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EXPANSIONS IN EVEN GENERALIZED SPHERICAL HARMONICS IN $\mathbb{R}^{k+1}$

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RÉSUMÉ. On démontre que les produits de polynômes de Jacobi, lorsque transformés des coordonnées sphériques aux coordonnées cartésiennes, forment une base orthonormée du sous-espace de $L^2_d(\mathbf{S}^k_+)$ composé des fonctions sphériques harmoniques généralisées d'ordre pair. Nous étudions ensuite les problèmes intérieurs et extérieurs de Dirichlet sur la sphère unitaire de dimension k.

ABSTRACT. It is shown that the products of Jacobi polynomials transformed from spherical coordinates system to cartesian coordinates system form an orthonormal basis of the subspace $\mathcal{K}^{k+1}_m$ of even generalized spherical harmonics in $L^2_d(\mathbf{S}^k_+)$. The interior and exterior Dirichlet problems on k-dimensional unit sphere are then investigated.

1. Introduction. We consider Dunkl's equation

$$\Delta_\rho u := \sum_{j=0}^{k} \mathcal{D}_j^2 u = 0, \tag{1}$$

where $\mathcal{D}_j$ is a modification of the jth partial derivative

$$\mathcal{D}_j u(x) := \partial_j u(x) + \Big(\rho_j - \frac{1}{2}\Big)\frac{u(x) - u(\sigma_j x)}{x_j}, \quad x = (x_0, \ldots, x_k) \in \mathbb{R}^{k+1},$$

and $\sigma_j : \mathbb{R}^{k+1} \to \mathbb{R}^{k+1}$ denotes the reflection

$$\sigma_j(x_0, \ldots, x_k) = (x_0, \ldots, x_{j-1}, -x_j, x_{j+1}, \ldots, x_k).$$

The parameters $\rho_0, \ldots, \rho_k$ in the functional-differential equation (1) are assumed to be positive. We consider these parameters to be fixed and usually do not indicate the dependence on $\rho_0, \ldots, \rho_k$ in our notations. Equation (1) reduces to the Laplace equation

$$\Delta u = 0 \tag{2}$$

when $\rho_0 = \rho_1 = \cdots = \rho_k = \frac{1}{2}$.

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It is well-known that an orthogonal basis in the space of spherical harmonics of a given degree n can be found by applying the method of separation of variables to equation (2) in spherical coordinates. We cannot directly apply the method of separation of variables to equation (1). We first restrict our attention to even functions u, that is, $u(x) = u(\sigma_j(x))$ for all $j = 0, \dots, k$. Then equation (1) reduces to the partial differential equation

$$\mathcal{L}u(x) := \Delta u(x) + \sum_{j=0}^{\infty} \Big(\frac{2\rho_j - 1}{x_j}\Big) \partial u_j(x) = 0\,. \tag{3}$$

We introduce the spherical coordinates $(\eta_0, \eta_1, \dots, \eta_k)$ in $\mathbb{R}_+^{k+1} = \{(x_0, \dots, x_k) : x_i > 0 \text{ for all } i\}$. For a given point $(x_0, \dots, x_k) \in \mathbb{R}_+^{k+1}$, its spherical coordinates, $\eta_0 > 0$ and $0 < \eta_i < 1$ for $1 \le i \le k$, are defined by

$$\begin{aligned} \eta_0 &= \sum_{i=0}^{k} x_i^2 \\ \eta_{j+1} &= \frac{x_j^2}{\sum_{s=j}^{k} x_s^2}\,, \quad 0 \le j \le k-1. \end{aligned} \tag{4}$$

Then the cartesian coordinates are given by

$$\begin{aligned} x_0^2 &= \eta_0 \eta_1 \\ x_p^2 &= \eta_0 \eta_{p+1} \prod_{s=1}^{p} (1 - \eta_s)\,, \quad 1 \le p \le k-1 \\ x_k^2 &= \eta_0 \prod_{s=1}^{k} (1 - \eta_s). \end{aligned} \tag{5}$$

This paper is organized as follows. In section 2, the operator $\mathcal{L}$ is transformed from cartesian coordinates to spherical coordinates. The operator $\mathcal{L}$ is then separated to $k+1$ differential equations. In section 3, we show how separation of variables in spherical coordinates leads to expressions of generalized spherical harmonics in terms of Jacobi polynomials. In section 4, we show that the products of solutions of $k+1$ differential equations form indeed an orthonormal basis in the space $\mathcal{K}_m^{k+1}$ of even generalized spherical harmonics.

In section 5, we investigate the interior and exterior Dirichlet problems for the equation (3) on the k-dimensional sphere $\mathbf{S}^k = \{(x_0, \dots, x_k) \in \mathbb{R}^{k+1} : \sum_{i=0}^{k} x_i^2 = 1\}$ by infinite series expansions in even generalized spherical harmonics.

