Ann. Sci. Math. Québec, **17** (1), 1993, 65–97.

ON A CONSTRUCTION OF ALGEBRAS STABLY EQUIVALENT TO SELFINJECTIVE SPECIAL BISERIAL ALGEBRAS

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RÉSUMÉ. On considère des systèmes maximaux de blocs othogonaux stables pour des algèbres auto-injectives, spéciales et bissérielles. Si A est une telle algèbre, qui n'est pas une algèbre locale du type de Nakayama, chacun de ses systèmes définit une algèbre auto-injective stablement équivalente à A. Voir le résumé substantiel en français à la fin de l'article.

ABSTRACT. Maximal systems of orthogonal stable bricks for selfinjective special biserial algebras are studied. It is shown that every such a system over a selfinjective special biserial algebra A which is not a local Nakayama algebra produces a selfinjective algebra that is stably equivalent to A.

The study of stable equivalences of finite-dimensional algebras over an algebraically closed field K has its sources in modular representation theory of finite groups. Problems of stable equivalences were considered in [4, 7, 8, 15, 16, 17, 21, 22, 24, 25]. R. Martinez-Villa in [17] indicated that the most important algebras for many problems concerning stable equivalences are selfinjective algebras. Ch. Riedtmann gave in [24, 25] (see also [8]) a classification of algebras stably equivalent to selfinjective algebras of finite representation type. But the problem of a classification in representation-tame cases is far from a satisfactory solution.

Recently a new important problem of classifying of derived equivalent algebras appeared (see [14]) that is equivalent in many cases to classifying stably equivalent selfinjective algebras of infinite dimension.

It was introduced a notion of a maximal system of orthogonal stable bricks (see Section 3 for a definition) in [21] that was applied successfully in the proof of the fact that the class of selfinjective special biserial algebras is closed under stable equivalence, where two algebras A, B are stably equivalent if there is an equivalence $\Phi\colon \underline{\text{mod}}\text{–}A \to \underline{\text{mod}}\text{–}B$ of their stable categories of finite-dimensional modules. In [22] this notion was applied to a classification of the algebras that are stably equivalent to trivial extensions of tame hereditary algebras of extended Dynkin type $\tilde{\mathbf{A}}_n$. On the other hand the problem how to construct all algebras that are stably equivalent to a given selfinjective algebra is still open. The main aim of the paper is to give such a construction for selfinjective special biserial algebras. Moreover this construction seems to have a general character. It can be applied to other classes of selfinjective algebras and it shows new properties and new structures on stable categories of finite-dimensional selfinjective algebras.

Throughout the paper we shall fix an algebraically closed field K.

The paper is organized in the following way.

We recall a notion of a locally bounded K–category and some standard notations in Section 1.

Reçu le 20 mars 1992 et, sous forme définitive, le 3 décembre 1992.

Supported by Polish Sc. Grant KBN No. 1222/2/91

Section 2 is about Galois coverings of finite-dimensional K–algebras. There are recalled all known facts that will be used in the paper.

Special biserial algebras are defined in Section 3. There are also given two useful lemmas from [21].

Maximal systems of orthogonal stable bricks are defined in Section 4. There is also recalled a notion of s-projective modules and their s-radicals.

Section 5 is about s-socles and s-tops. In this section there is proved that every finite-dimensional module of the first kind (with respect to a fixed Galois covering) has a finite nonzero s-top and a finite nonzero s-socle (see Corollary 1).

The same is proved for modules of the second kind in Section 6.

Sections 7, 8 are devoted for proving that modules of the first kind have their s-radicals. The notion of an s-radical is generalized for arbitrary modules in Section 8.

There is introduced a notion of an s-support for modules of the first kind in Section 9. Moreover shapes of s-supports are studied. s-supports of τ–shifts of s-projective modules are studied in Section 10.

Section 11 is devoted for a description of indecomposable modules of the second kind in terms of primitive families of s-local modules. The obtained description allows to define s-supports for modules of the second kind.

There is given a standard construction of a selfinjective special biserial algebra in Section 12. This construction shows that from a fixed maximal system of orthogonal stable bricks over a special biserial selfinjective algebra one can produce a selfinjective special biserial algebra.

