\
&\le C_2 \big\|(\partial_3 u_1)_r\big\|^2_{L^2(\triangle_r)} \\
&\quad + \bigg\|\frac{(\partial_3 u_1(x))_r \partial_k a_r(x) - (\partial_3 u_1(y))_r \partial_k a_r(y)}{|x-y|^2}\bigg\|^2_{L^2(\triangle_r\times\triangle_r)}
\end{aligned}$$

where $C_2 = \sup_{\triangle_r, k=1,2} |\partial_k a_r|$, and, there exists $C_3 > 0$ such that

$$\begin{aligned}
&\bigg\|\frac{(\partial_3 u_1(x))_r \partial_k a_r(x) - (\partial_3 u_1(y))_r \partial_k a_r(y)}{|x-y|^2}\bigg\|^2_{L^2(\triangle_r\times\triangle_r)} \\
&\qquad \le C_3 \Bigg(\bigg\|\frac{(\partial_k a_r(x) - \partial_k a_r(y))(\partial_3 u_1(y))_r}{|x-y|^2}\bigg\|^2_{L^2(\triangle_r\times\triangle_r)} \\
&\qquad\qquad + \bigg\|\frac{((\partial_3 u_1(x) - \partial_3 u_1(y))_r)\partial_k a_r(x)}{|x-y|^2}\bigg\|^2_{L^2(\triangle_r\times\triangle_r)} \Bigg).
\end{aligned}$$

It becomes that

$$\begin{aligned}
&\bigg\|\frac{(\partial_3 u_1(x))_r \partial_k a_r(x) - (\partial_3 u_1(y))_r \partial_k a_r(y)}{|x-y|^2}\bigg\|^2_{L^2(\triangle_r\times\triangle_r)} \\
&\qquad \le C_3 \sup_{\triangle_r} \big|(\partial_3 u_1)_r\big| \bigg\|\frac{\partial_k a_r(x) - \partial_k a_r(y)}{|x-y|^2}\bigg\|^2_{L^2(\triangle_r\times\triangle_r)} \\
&\qquad\qquad + C_2 C_3 \bigg\| \left(\frac{\partial_3 u_1(x) - \partial_3 u_1(y)}{|x-y|^2}\right)_r \bigg\|^2_{L^2(\triangle_r\times\triangle_r)}.
\end{aligned}$$

For $i = 1, 2, 3$, $\sup_{\Gamma_1} |g_i| < +\infty$, using Lemma 3.1 we obtain $\sup_{\triangle_r} |(\partial_3 u_1)_r| < +\infty$, and with (3)

$$\left(\frac{\partial_3 u_1(x) - \partial_3 u_1(y)}{|x-y|^2}\right)_r \in L^2(\triangle_r \times \triangle_r).$$

Since $a_r \in C^\infty(\triangle_r)$, then $\partial_k a_r \in H^{1/2}(\triangle_r)$ and

$$\frac{\partial_k a_r(x) - \partial_k a_r(y)}{|x-y|^2} \in L^2(\triangle_r \times \triangle_r).$$

We conclude that $\nabla((u_1)_r) \in (H^{1/2}(\triangle_r))^2$, i.e. $(u_1)_{|\Gamma_1} \in H^{3/2}(\Gamma_1)$, or again $u_{|\Gamma_1} \in (H^{3/2}(\Gamma_1))^3$. Summarizing the previous results, with the fact that $u_{|\Gamma_0} = 0$, we have

$$\|u\|^2_{(H^{3/2}(\Gamma))^3} = \sum_{r=1}^{m} \|u\|^2_{(H^{3/2}(\triangle_r))^3} = \sum_{r=1}^{m'} \|u\|^2_{(H^{3/2}(\triangle_r))^3},$$

and consequently, we have establish that $u \in (H^{3/2}(\Gamma))^3$.