2. Transformation and separation of the operator $\mathcal{L}$ in spherical coordinates. We use relations (4) and chain rule to rewrite the equation $\mathcal{L}u = 0$ in spherical coordinates as $\mathcal{L}v = 0$ where $v(\eta_0, \eta_1, \dots, \eta_k) = u(x_0, x_1, \dots, x_k)$:

$$\sum_{j=0}^{k} \sum_{i=0}^{k} \left\{ \frac{\partial v}{\partial \eta_i} \frac{\partial^2 \eta_i}{\partial {x_j}^2} + \sum_{\ell=0}^{k} \frac{\partial \eta_i}{\partial x_j} \frac{\partial \eta_\ell}{\partial x_j} \frac{\partial^2 v}{\partial \eta_\ell \eta_i} + \frac{2\rho_j - 1}{x_j} \frac{\partial v}{\partial \eta_i} \frac{\partial \eta_i}{\partial x_j} \right\} = 0 \tag{6}$$

where

$$\sum_{j=0}^{k} \frac{\partial \eta_i}{\partial x_j \frac{\partial \eta_\ell}{\partial x_j}} = 0 \quad \text{if } i \neq \ell.$$

We look for the solutions to (6) in the form $v(\eta_0, \eta_1, \dots, \eta_k) = v_0(\eta_0) v_1(\eta_1) \cdots v_k(\eta_k)$.

According to Schmidt and Wolf [11], separation of variables for (6) leads to the following $k + 1$ ordinary differential equations:

$$\eta_0^2 v_0'' + \left(\sum_{i=0}^{k} \rho_i \right) \eta_0 v_0' + \lambda_{k-1} v_0 = 0, \tag{7}$$

$$\eta_j (1 - \eta_j) \left\{ v_j'' + \left(\frac{\rho_{j-1}}{\eta_j} - \frac{\sum_{t=j}^{k} \rho_t}{1 - \eta_j} \right) v_j' \right\} + (-1)^j \left(\lambda_{k-j} + \frac{\lambda_{k-j-1}}{1 - \eta_j} \right) v_j = 0 \tag{8}$$

with $v_j = v_j(\eta_j)$ for $1 \leq j \leq k$ and where $\lambda_0, \dots, \lambda_{k-1}$ are separation parameters. We set $\lambda_{-1} = 0$.

3. Separated solutions of the operator $\mathcal{L}$. The parameters $\lambda_0, \lambda_1, \dots, \lambda_{k-1}$ will be chosen in such a way that (7) and (8) admit solutions which are nonzero polynomials.

For $0 \leq i \leq k - 1$, $m_i \in \mathbb{Z}$ (the set of integers) and $0 \leq m_{k-1} \leq \cdots \leq m_0$, we set

$$\begin{aligned}
\lambda_{k-1} &= -m_0 \left(m_0 + \sum_{t=0}^{k} \rho_t - 1 \right), \quad m_0 \geq 0 \\
\lambda_{k-2} &= -m_1 \left(m_1 + \sum_{t=1}^{k} \rho_t - 1 \right), \quad m_0 \geq m_1 \\
\lambda_{k-3} &= -m_2 \left(m_2 + \sum_{t=2}^{k} \rho_t - 1 \right), \quad m_1 \geq m_2 \\
&\vdots \\
\lambda_1 &= -m_{k-2} \left(m_{k-2} + \sum_{t=k-2}^{k} \rho_t - 1 \right), \quad m_{k-3} \geq m_{k-2} \\
\lambda_0 &= -m_{k-1} \left(m_{k-1} + \sum_{t=k-1}^{k} \rho_t - 1 \right), \quad m_{k-2} \geq m_{k-1} \geq 0\,.
\end{aligned}$$

Then the Euler equation (7) has the polynomial solution

$$v_0(\eta_0) = \eta_0^{m_0}. \tag{9}$$

For each $1 \leq j \leq k-1$, $k \geq 2$, equation (8) is hypergeometric. Its polynomial solution is an hypergeometric function in η_j. In here we identify them as Jacobi polynomial in the variable $1 - 2\eta_j$. Specifically we have, for $j = 1, \dots, k - 1$, $k \geq 2$,

$$v_j(\eta_j) = (1 - \eta_j)^{m_j} P_{m_{j-1} - m_j}^{(\rho_{j-1} - 1,\, 2m_j + \sum_{t=j}^{k} \rho_t - 1)} (1 - 2\eta_j). \tag{10}$$

For $j = k$, the solution of (8) is the Jacobi polynomial:

$$v_k(\eta_k) = P_{m_{k-1}}^{(\rho_{k-1}-1,\,\rho_k-1)}(1-2\eta_k). \tag{11}$$

If we set $m_k = 0$, then (11) is a special case of (10). Therefore a solution of (6) is the product

$$v(\eta_0, \eta_1, \ldots, \eta_k) = \prod_{i=0}^{k} v_i(\eta_i). \tag{12}$$

For $m = (m_0, m_1, \ldots, m_{k-1})$, we introduce the following notations:

$$\begin{aligned} v_0(\eta_0) &= J_{m,0}^{m_0}(\eta_0) \\ v_i(\eta_i) &= J_{m,i}^{m_{i-1}}(\eta_i) \text{ for } 1 \le i \le k, \\ J^m(\eta) &= \prod_{i=1}^{k} J_{m,i}^{m_{i-1}}(\eta_i). \end{aligned} \tag{13}$$

By (13) the polynomial $J_{m,i}^{m_{i-1}}(\eta_i)$ is of degree m_{i-1} in the variables η_i for $i = 1, \ldots, k$.