Section 13 is devoted for a useful description of stable morphisms between modules of the first kind.

There are studied supports of indecomposable modules over the constructed algebras in Section 14.

Section 15 shows that the constructed algebras are stably equivalent to algebras over that we consider maximal systems of orthogonal stable bricks (see Theorem 1). Moreover, under some assumptions, every stable equivalence of two selfinjective special biserial algebras is induced by a stable equivalence of subcategories of their modules of the first kind (see Theorem 2).

1. Preliminaries. Recall from [9,12] that a K–category $\mathcal{R}$ is a category that has a structure of K–linear spaces on the sets $\mathcal{R}(x,y)$ of morphisms from every object x to every object y and compositions of morphisms are K–bilinear. A K–category $\mathcal{R}$ is said to be *locally bounded* if it satisfies the following conditions:

(a) Different objects are not isomorphic.
(b) For any object x in $\mathcal{R}$ its endomorphism algebra $\mathcal{R}(x,x)$ is local.
(c) For every object x in $\mathcal{R}$ we have:

$$\sum_{y\in\mathcal{R}} \dim_K \mathcal{R}(x,y) < \infty \quad \text{and} \quad \sum_{y\in\mathcal{R}} \dim_K \mathcal{R}(y,x) < \infty.$$

It is well-known that every basic finite-dimensional K–algebra is a locally bounded K–category.

Let A be a finite-dimensional K-algebra over a fixed algebraically closed field K. A is assumed to be basic connected with an identity element. Let mod–A denote the category of all finite-dimensional right A–modules. As usual, $\underline{\text{mod}}$–A denotes the stable category of mod–A. We denote by MOD–A the category of all right A-modules, and by (ind–A)/$\cong$ the set of the isomorphism classes of the indecomposable objects in mod–A.

Recall that a quiver Q is a pair (Q_0, Q_1), where Q_0 is a set of vertices and Q_1 is a set of arrows between vertices from Q_0. A *relation* between vertices $x, y \in Q_0$ is a linear combination

$\rho = \sum_{i=1}^{m} \lambda_i w_i$ where, for each $1 \le i \le m$, $\lambda_i \in K^* = K \setminus \{0\}$ and w_i is a path from x to y that is a composition of at least two arrows. A set of relations in Q generates an ideal I in the path algebra (category) KQ of Q. A pair (Q, I) is said to be a *bound quiver.* It is well-known that for every basic algebra A (more general for every locally bounded K-category) there is a bound quiver (Q_A, I_A) such that there is an isomorphism $A \cong KQ_A/I_A$ which is called a *presentation* of A (see [5, 11]).

For each vertex x in Q_A, we shall denote by S_x the corresponding simple A-module, by P_x (resp. E_x) its projective cover (resp. injective envelope).

We shall use freely all properties of the Auslander–Reiten translation τ and of the Auslander–Reiten quiver Γ_A of an algebra A. All informations concerning these notions can be found in [2, 3].

2. Galois coverings. Let $\mathcal{R}$, $\mathcal{S}$ be K–categories. A K–linear functor $F\colon \mathcal{R} \to \mathcal{S}$ is said to be a *covering functor* [12] if the induced maps $\bigoplus_{Fy=a} \mathcal{R}(x, y) \to \mathcal{S}(Fx, a)$ and $\bigoplus_{Fy=a} \mathcal{R}(y, x) \to \mathcal{S}(a, Fx)$ are K–isomorphisms for all $x \in \mathcal{R}$ and $a \in \mathcal{S}$.