The operator $-\mathrm{div}(\epsilon(\cdot))$ is strongly elliptic, $f \in (H^1(\Omega))^3$ and $u \in (H^1(\Omega))^3$, we deduce that (see [4, page 166]) $u \in (H^2(\Omega))^3$ and the desired result is obtained. □

The following theorem, which gives more regularity than the previous one, will enable us to study the nonlinear mixed problem (P).

Theorem 3.2. *Let $f \in (H^1(\Omega))^3$ and $g \in (H^{1/2}(\Gamma_1))^3$. Assume that Γ_1 satisfies the condition of Lemma 3.1,* $\sup_{\Gamma_1} |g| < +\infty$, $\nabla g \in (H^{1/2}(\Gamma_1))^9$ *and* $\sup_{\Gamma_1} |\nabla g| < +\infty$. *Then the solution $u \in \mathbf{V}_1$ of the linear problem (P_ℓ) is in $(H^3(\Omega))^3$.*

Proof. Let $u = (u_1, u_1, u_1) \in \mathbf{V}_1$ be the solution of (P_ℓ). First, we will prove that $u_{|\Gamma_1} \in (H^{5/2}(\Gamma_1))^3$. In fact, one has

$$\|u\|^2_{(H^{5/2}(\Gamma_1))^3} = \sum_{i=1}^{3} \|u_i\|^2_{H^{5/2}(\Gamma_1)} = 3 \|u_1\|^2_{H^{5/2}(\Gamma_1)}$$

and, a simple transformation gives

$$\|u_1\|^2_{H^{5/2}(\Gamma_1)} = \sum_{r=1}^{m'} \left(\|(u_1)_r\|^2_{H^1(\triangle_r)} + \sum_{|\alpha|=2} \|\partial^\alpha((u_1)_r)\|^2_{H^{1/2}(\triangle_r)} \right).$$

According to the proof of Theorem 3.1, we have $(u_1)_r \in H^1(\triangle_r)$.

Now, we show that for $|\alpha| = 2$, $\partial^\alpha((u_1)_r) \in H^{1/2}(\triangle_r)$, i.e. $\partial_{kl}((u_1)_r) \in H^{1/2}(\triangle_r)$ for all $k, \ell = 1, 2$. Using (5), an elementary check yields:

$$\begin{aligned}\|\partial_{k\ell}((u_1)_r)\|^2_{H^{1/2}(\triangle_r)} = \|&(\partial_{k\ell}\, u_1)_r + (\partial_{3\ell}\, u_1)_r \partial_{k\ell}\, a_r + (\partial_{33} u_1)_r \partial_{k\ell}\, a_r \\ &+ (\partial_{3k} u_1)_r \partial_\ell\, a_r + (\partial_3 u_1)_r \partial_{k\ell}\, a_r\|^2_{H^{1/2}(\triangle_r)},\end{aligned}$$

from which we deduce, there exists $C > 0$ such that

$$\begin{aligned}\|\partial_{k\ell}((u_1)_r)\|^2_{H^{1/2}(\triangle_r)} \le C\Big(&\|(\partial_{kl} u_1)_r\|^2_{H^{1/2}(\triangle_r)} + \|(\partial_{3l} u_1)_r \partial_{kl} a_r\|^2_{H^{1/2}(\triangle_r)} \\ &+ \|(\partial_{33} u_1)_r \partial_{k\ell}\, a_r\|^2_{H^{1/2}(\triangle_r)} + \|(\partial_{3k} u_1)_r \partial_\ell\, a_r\|^2_{H^{1/2}(\triangle_r)} \\ &+ \|(\partial_3 u_1)_r \partial_{k\ell}\, a_r\|^2_{H^{1/2}(\triangle_r)}\Big).\end{aligned}$$