Lemma 3.1. *The products $J_{m,0}^{m_0}(\eta_0) \prod_{i=1}^{k} J_{m,i}^{m_{i-1}}(\eta_i)$ are homogeneous polynomials in Cartesian coordinates $(x_0, \ldots, x_k)$ of degree m_0 in variables $x_0^2, \ldots, x_k^2$.*

Proof. Let

$$q_1 = m_0 - m_1 \ge 0, \ldots, q_{k-1} = m_{k-2} - m_{k-1} \ge 0, q_k = m_{k-1}$$

and, for $j = 1, \ldots, k$,

$$b_j = m_{j-1} + m_j + \sum_{t=j-1}^{k} \rho_t - 1, \quad c_j = \rho_{j-1}, \quad \ell_j = \frac{(-1)^{q_j}(\rho_{j-1})_{q_j}}{\Gamma(q_j+1)} .$$

Then we may use the properties of hypergeometric functions to rewrite equations (10) and (11) as follows:

$$\begin{aligned} J_{m,j}^{m_{j-1}}(\eta_j) &= \ell_j(1-\eta_j)^{m_j} {}_2F_1(m_j - m_{j-1}, m_j + m_{j-1} \sum_{t=j-1}^{k} \rho_t - 1; \eta_j) \\ &= \ell_j(1-\eta_j)^{m_j} {}_2F_1(-q_j, b_j; c_j; \eta_j) \\ &= \ell_j(1-\eta_j)^{m_j} \sum_{p_j=0}^{q_j} \frac{(-q_j)_{p_j}(b_j)_{p_j}}{(c_j)_{p_j}\Gamma(p_j+1)} \eta_j^{p_j} \end{aligned}$$

where ${}_2F_1(a, b; c; x)$ denotes the hypergeometric series where the suffixes 2 and 1 are two parameters of the type a and one of the type c. The product $J_{m,0}^{m_0}(\eta_0) J^m(\eta_i)$ is computed as follows:

$$\begin{aligned}
f(\eta) &= \eta_0^{m_0} \prod_{j=1}^{k} \ell_j (1-\eta_j)^{m_j} {}_2F_1(-q_j, b_j; c_j; \eta_j) \\
&= \eta_0^{m_0} \prod_{j=1}^{k} \left[\ell_j (1-\eta_j)^{m_j} \sum_{p_j=0}^{q_j} \frac{(-q_j)_{p_j} (b_j)_{p_j}}{(c)_{p_j} \Gamma(p_j+1)} \, \eta_j^{p_j} \right] \\
&= \prod_{j=1}^{k} \left[\ell_j (1-\eta_j)^{m_j} \sum_{p_j=0}^{q_j} \frac{(-q_j)_{p_j} (b_j)_{p_j}}{(c_j)_{p_j} \Gamma(p_j+1)} \, \eta_j^{p_j} \eta_0^{m_0} \right] \\
&= \prod_{j=1}^{k} \left[\sum_{p_j=0}^{q_j} \frac{\ell_j (-q_j)_{p_j} (b_j)_{p_j}}{(c_j)_{p_j} \Gamma(p_j+1)} \, \eta_j^{p_j} (1-\eta_j)^{m_j} \eta_0^{m_0} \right] \\
&= \prod_{s=1}^{k} \left[\sum_{p_s=0}^{q_s} \underbrace{\left(\prod_{j=1}^{k} \frac{\ell_j (-q_j)_{p_j} (b_j)_{p_j}}{(c_j)_{p_j} \Gamma(p_j+1)} \right)}_{A_p} \eta_s^{p_s} (1-\eta_s)^{m_s} \eta_0^{m_0} \right] \\
&= \prod_{s=1}^{k} \sum_{p_s=0}^{q_s} A_p \, \eta_0^{m_0} \, \eta_s^{p_s} (1-\eta_s)^{m_s} \, .
\end{aligned}$$

We set $z_j = x_j^2$ for $j = 0, \ldots, k$, and use the relations (4) to rewrite the product $f(\eta)$ as