Let (Q, I) be a connected bound quiver. A *minimal relation* in I is a relation $\rho = \sum_{i=1}^{m} \lambda_i w_i$ between vertices x, $y \in Q_0$ such that for each nonempty proper subset $T \subset \{1, \dots, m\}$ we have $\sum_{i \in T} \lambda_i w_i \notin I$ (see [18]). Let x_0 be a fixed vertex of Q. Then $\Pi_1(Q, x_0)$ denotes the fundamental group of the quiver Q with the base point x_0 [19], i.e. the set of formal walks whose sources and whose sinks coincide to x_0 with an ordinary composition. Recall that a *walk* in the quiver Q is a formal composition of arrows and their formal inverses. Let $N(Q, x_0, m(I))$ be the subgroup in $\Pi_1(Q, x_0)$ that is generated by all elements of the form $\gamma^{-1}u^{-1}v\gamma$, where γ is a walk from x_0 to x, and u, v are paths from x to y such that in the set $m(I)$ of minimal relations in I there is $\rho = \sum_{i=1}^{m} \lambda_i w_i$ with $w_1 = u$, $w_2 = v$, $m \ge 2$ (see [13,20]). Consequently $N(Q, x_0, m(I))$ is a normal subgroup in $\Pi_1(Q, x_0)$ and the group $\Pi(Q, I) = \Pi_1(Q, x_0)/N(Q, x_0, m(I))$ is called a *fundamental group of the bound quiver* (Q, I). In fact if (Q, I) is connected then for different choices of the base point one obtains the same group (up to isomorphism).

Let $A = KQ_A/I_A$ for a bound quiver (Q_A, I_A) and let $x_0 \in Q_A$ be a fixed vertex. Suppose that $\mathcal{W}$ is a topological universal cover of Q_A with the base point x_0. Following [19] it is known that there is a natural map $q\colon \mathcal{W} \to Q_A$ given by the action of $\Pi_1(Q_A, x_0)$. Consequently we define $\widetilde{Q}_A$ as an orbit quiver $\mathcal{W}/N(Q_A, x_0, m(I))$ and a map $\pi\colon \widetilde{Q}_A \to Q_A$ is given by the action of the group $\Pi(Q_A, I_A)$. The map π yields a Galois covering $\pi\colon K\widetilde{Q}_A \to KQ_A$ of path categories [13,20] and we obtain a Galois covering $F\colon K\widetilde{Q}_A/\tilde{I}_A \to KQ_A/I_A$ with the group $\Pi(Q_A, I_A)$, where $\tilde{I}_A$ is an ideal in $K\widetilde{Q}_A$ that is generated by all elements u such that $\pi(u) \in I_A$. The locally bounded K-category $\tilde{A} = K\widetilde{Q}_A/\tilde{I}_A$ is said to be a *universal Galois cover* of A [18] determined by the presentation $A \cong KQ_A/I_A$.

Recall (see [1, 23]) that a locally bounded K–category $\mathcal{R}$ is said to be *simply connected* if it is triangular (its quiver has no oriented cycles) and for any presentation $\mathcal{R} \cong KQ/I$ as a path category, the fundamental group $\Pi(Q, I)$ of the bound quiver (Q, I) is trivial. An algebra A is said to be *standard* [1] if there is a Galois covering $\tilde{A} \to A$ with $\tilde{A}$ simply connected.

Every Galois covering $F\colon K\widetilde{Q}_A/\tilde{I}_A \to KQ_A/I_A$ induces a functor

$$F_\bullet\colon \text{MOD–}KQ_A/I_A \to \text{MOD–}K\widetilde{Q}_A/\tilde{I}_A$$

which attaches the module $N \circ F^{\mathrm{op}}$ to a KQ_A/I_A–module N. Moreover, there exists a functor

$$F_\lambda\colon \text{MOD–}K\widetilde{Q}_A/\tilde{I}_A \to \text{MOD–}KQ_A/I_A$$