Similarly to the proof of Theorem 3.1, we establish that

$$(\partial_{kl} u_1)_r \in H^{1/2}(\triangle_r), \;\; (\partial_{3l} u_1)_r \partial_{kl} a_r \in H^{1/2}(\triangle_r), \;\; (\partial_{33} u_1)_r \partial_{kl} a_r \in H^{1/2}(\triangle_r),$$
$$(\partial_{3k} u_1)_r \partial_l a_r \in H^{1/2}(\triangle_r) \;\text{ and }\; (\partial_3 u_1)_r \partial_{kl} a_r \in H^{1/2}(\triangle_r).$$

In summary, we have showed that $\partial_{k\ell}((u_1)_r) \in H^{1/2}(\triangle_r)$, that is for $|\alpha| = 2$, $(\partial^\alpha((u_1)_r)) \in H^{1/2}(\triangle_r)$, and consequently $u_1 \in H^{5/2}(\Gamma_1)$. Since $u_{|\Gamma_0} = 0$, we have $\|u\|_{(H^{5/2}(\Gamma_1))^3} = \|u\|_{(H^{5/2}(\Gamma))^3}$, and then $u \in (H^{5/2}(\Gamma))^3$.

The operator $u \longrightarrow -\text{div}(\nabla u)$ is strongly elliptic, $u \in (H^{5/2}(\Gamma))^3$, $u \in (H^1(\Omega))^3$ and $f \in (H^1(\Omega))^3$, we deduce (see [4, page 166]) that $u \in (H^3(\Omega))^3$, and Theorem 3.2 is proved. □

4. Existence and uniqueness solution for the problem (P). As already pointed, our aim in this section is to study the nonlinear mixed problem (P). Combining the results of the previous section, we shall prove the next theorem which is the original motivation of this paper.

Theorem 4.1. *Let $f \in (H^1(\Omega))^3$ and $g \in (H^{1/2}(\Gamma_1))^3$. Assume that Γ_1 verifies the condition of Lemma 3.1, $\nabla g \in (H^{1/2}(\Gamma_1))^9$, $\sup_{\Gamma_1} |g| < +\infty$ and $\sup_{\Gamma_1} |\nabla g| < +\infty$. If moreover $\|f\|_{(H^1(\Omega))^3}$ and $\|g\|_{(H^{3/2}(\Gamma_1))^3}$ are small, then the nonlinear mixed problem (P) has one and only one solution $u \in \mathbf{V}_2 := \mathbf{V}_1 \cap (H^3(\Omega))^3$ of small norm.*

Proof. Let us put that $A_1 u = (I + \nabla u)E(\nabla u)$. According to [2] with the continuous injection $H^2(\Omega) \hookrightarrow C^\circ(\overline{\Omega})$, we deduce that the following linear operator

$$\begin{aligned} \tilde{A}_1 : \mathbf{V}_2 &\longrightarrow (H^1(\Omega))^3 \\ u &\longrightarrow -\mathrm{div}(A_1 u) \end{aligned}$$

is defined and infinitely Fréchet differentiable, and $\tilde{A}_1(0) = 0$.

Now, let $\tilde{A}_2 u = A_1 u \cdot \vec{n}$ where

$$\begin{array}{rcccl} \tilde{A}_2 : \mathbf{V}_2 & \longrightarrow & \left(H^2(\Omega)\right)^3 & \longrightarrow & (H^{3/2}(\Gamma_1))^3 \\ u & \longrightarrow & A_1 u & \longrightarrow & A_1 u \cdot \vec{n} \end{array}$$

By the same arguments as previous, the operator A_1 is defined and infinitely Fréchet differentiable, the linear operator " trace" is continuous, then the operator $\tilde{A}_2$ is defined and infinitely Fréchet differentiable with $\tilde{A}_2(0) = 0$.

We put $Du = (\tilde{A}_1 u, \tilde{A}_2 u)$, where $D'(u)(0) = (-\mathrm{div}(\epsilon(u), \epsilon(u) \cdot \vec{n})$ is a continuous linear operator. Due to Theorem 3.2, the linear problem (P_ℓ) has one and only one solution $u \in \mathbf{V}_2$ and so $D'(u)(0)$ is bijective.