$$\begin{aligned}
f(z) = \sum_{p_1=0}^{q_1} \cdots \sum_{p_k=0}^{q_k} A_p \, z_0^{p_1} \cdots z_{k-1}^{p_k} \left(\sum_{s=0}^{k} z_s \right)^{m_0 - m_1 - p_1} & \times \\
\left(\sum_{s=1}^{k} z_s \right)^{m_1 - m_2 - p_2} \times \cdots \times \left(\sum_{s=k-1}^{k} z_s \right)^{m_{k-1} - m_k - p_k} & \\
= \sum_{p_1=0}^{q_1} \cdots \sum_{p_k=0}^{q_k} A_p \, z_0^{p_1} \left(\sum_{s=0}^{k} z_s \right)^{m_0 - m_1 - p_1} & \times \\
z_1^{p_2} \left(\sum_{s=1}^{k} z_s \right)^{m_1 - m_2 - p_2} \times \cdots \times z_{k-1}^{p_k} \left(\sum_{s=k-1}^{k} z_s \right)^{m_{k-1} - m_k - p_k} & \\
= \sum_{p_k=0}^{q_k} \cdots \sum_{p_2=0}^{q_2} \left(\sum_{p_1=0}^{q_1} A_p z_0^{p_1} \left(\sum_{s=0}^{k} z_s \right)^{m_0 - m_1 - p_1} \right) & \times \\
z_1^{p_2} \left(\sum_{s=1}^{k} z_s \right)^{m_1 - m_2 - p_2} \times \cdots \times z_{k-1}^{p_k} \left(\sum_{s=k-1}^{k} z_s \right)^{m_{k-1} - m_k - p_k} & .
\end{aligned}$$

We compute $f(rz)$

$$
\begin{aligned}
f(rz) &= \sum_{p_1=0}^{q_1}\cdots\sum_{p_k=0}^{q_k} A_p (rz_0)^{p_1}\left(\sum_{s=0}^{k}(rz_s)\right)^{m_0-m_1-p_1}(rz_1)^{p_2}\times \\
&\qquad \left(\sum_{s=1}^{k}(rz_s)\right)^{m_1-m_2-p_2}\times\cdots\times(rz_{k-1})^{p_k}\left(\sum_{s=k-1}^{k}(rz_s)\right)^{m_{k-1}-m_k-p_k} \\
&= r^{m_0}\sum_{p_k=0}^{q_k}\cdots\sum_{p_2=0}^{q_2}\left(\sum_{p_1=0}^{q_1}A_p z_0^{p_1}\left(\sum_{s=0}^{k}z_s\right)^{m_0-m_1-p_1}\right)\times \\
&\qquad z_1^{p_2}\left(\sum_{s=1}^{k}z_s\right)^{m_1-m_2-p_2}\times\cdots\times z_{k-1}^{p_k}\left(\sum_{s=k-1}^{k}z_s\right)^{m_{k-1}-m_k-p_k} \\
&= r^{m_0}f(z).
\end{aligned}
$$

Hence the polynomial $f(z)$ is homogeneous of degree m_0 in the variables $z_0,\dots,z_k$. That concludes the proof. □

4. Orthogonality. We are going to show that the system of products

$$v_1(\eta_1)v_2(\eta_2)\cdots v_k(\eta_k),\quad \eta_j\in(0,1)$$

of polynomials is mutually orthogonal over the subset

$$Q=\underbrace{(0,1)\times(0,1)\times(0,1)\times\cdots\times(0,1)}_{k\ \text{times}}=(0,1)^k \tag{14}$$

of the Cartesian space $\mathbb{R}^k$ with respect to the continuous, integrable, and positive weight function

$$w(\eta_1,\dots,\eta_k)=\eta_k^{\rho_{k-1}-1}\big(1-\eta_k\big)^{\rho_k-1}\prod_{\ell=1}^{k-1}\eta_\ell^{\rho_{\ell-1}-1}\big(1-\eta_\ell\big)^{\sum_{t=\ell}^{k}\rho_t-1} \tag{15}$$

where $w(\eta_1,\dots,\eta_k)=w_k(\eta_k)w_1(\eta_1)\cdots w_{k-1}(\eta_{k-1})$.

Let $L_w^2(Q)$ be the space of measurable functions $f:Q\to\mathbb{R}$ with

$$\int_Q w(\eta)f^2(\eta)\,\mathrm{d}\eta<\infty,\quad \eta=(\eta_1,\dots,\eta_k).$$

The space $L_w^2(Q)$ is equipped with the inner product

$$\langle f,g\rangle_w=\int_Q w(\eta)f(\eta)g(\eta)\,\mathrm{d}\eta \tag{16}$$

and norm

$$\|f\|_w=\Big(\int_Q w(\eta)f(\eta)^2\,\mathrm{d}\eta\Big)^{\frac12}=\big(\langle f,f\rangle_w\big)^{\frac12} \tag{17}$$

turning it into a Hilbert space.