[6, 9, 12] that is left adjoint to $F_\bullet$, and F_λ induces an injection of $((\text{ind–}K\widetilde{Q}_A/\tilde{I}_A)/\cong)/\Pi(Q_A, I_A)$, the set of $\Pi(Q_A, I_A)$–orbits of $(\text{ind–}K\widetilde{Q}_A)/\tilde{I}_A/\cong$, into the set $(\text{ind–}KQ_A/I_A)/\cong$. We shall denote by $\mathrm{mod}_1\text{–}KQ_A/I_A$ the full subcategory of $\mathrm{mod}\text{–}KQ_A/I_A$ formed by all modules of the form $F_\lambda(\widetilde{M})$, where $\widetilde{M}$ is an object of $\mathrm{mod}\text{–}K\widetilde{Q}_A/\tilde{I}_A$. Modules from $\mathrm{mod}_1\text{–}KQ_A/I_A$ are called *modules of the first kind* (with respect to the covering F). We shall denote by $\mathrm{mod}_2\text{–}KQ_A/I_A$ the full subcategory of $\mathrm{mod}\text{–}KQ_A/I_A$ formed by all modules that do not have direct summands from $\mathrm{mod}_1\text{–}KQ_A/I_A$. Modules from $\mathrm{mod}_2\text{–}KQ_A/I_A$ are called modules of the second kind (with respect to the covering F).

For every $K\widetilde{Q}_A/\tilde{I}_A$–module $M \in \mathrm{mod}\text{–}K\widetilde{Q}_A/\tilde{I}_A$ its *support* is a full subcategory $\mathrm{supp}(M)$ of $K\widetilde{Q}_A/\tilde{I}_A$ formed by all objects $x \in K\widetilde{Q}_A/\tilde{I}_A$ such that $M(x) \neq 0$.

3. Special biserial algebras. Let A be a finite-dimensional K–algebra (locally bounded K–category). A is said to be *biserial* [10] if the radical of any indecomposable left or right projective A–module is a sum of at most two uniserial submodules whose intersection is simple or zero. A is said to be *special biserial* [26] if it is isomorphic to KQ_A/I_A, where the bound quiver (Q_A, I_A) satisfies the following conditions:

(i) Every vertex of Q_A is a source of at most two arrows and a sink of at most two arrows.
(ii) For every arrow α of Q_A there are at most one arrow β and at most one arrow γ such that $\alpha\beta, \gamma\alpha \notin I_A$.

It was proved in [26] that every special biserial K–algebra A is biserial. This class of algebras was studied in [9, 23, 26, 27]. We are interested in selfinjective special biserial algebras. The main result of [23] shows that the class of selfinjective special biserial algebras coincides to the class of standard selfinjective biserial algebras. Moreover we have a full description of indecomposable A–modules in [9, 27]. In particular indecomposable A–modules of the first kind are of the forms $F_\lambda(M)$, where M are indecomposable $\tilde{A}$–modules of finite dimension whose $\mathrm{supp}(M)$ are path categories KQ_M, Q_M are relation-free quivers and their underlying graphs are of Dynkin type $\mathbf{A}_n$. Moreover, every indecomposable A–module N of the second kind is of τ–period 1, i.e. $\tau(N) \cong N$.

Following [6] we know that F_λ preserves simple objects and projectives objects. Consequently F_λ preserves factorization of morphisms through projective objects. There is given a reduction of studying of $\underline{\mathrm{mod}}\text{–}A$ to studying of $\underline{\mathrm{mod}}\text{–}\tilde{A}$ in [21]. We shall use this reduction. Moreover, we have the following two important lemmas that were proved in [21].

Lemma 1. *Let $A \cong KQ_A/I_A$ be a selfinjective special biserial K-algebra. Let M, N be two indecomposable finite-dimensional $K\widetilde{Q}_A/\tilde{I}_A$-modules whose supports are of the forms*

$$- \cdots \leftarrow r_0 \rightarrow \cdots \rightarrow x \rightarrow \cdots \rightarrow r_1 \leftarrow \cdots -$$
$$- \cdots \leftarrow x \rightarrow \cdots \rightarrow r_1 \rightarrow \cdots \rightarrow r_1' \leftarrow \cdots -$$

respectively. Let $f\colon N \to M$ be a morphism that is a composition of an epimorphism $f_1\colon N \to X$ and a monomorphism $f_2\colon X \to M$, where X is an indecomposable $K\widetilde{Q}_A/\tilde{I}_A$–module whose support is of the form $x \to \cdots \to r_1$. Let $\underline{f}$ denote the coset of f in $\underline{\mathrm{mod}}$–A. Then the following implications hold:

(a) *If P_{r_0} is uniserial, then $\underline{f} \neq 0$ iff the path*

$$r_0 \rightarrow \cdots \rightarrow x \rightarrow \cdots \rightarrow r_1 \rightarrow \cdots \rightarrow r_1'$$

does not contain a subpath of the form

$$r_0 \rightarrow \cdots \rightarrow x \rightarrow \cdots \rightarrow r_1 \rightarrow \cdots \rightarrow y$$

which is the support of P_{r_0}.