By virtue of the closed graph theorem, we can deduce that

$$D'(u)(0) \in \mathrm{isom}(\mathbf{V}_2, (H^1(\Omega))^3 \times (H^{3/2}(\Gamma_1))^3) \,.$$

According to the implicit function theorem, there exist a neighbourhood W_1 of 0 in $\mathbf{V}_2$ and a neighbourhood W_2 of 0 in $(H^1(\Omega))^3 \times (H^{3/2}(\Gamma_1))^3$ such that for every $(f_1, g_1) \in W_2$ the problem $Du = (f_1, g_1)$ has one and only one solution $u \in W_1$.

Since $\|f\|_{(H^1(\Omega))^3}$ and $\|g\|_{(H^{3/2}(\Gamma_1))^3}$ are small, we conclude that the problem $Du = (f, g)$ has one and only one solution $u \in \mathbf{V}_2$ of small norm, i.e. the nonlinear mixed problem (P) has a unique solution $u \in \mathbf{V}_2$ of small norm. This completes the proof. □

An immediate consequence of Theorem 4.1 is the next corollary.

Corollary 4.1. *Let Γ_1 as in Lemma 3.1 and suppose that $f = 0$ and $g = 0$. Then $u = 0$ is the unique solution in $\mathbf{V}_2$ for the problem (P).*

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Résumé substantiel en français. Soit $\Omega \subset \mathbb{R}^3$ un ouvert borné de frontière $\Gamma = \partial\Omega$ suffisamment régulière. On suppose que $\Gamma = \Gamma_0 \cup \Gamma_1$ avec Γ_0 et Γ_1 deux portions de Γ telles que $\Gamma_0 \cap \Gamma_1 = \emptyset$.

Soient $f \in (H^1(\Omega))^3$ et $g \in (H^{3/2}(\Gamma_1))^3$ donnés. On considère le problème mixte (non linéaire) suivant :

Trouver $u \in (H^3(\Omega))^3$ tel que

$$(P) \quad \begin{cases} -\text{div}((I+\nabla u)(E(\nabla u))) = f & \text{dans } \Omega, \\ ((I+\nabla u)(E(\nabla u))) \cdot \vec{n} = g & \text{sur } \Gamma_1, \\ u = 0 & \text{sur } \Gamma_0 \end{cases}$$

où

$$E(\nabla u) = 1/2((\nabla u)^t(\nabla u) + (\nabla u)^t + \nabla u)$$

est le tenseur (non linéaire) de Green-Saint-Venant et $\vec{n}$ est le vecteur normal exterieur à Γ. Le problème mixte linéarisé de (P) est :

Trouver $u \in (H^3(\Omega))^3$ satisfaisant

$$(P_\ell) \quad \begin{cases} -\text{div}(\epsilon(u)) = f & \text{dans } \Omega, \\ (\epsilon(u)) \cdot \vec{n} = g & \text{sur } \Gamma_1, \\ u = 0 & \text{sur } \Gamma_0 \end{cases}$$

où

$$\epsilon(u) = (\epsilon_{ij}(u))_{1\leq i,j\leq 3} = 1/2(\nabla u + (\nabla u)^t),$$

est le tenseur linéaire de Green-Saint-Venant.

Les problèmes (P) et (P_ℓ) occupent une place importante en élasticité tridimensionnelle. Dans [2], P. G. Ciarlet a montré que le problème mixte non linéaire (P) avec condition de Dirichlet (c'est-à-dire $\Gamma_1 = \emptyset$) admet, au voisinage de 0, une solution unique lorsque la force de volume f est de norme assez petite. Un tel problème avec conditions mixtes (de Dirichlet et Neuman) est, jusqu'à présent, encore ouvert.

Le but de ce travail est de montrer, d'abord, que le problème mixte linéaire (P_ℓ) admet une solution unique régulière, et d'en déduire ensuite l'existence (locale), l'unicité et la régularité de solution du problème mixte non linéaire (P).