Theorem 4.1. *If $m = (m_0, \ldots, m_{k-1})$ and $\ell = (\ell_0, \ldots, \ell_{k-1})$ are two distinct multi-indices, then*

$$\int_Q w(\eta) J^m(\eta) J^\ell(\eta)\, \mathrm{d}\eta = 0\,. \tag{18}$$

Proof. For any multi-indices m and ℓ, we have

$$\begin{aligned}\int_Q w(\eta) J^m(\eta) J^\ell(\eta)\, \mathrm{d}\eta &= \int \cdots \int_{(0,1)^k} \prod_{i=1}^k w_i(\eta_i) \prod_{i=1}^k J_{m,i}^{m_{i-1}}(\eta_i) \prod_{i=1}^k J_{\ell,i}^{\ell_{i-1}}(\eta_i)\, \mathrm{d}\eta \\ &= \prod_{i=1}^k \int_0^1 w_i(\eta_i) J_{m,i}^{m_{i-1}}(\eta_i) J_{\ell,i}^{\ell_{i-1}}(\eta_i)\, \mathrm{d}\eta_i\,.\end{aligned}$$

If $m \neq \ell$, let j be the largest integer in $\{1, \ldots, k\}$ for which $m_{j-1} \neq \ell_{j-1}$. By the properties of Jacobi polynomials, the integral involving η_j is then equal to zero. This completes the proof. $\square$

For $0 \leq m_{k-1} \leq m_{k-2} \leq \cdots \leq m_0$, the function $J_{m,0}^{m_0}(\eta_0) \prod_{i=1}^k J_{m,i}^{m_{i-1}}(\eta_i)$ when transformed to coordinates $x_0, x_1, \ldots, x_k$ is an homogeneous polynomial in $x_0^2, x_1^2, \ldots, x_k^2$ of degree m_0 (Lemma 3.1) and it solves $\mathcal{L}u = 0$. Thus it is in $\mathcal{K}_{m_0}^{k+1}$, the space of even generalized harmonics.

For a fixed m_0, let M_{m_0} be the set of multi-indices $m = (m_1, \ldots, m_{k-1})$ of nonnegative integers such that $m_{k-1} < m_{k-2} \leq \cdots \leq m_0$. Given m_0, Theorem 4.1 shows that the system $\left\{ J^m(\eta),\, m \in M_{m_0} \right\}$ is linearly independent. Hence, since the dimension of $\mathcal{K}_{m_0}^{k+1}$ is equal to the cardinality $\binom{m_0+k-1}{k-1}$ of the set M_{m_0}, this system forms a basis.

We define the constant $p_m = \prod_{j=1}^k p_{m_{j-1}, m_j}$ (where $m_k = 0$) by

$$\begin{aligned} p_{m_{j-1}, m_j} = &\left(\frac{1}{2m_{j-1} + \rho_{j-1} + \sum_{t=j}^k \rho_t - 1} \right) \\ &\times \left(\frac{\Gamma(m_{j-1} - m_j + \rho_{j-1})}{\Gamma(m_{j-1} - m_j + 1)} \right) \\ &\times \left(\frac{\Gamma(m_{j-1} + m_j + \sum_{t=j}^k \rho_t)}{\Gamma(m_{j-1} + \rho_{j-1} + m_j + \sum_{t=j}^k \rho_j - 1)} \right)\end{aligned}$$

when $j = 1, \ldots, k-1$ and

$$p_{m_{k-1}, m_k} = \frac{\Gamma(m_{k-1} + \rho_{k-1}) \Gamma(m_{k-1} + \rho_k)}{(2m_{k-1} + \rho_{k-1} + \rho_k - 1)\Gamma(m_{k-1} + 1)\Gamma(m_{k-1} + \rho_{k-1} + \rho_k - 1)}\,.$$

For every $j = 1, \ldots, k$ we have

$$\int_0^1 w_j(\eta_j) (J_{m,j}^{m_{j-1}}(\eta_j))^2\, \mathrm{d}\eta_j = p_{m_{j-1}, m_j} \tag{19}$$

so

$$\int_Q w(\eta) (J^m(\eta))^2\, \mathrm{d}\eta = p_m\,. \tag{20}$$

Then the product

$$\tilde{J}^m(\eta) = \prod_{j=1}^{k} \left[\frac{J_{m,j}^{m_{j-1}}(\eta_j)}{(p_{m_{j-1},m_j})^{1/2}} \right] = \prod_{j=1}^{k} \tilde{J}_{m,j}^{m_{j-1}}(\eta_j)$$

is normalized such that

$$\int_Q w(\eta)(\tilde{J}^m(\eta))^2 \mathrm{d}\eta = 1. \tag{21}$$

Therefore this system of products $\{ \tilde{J}^m(\eta),\, m \in M_{m_0} \}$ is orthonormal in $L_w^2(Q)$.

Theorem 4.2. *The set of the functions* $\{ \tilde{J}^m(\eta),\, m \in M_{m_0} \}$ *is complete in* $L_w^2(Q)$.