(b) *If P_{r_0} is not uniserial, then $\underline{f} \neq 0$ implies either the path $r_1 \to \cdots \to r_1'$ does not contain a vertex z with $S_z \cong$ s-soc(P_{r_0}), or it contains such a vertex z and thus $z = r_1'$,* supp(M) *is of the form*

$$- \cdots \to r_{-1} \leftarrow \cdots \leftarrow y \leftarrow \cdots \leftarrow r_0 \to \cdots \to x \to \cdots \to r_1 \leftarrow \cdots -$$

and supp(N) *is of the form*

$$- \cdots \leftarrow x \to \cdots \to r_1 \to \cdots \to r_1' \leftarrow \cdots \leftarrow r_{-1} \leftarrow \cdots \leftarrow y \to \cdots -$$

where

$$\begin{array}{ccc} r_0 & \to \cdots \to x \to \cdots \to & r_1 \\ \downarrow & & \downarrow \\ \vdots & & \cdot \\ \downarrow & & \cdot \\ y & & \cdot \\ \downarrow & & \cdot \\ \vdots & & \cdot \\ \downarrow & & \downarrow \\ r_{-1} & \longrightarrow \cdots \longrightarrow & r_1' \end{array}$$

is the support of P_{r_0}.

Lemma 2. *Let $A \cong KQ_A/I_A$ be a selfinjective special biserial K-algebra. Let M, N be two indecomposable finite-dimensional $K\widetilde{Q}_A/\widetilde{I}_A$–modules whose supports are of the forms:*

$$- \cdots \to r_{-1} \leftarrow \cdots \leftarrow y \leftarrow \cdots \leftarrow r_0 \to \cdots \to x \to \cdots \to r_1 \leftarrow \cdots -$$
$$- \cdots \leftarrow y \to \cdots \to r_{-1} \to \cdots \to r_0' \leftarrow \cdots \leftarrow r_1 \leftarrow \cdots \leftarrow x \to \cdots -$$

respectively, such that the paths

$$r_0 \to \cdots \to y \to \cdots \to r_{-1} \to \cdots \to r_0'$$
$$r_0 \to \cdots \to x \to \cdots \to r_1 \to \cdots \to r_0'$$

do not belong to $\widetilde{I}_A$ and their difference belongs to $\widetilde{I}_A$. Let $f\colon N \to M$ be a morphism that is a composition of an epimorphism $f_1\colon N \to X$ and a monomorphism $f_2\colon X \to M$, where X is an indecomposable $K\widetilde{Q}_A/\widetilde{I}_A$–module whose support is of the form $x \to \cdots \to r_1$. Let $g\colon N \to M$ be a morphism that is a composition of an epimorphism $g_1\colon N \to Y$ and a monomorphism $g_2\colon Y \to M$, where Y is an indecomposable $K\widetilde{Q}_A/\widetilde{I}_A$–module whose support is of the form $y \to \cdots \to r_{-1}$. Then $\lambda\underline{f} = \underline{g}$ for some $\lambda \in K^*$.

4. Systems of orthogonal stable bricks.

We start this section with recalling a notion of a system of orthogonal stable bricks over a selfinjective K-algebra that was used succesfully in [21,22].

Let B be a selfinjective K–algebra. An indecomposable B–module M in $\underline{\text{mod}}$–B is said to be a *stable B–brick* if its endomorphism ring $\underline{\text{End}}_B(M)$ is isomorphic to K. A family $\{M_j\}_{j \in J}$ of stable B–bricks is said to be a *system of orthogonal stable B–bricks* if the following conditions are satisfied:

(1) M_j is not of τ-period 1 for every $j \in J$.
(2) For any two different $i, j \in J$; $\underline{\text{Hom}}_B(M_i, M_j) = 0 = \underline{\text{Hom}}_B(M_j, M_i)$.