Avant d'enoncer nos résultats, nous avons besoin de préciser quelques notations utiles par la suite.

Soient

$$\mathbf{V} = \{v \in (H^1(\Omega))^3 \mid v = 0 \text{ sur } \Gamma_0\}$$

$$\mathbf{V}_1 = \{v = (v_1, v_1, v_1) \in (H^1(\Omega))^3 \mid v = 0 \text{ sur } \Gamma_0\}.$$

Il est clair que $\mathbf{V}_1 \subset \mathbf{V}$ et que $\mathbf{V}_1$, $\mathbf{V}$ sont deux sous-espaces fermés de $(H^1(\Omega))^3$. Nous avons besoin de faire l'hypothèse suivante :

(H) Pour $g \in (H^{1/2}(\Gamma_1))^3$ et $u = (u_1, u_1, u_1) \in \mathbf{V}_1$ tel que $\epsilon(u) \cdot \vec{n} = g$, on supposera que le vecteur normal exterieur $\vec{n} = (n_1, n_2, n_3)$ satisfait $n_1 + n_2 + n_3 \neq 0$ sur Γ_1.

Notre premier résultat fondamental est le suivant :

Théorème 3.2. *Soient $f \in (H^1(\Omega))^3$ et $g \in (H^{1/2}(\Gamma_1))^3$. Supposons que l'hypothèse (H) soit satisfaite et que, $\sup_{\Gamma_1} |g| < +\infty$, $\nabla g \in (H^{1/2}(\Gamma_1))^9$ et $\sup_{\Gamma_1} |\nabla g| < +\infty$. Alors le problème mixte linéaire (P_ℓ) admet une solution unique dans $u \in \mathbf{V}_2 := \mathbf{V}_1 \cap (H^3(\Omega))^3$.*

Utilisant le théorème des fonctions implicites, le résultat précédent nous permet de déduire le suivant :

Théorème 4.1. *Soient* $f \in (H^1(\Omega))^3$ *et* $g \in (H^{1/2}(\Gamma_1))^3$*. Outre l'hypothèse (H), on suppose que,* $\nabla g \in (H^{1/2}(\Gamma_1))^9$, $\sup_{\Gamma_1} |g| < +\infty$ *et* $\sup_{\Gamma_1} |\nabla g| < +\infty$*. Si de plus* $\|f\|_{(H^1(\Omega))^3}$ *et* $\|g\|_{(H^{3/2}(\Gamma_1))^3}$ *sont assez petits, alors le problème mixte non linéaire* (P) *admet une solution et une seule* $u \in \mathbf{V}_2 := \mathbf{V}_1 \cap (H^3(\Omega))^3$ *de norme assez petite.*

References

1. J. M. Ball, *Convexity conditions and existence theorems in nonlinear elasticity*, Arch. Rational Mech. Anal. **63** (1976/77), 337–403.
2. P. G. Ciarlet, *Mathematical elasticity. Vol. I. Three dimensional elasticity*, Studies in Mathematics and its Applications, 20, North-Holland, Amsterdam, 1988.
3. P. G. Ciarlet and G. Geymonat, *Sur les lois de comportement en élasticité non linéaire compressible*, C. R. Acad. Sci. Paris, Sér. II **295** (1982), 423–426.
4. J. L. Lions and E. Magenes, *Problèmes aux limites non homogènes et applications*, Travaux et Recherches Mathématiques, No. 18, vol. **2**, Dunod, Paris, 1968.
5. J. E. Marsden and T. J. R. Hughes, *Mathematical Foundations of Elasticity*, Prentice-Hall, Englewood Cliffs, 1983.
6. J. Nečas, *Les méthodes directes en théorie des équations elliptiques*, Masson, Paris, 1967.
7. M. Raïssouli, *Sur un probléme mixte de Saint-Venant en élasticité non linéaire multidimensionnelle*, Thèse de doctorat, Université Paul Sabatier, Toulouse, 1986.

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