Proof. Let $f \in L_w^2(Q)$ be given with $\langle f, \tilde{J}^m(\eta) \rangle = 0$ for all multi-indices $m = (m_0, \ldots, m_{k-1}) \in \mathbb{N}_0^k$ of nonnegative integers. Then

$$0 = \langle f, \tilde{J}^m(\eta) \rangle = \int_Q w(\eta) f(\eta) \tilde{J}^m(\eta)\, \mathrm{d}\eta = \int_0^1 w_k(\eta_k) \tilde{J}_{m,k}^{m_{k-1}}(\eta_k) g_1(\eta_k)\, \mathrm{d}\eta_k$$

where

$$g_1(\eta_k) = \int_{(0,1)^{k-1}} \Big(\prod_{i=1}^{k-1} w_i(\eta_i) \tilde{J}_{m,i}^{m_{i-1}}(\eta_i) \Big) f(\eta_1, \ldots, \eta_k)\, \mathrm{d}\eta_1 \cdots \mathrm{d}\eta_{k-1}\,.$$

Let $m_0, m_1, \ldots, m_{k-2}$ be fixed. Since $\tilde{J}_{m,k}^{m_{k-1}}(\eta_k),\, m_{k-1} \in \mathbb{N}_0$ is complete in $L_{w_k}^2(0,1)$, we obtain $g_1(\eta_k) = 0$ for a.e. $\eta_k \in (0,1)$. Now we let $m_0, \ldots, m_{k-3}$ and m_{k-1} be fixed. Using the fact that $\tilde{J}_{m,k-1}^{m_{k-2}}(\eta_{k-1}),\, m_{k-2} \in \mathbb{N}_0$ is complete in $L_{w_{k-1}}^2(0,1)$, we obtain in a similar way that

$$g_2(\eta_{k-1}, \eta_k) = \int_{(0,1)^{k-2}} \Big(\prod_{i=1}^{k-2} w_i(\eta_i) \tilde{J}_{m,i}^{m_{i-1}}(\eta_i) \Big) f(\eta_1, \ldots, \eta_k)\, \mathrm{d}\eta_1 \cdots \mathrm{d}\eta_{k-2} = 0$$

for a.e. $(\eta_{k-1}, \eta_k) \in (0,1)^2$. Proceeding this way, we obtain $f = 0$ a.e. This completes the proof. □

Let $\mathbf{S}_+^k$ be the positive part of the unit sphere $\mathbf{S}^k$ in $\mathbb{R}^{k+1}$. $\mathbf{S}_+^k$ is mapped bijectively onto $(\eta_1, \ldots, \eta_k) \in Q$. The surface element $\mathrm{d}S$ can be computed as follows. We use relations (5) to calculate partial derivatives of $x_0, x_1, \ldots, x_k$ with respect to $\sqrt{\eta_0}, \eta_1, \ldots, \eta_k$. Then we construct the $k+1$ by $k+1$ Jacobian matrix. For $\eta_0 = 1$, the surface element $\mathrm{d}S$ is given by

$$\mathrm{d}S = \frac{1}{2^k} \prod_{i=1}^{k} {\eta_i}^{-\frac{1}{2}} (1 - \eta_i)^{\frac{k-i-1}{2}}\, \mathrm{d}\eta_1 \mathrm{d}\eta_2 \cdots \mathrm{d}\eta_k. \tag{22}$$

Lemma 4.1.

$$\int_Q \mathrm{d}S = \frac{\pi^{\frac{k+1}{2}}}{2^k\,\Gamma\left(\frac{k+1}{2}\right)}\,. \tag{23}$$

Proof.

$$\begin{aligned}\int_Q \mathrm{d}S &= \frac{1}{2^k}\int_0^1\cdots\int_0^1\prod_{i=1}^k \eta_i{}^{-\frac{1}{2}}(1-\eta_i)^{\frac{k-i-1}{2}}\,\mathrm{d}\eta_1\mathrm{d}\eta_2\cdots\mathrm{d}\eta_k\\ &= \frac{1}{2^k}\prod_{i=1}^k\int_0^1 \eta_i{}^{-\frac{1}{2}}(1-\eta_i)^{\frac{k-i-1}{2}}\,\mathrm{d}\eta_i\\ &= \frac{1}{2^k}\prod_{i=1}^k\frac{\Gamma\left(\frac{1}{2}\right)^k\Gamma\left(\frac{k-i+1}{2}\right)}{\Gamma\left(\frac{k-i}{2}+1\right)}\\ &= \frac{\Gamma\left(\frac{1}{2}\right)^{k+1}}{2^k\,\Gamma\left(\frac{k+1}{2}\right)}\\ &= \frac{\pi^{\frac{k+1}{2}}}{2^k\,\Gamma\left(\frac{k+1}{2}\right)}\,.\end{aligned}$$

This completes the proof. □

Let $w(\eta_1,\eta_2,\ldots,\eta_k)$ be the weight function as described in (15). Introduce another weight function

$$d(x_0,x_1,\ldots,x_k) = 2^k x_0^{2\rho_0-1}x_1^{2\rho_1-1}\cdots x_k^{2\rho_k-1}. \tag{24}$$

Then

$$d(x_0,x_1,\ldots,x_k)\,\mathrm{d}S = w(\eta_1,\eta_2,\ldots,\eta_k)\,\mathrm{d}\eta_1\mathrm{d}\eta_2\cdots\mathrm{d}\eta_k\,. \tag{25}$$

Let $f_1(x_0,\ldots,x_k)$, $f_2(x_0,\ldots,x_k)$ be two functions in $L^2_d(\mathbf{S}^k_+)$ and let $g_1(\eta_1,\ldots,\eta_k)$, $g_2(\eta_1,\ldots,\eta_k)$ be their representations in spherical coordinates. Then

$$\int_{\mathbf{S}^k_+} d(x)f_1(x)f_2(x)\,\mathrm{d}S = \int_Q w(\eta)g_1(\eta)g_2(\eta)\,\mathrm{d}\eta\,. \tag{26}$$

Therefore, for a fixed m_0, the system $\left\{\tilde{J}^m(\eta_1,\ldots,\eta_k),\ m\in M_{n_0}\right\}$, when transformed in the Cartesian coordinates $(x_0,x_1,\ldots,x_k)$, forms an orthonormal basis of the subspace $\mathcal{K}^{k+1}_{m_0}$ in $L^2_d(\mathbf{S}^k_+)$.