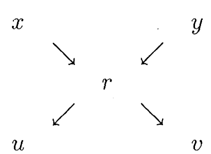
A system of orthogonal stable B–bricks $\{M_j\}_{j\in J}$ is called *maximal* if for every indecomposable B–module N that is neither projective nor of τ–period 1 there exists $j_0 \in J$ such that $\underline{\mathrm{Hom}}_B(M_{j_0}, N) \neq 0$ and there exists $j_1 \in J$ such that $\underline{\mathrm{Hom}}_B(N, M_{j_1}) \neq 0$.

We are interested in maximal systems of orthogonal B–bricks whose cardinalities coincide with the cardinality of isoclasses of the simple B–modules. We shall consider only such maximal systems without additional comments.

Let A be a special biserial selfinjective K–algebra that is not a local Nakayama algebra. Let $\mathcal{M}_A = \{M_1, \dots, M_n\}$ be a maximal system of orthogonal stable A-bricks. Let us fix a Galois covering functor $F\colon \tilde{A} \to A$ with $\tilde{A}$ to be simply connected. We know by definition that all $M_i \in \mathcal{M}_A$ are A–modules of the first kind with respect to any Galois covering functor, because they are not of τ–period 1. Therefore any $M_i \in \mathcal{M}_A$ is of the form $F_\lambda(\widetilde{M_i})$ and $\mathrm{supp}(\widetilde{M_i})$ is one of the following forms:

(i) $r_0 \to \cdots \to r_1 \leftarrow \cdots \leftarrow r_2 \to \cdots \leftarrow r_{t_i} \to \cdots \to r_{t_i+1}, t_i \geq 0$
(ii) $r_0 \leftarrow \cdots \leftarrow r_1 \to \cdots \to r_2 \leftarrow \cdots \to r_{t_i} \leftarrow \cdots \leftarrow r_{t_i+1}, t_i \geq 0$
(iii) $r_0 \to \cdots \to r_1 \leftarrow \cdots \leftarrow r_2 \to \cdots \to r_{t_i} \leftarrow \cdots \leftarrow r_{t_i+1}, t_i \geq 1$
(iv) $r_0 \leftarrow \cdots \leftarrow r_1 \to \cdots \to r_2 \leftarrow \cdots \leftarrow r_{t_i} \to \cdots \to r_{t_i+1}, t_i \geq 1$.

We state some conventions concerning notations of supports of indecomposable $\tilde{A}$–modules. If P_x is an indecomposable projective $\tilde{A}$–module then $S_{x'}$ denotes its socle. S_{x^*} denotes the top of E_x. If $\mathrm{supp}(X)$ is of the form $r_0 \to \cdots \to r_1 \leftarrow \cdots -$ (where $-$ means an arrow that can be $\to$ or $\leftarrow$) then $r_{-1} \to \dots r'_0$ means either the nonzero path connecting r_{-1} with r'_0, where $S_{r_{-1}}$ is the direct summand in $\text{s-top}(\text{s-rad}(P_{r_0}))$ and $r_{-1} \notin (r_0 \to \cdots \to r_1)$, if P_{r_0} is not uniserial or $(r_{-1} \to \cdots \to r'_0) = r'_0$ if P_{r_0} is uniserial. If $\mathrm{supp}(X)$ is of the form $- \cdots \to r_t \leftarrow \cdots \leftarrow r_{t+1}$ then $r'_{t+1} \leftarrow \cdots \leftarrow r_{t+2}$ has a similar meaning. If $\mathrm{supp}(X)$ is of the form $r_0 \leftarrow \cdots \leftarrow r_1 \to \cdots -$ then $r_{-1} \leftarrow \cdots \leftarrow r_0$ means either the nonzero path that connects r_0 with r_{-1}, where $S_{r_{-1}}$ is a direct summand in $\text{s-soc}(P_{r_0}/\,\text{s-soc}(P_{r_0}))$ and $r_{-1} \notin (r_0 \to \cdots \to r'_1)$, if P_{r_1} is not uniserial or $S_{r_{-1}} \cong \text{s-soc}(P_{r_1}/\,\text{s-soc}(P_{r_1}))$ if P_{r_1} is uniserial. If $\mathrm{supp}(X)$ is of the form $- \cdots \leftarrow r_t \to \cdots \to r_{t+1}$ then $r_{t+1} \to \cdots \to r_{t+2}$ has a similar meaning. Moreover, if r is a vertex in $\widetilde{Q}_A$ whose neighbourhood is of the form



then we shall denote $y = r^+$, $x = r^-$, $u = r_-$, $v = r_+$.