5. Expansions in infinite series and Dirichlet problems. Let $h\in L^2_w(Q)$. We expand the function h in a Fourier series in terms of the product $\tilde{J}^m$:

$$h(\eta) \sim \sum_{m\in M} A_m\tilde{J}^m(\eta) \tag{27}$$

$$\equiv \sum_{m_0=0}^{\infty}\ \sum_{m_1,m_2,\ldots,m_{k-1}} A_{m_0,\ldots,m_{k-1}}\prod_{i=1}^k \tilde{J}^{m_{i-1}}_{m,i}(\eta_i) \tag{28}$$

where

$$\sum_{m_1,\ldots,m_{k-1}} = \sum_{m_1=0}^{m_0} \sum_{m_2=0}^{m_1} \cdots \sum_{m_{k-2}=0}^{m_{k-1}} \tag{29}$$

and also where the Fourier coefficients are given by

$$A_m = \int_Q w(\eta) h(\eta) \prod_{i=1}^{k} \tilde{J}_{m,i}^{m_{i-1}}(\eta_i)\, \mathrm{d}\eta. \tag{30}$$

We let the constant

$$B_m = \frac{A_m}{p_m^{\frac{1}{2}}}. \tag{31}$$

Then (28) becomes

$$h(\eta) \sim \sum_{m \in M}^{\infty} B_m \prod_{i=1}^{k} J_{m,i}^{m_{i-1}}(\eta_i). \tag{32}$$

5.1. The interior Dirichlet problem. Let $\rho = (\rho_0, \rho_1, \ldots, \rho_k)$ and $|\rho|_1 = \rho_0 + \rho_1 + \cdots + \rho_k$. Define

$$\hat{J}_{m,0}^{m_0}(\eta_0) = C\eta_0^{m_0} + D\eta_0^{-(m_0+|\rho|_1-1)}\,. \tag{33}$$

By (13), the separated solutions of $\mathcal{L}v = 0$ in spherical coordinates are

$$v(\eta_0)v(\eta_1)\cdots v(\eta_k) = \tilde{J}^m(\eta) = \hat{J}_{m,0}^{m_0}(\eta_0) \prod_{i=1}^{k} J_{m,i}^{m_{i-1}}(\eta_i)\,.$$

We want the solution of the Dirichlet problem to be bounded, so we set $D = 0$. We consider the following Dirichlet problem:

Let $\mathcal{B}$ be the unit ball. Given an even function $f : \mathbf{S}^k \to \mathbb{R}$ such that $f \in L_d^2(\mathbf{S}_+^k)$, find $v \in C^2(\mathcal{B})$ that solves

$$\mathcal{L}v = 0 \text{ in } \mathcal{B}, \tag{34}$$

and that satisfies the boundary condition

$$v(tx) \to f(x) \text{ as } 1 > t \to 1 \text{ in } L_d^2(\mathbf{S}_+^k). \tag{35}$$

We find the solution as follows. We let

$$f \sim \sum_{n=0}^{\infty} f_n$$

be the expansion of f in generalized spherical harmonics f_n of degree n. We assume that the boundary function f is even ($f(\sigma_j x) = f(x)$ for all j). Then $f_n = 0$ for odd n and f_n is a harmonic polynomial in $x_0^2, \ldots, x_k^2$ of degree $m_0 = n/2$. Define

$$v(x) = \sum_{n=0}^{\infty} r^n f_n\left(\frac{x}{r}\right), \quad x \in \mathcal{B} \tag{36}$$

where $r = |x|$, and $f_n(x) = r^n f_n(x/r)$. The series (36) converges in $\mathcal{B}$, $v \in C^2(\mathcal{B})$ and satisfies $\mathcal{L}v = 0$.

We find f_n as follows. We express f in spherical coordinates as $\hat{f}(\eta_1, \ldots, \eta_k)$. Recall that $\{\tilde{J}^m(\eta),\ m \in M_{m_0}\}$, m_0 fixed, is an orthonormal basis in $\hat{\mathcal{K}}^{k+1}_{m_0}$. Thus, if $n = 2m_0$ is even,

$$\hat{f}_n(\eta) = \sum_{m_1,\ldots,m_{k-1}} C_m \tilde{J}^m(\eta)$$

where $m = (m_0, \ldots, m_{k-1})$. Hence

$$\hat{f} \sim \sum_{m_0=0}^{\infty} \sum_{m_1,\ldots,m_{k-1}} C_m \tilde{J}^m(\eta)$$

where the Fourier coefficients are

$$C_m = \int_{\mathbf{S}^k_+} w(\eta)\hat{f}(\eta) \prod_{i=1}^{k} J^{m_{i-1}}_{m,i}(\eta_i)\, \mathrm{d}\eta.$$

Therefore the solution of the interior Dirichlet problem is

$$v(P) = \sum_{m_0=0}^{\infty} \sum_{m_1,\ldots,m_{k-1}} C_m \eta_0^{m_0} \prod_{i=1}^{k} \tilde{J}^{m_{i-1}}_{m,i}(\eta_i),\ P \in \mathcal{B} \tag{37}$$

where $\eta_0, \ldots, \eta_k$ are the spherical coordinates of P.

5.2. The exterior Dirichlet problem. Let $\mathcal{B}_e = \mathbb{R}^{k+1} - \overline{\mathcal{B}}$ where $\mathcal{B}$ is the open unit ball. We consider the following exterior Dirichlet problem: Find $v \in C^2(\mathcal{B}_e)$ that satisfies

$$\mathcal{L}v(P) = 0\,,\ P \in \mathcal{B}_e$$

$$v(P) = f(P)\,,\ P \in \partial\mathcal{B}_e \tag{38}$$

$$\lim_{\eta_0 \to \infty} v(P) = 0 \tag{39}$$

with a given even boundary function $f \in L^2_d(\mathcal{B}_e)$. We require that $|v(P)| \leq M =$ constant.

From equations (33), we want $|v(P)| \leq M$ as $\eta_0 \to \infty$. So we let $C = 0$. Then

$$v(P) = \sum_{m_0=0}^{\infty} \sum_{m_1,\ldots,m_{k-1}} D_{m_0,\ldots,m_{k-1}}\, \eta_0^{-(m_0+|\rho|_1-1)} \prod_{i=1}^{k} J^{m_{i-1}}_{m,i}(\eta_i)\,. \tag{40}$$

We use the given boundary conditions (38) to find the Fourier coefficients

$$D_m = \int_{\partial\mathcal{B}} w(P) f(P) \prod_{i=1}^{k} J^{m_{i-1}}_{m,i}(\eta_i)\, \mathrm{d}(\partial\mathcal{B}). \tag{41}$$

Then the solution of the exterior Dirichlet problem is

$$v(P) = \sum_{m_0=0}^{\infty} \sum_{m_1,m_2,\ldots,m_{k-1}} \tilde{D}_m\, \eta_0^{-(m_0+|\rho|_1-1)}\, \tilde{J}^m(\eta), \quad P \in \mathcal{B}_e \tag{42}$$

with

$$\tilde{D}_m = \int_{\partial\mathcal{B}} w(P) f(P) \tilde{J}^m(\eta)\, \mathrm{d}(\partial\mathcal{B}) .$$

Let

$$g_n(x) = \eta_0^{-(\frac{n}{2}+|\rho|-1)} f_n\Big(\frac{x}{r}\Big), \quad |x| = r > 0. \tag{43}$$

Then

$$v(x) = \sum_{n=0}^{\infty} g_n(x), \quad x \in \mathcal{B}_e . \tag{44}$$

The series (44) is a solution of the exterior Dirichlet problem and converges uniformly on compact subsets of $\mathcal{B}_e$.

Résumé substantiel en français. Nous considérons l'équation aux dérivées partielles suivante :

$$\mathcal{L}u(x) := \Delta u(x) + \sum_{j=0}^{\infty} \Big(\frac{2\rho_j - 1}{x_j}\Big) \partial u_j(x) = 0 \tag{3}$$

pour des fonctions u qui sont paires. Nous récrivons l'équation $\mathcal{L}u = 0$ en coordonnées sphériques $\eta_0, \ldots, \eta_k$ sous la forme $\mathcal{L}v = 0$ avec $v(\eta_0, \ldots, \eta_k) = u(x_0, \ldots, x_k)$. En appliquant la méthode de séparation des variables, on obtient un système de $k + 1$ équations différentielles ordinaires.

Nous expliquons ensuite comment la séparation des variables en coordonnées sphériques nous donne des expressions pour les fonctions harmoniques sphériques généralisées en termes des polynômes de Jacobi. Nous démontrons alors que les produits des solutions des $k + 1$ équations différentielles forment une base orthonormée de l'espace $\mathcal{K}_m^{k+1}$ de fonctions harmoniques sphériques généralisées.

Finalement, nous étudions les problèmes intérieur et extérieur de Dirichlet pour l'équation (3) sur la sphère

$$\mathbf{S}^k = \{(x_0, x_1, \ldots, x_k) \in \mathbb{R}^{k+1} : \sum_{i=0}^{k} x_i^2 = 1\}$$

de dimension k en développant les solutions en séries infinies de fonctions harmoniques sphériques généralisées d'ordre pair.

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