For a given maximal system of orthogonal stable A–bricks $\mathcal{M}_A = \{M_1, \dots, M_n\}$, an indecomposable A–module N that is not of τ–period 1 is said to be *s-projective with respect to* $\mathcal{M}_A$ if the following conditions are satisfied:

(1) $\underline{\mathrm{Hom}}_A(N, \bigoplus_{i=1}^n M_i) \cong K$.
(2) If $\underline{\mathrm{Hom}}_A(N, M_{i_0}) \neq 0$, then for every $0 \neq \underline{f}\colon X \to M_{i_0}$ and every $0 \neq \underline{g}\colon N \to M_{i_0}$ there is $\underline{h}\colon N \to X$ such that $\underline{fh} = \underline{g}$.

s-projective modules were studied in [21] and their supports are known. If we have a maximal system of orthogonal stable A–bricks $\mathcal{M}_A = \{M_1, \dots, M_n\}$ then we have a system of s-projective modules $\mathcal{N}_A = \{N_1, \dots, N_n\}$ with respect to $\mathcal{M}_A$. Moreover, $\underline{\mathrm{Hom}}_A(N_i, M_i) = K$ and $\underline{\mathrm{Hom}}_A(N_i, M_j) = 0$ for different $1 \leq i, j \leq n$.

Following [21] if N is an s-projective A–module with respect to a maximal system of orthogonal stable A–bricks $\mathcal{M}_A$, then A–module R is said to be an *s-radical* of N (we denote R by $\text{s-rad}(N)$)

if the following conditions are satisfied:

(1) R does not contain any projective direct summand.

(2) There is a projective or zero A–module P such that there exists a right minimal almost split morphism $R \oplus P \to N$ in mod–A.

It was proved in [21] that for each s-projective A–module N its s-radical is a direct sum of at most two indecomposable A–modules of the first kind.

5. s-tops and s-socles. Let Y be an A-module. Suppose that $\dim_K \underline{\mathrm{Hom}}_A(Y, M_i) = d_i$, $i = 1, \dots, n$. Then we say that $\bigoplus_{i=1}^n M_i^{d_i}$ is an *s-top* of Y and we denote it s-top(Y), where $M_i^{d_i}$ denotes a direct sum of d_i copies of M_i. We define s-socle of Y (that is denoted by $\text{s-soc}(Y)$) dually. In [21] it was proved that each direct summand in $\text{s-rad}(N)$ has an indecomposable s-top and an indecomposable s-socle, where N is s-projective.

The main aim of this section is to show for any special biserial selfinjective K–algebra A which is not a local Nakayama algebra that every A–module of the first kind has its s-top which is a direct sum of finitely many indecomposable modules from $\mathcal{M}_A$.

Throughout the paper we assume that the above fixed Galois covering $F\colon \tilde{A} \to A$ with $\tilde{A}$ simply connected is chosen in such a manner that $\tilde{I}_A$ is generated only by paths and differences of some paths,i.e. if $u - \lambda v \in \tilde{I}_A$ is a generator with $\lambda \in K^*$, then $\lambda = 1$. It is well-known that for special biserial algebras it is possible to choose such a set of generators of $\tilde{I}_A$.

Lemma 3. *Let A be a special biserial selfinjective K–algebra which is not a local Nakayama algebra. Let $F_\lambda(\widetilde{Y}_1) = Y_1$, $F_\lambda(\widetilde{Y}_2) = Y_2$ be two indecomposable A–modules of the first kind. Let $0 \neq F_\lambda(\underline{\tilde{f}}) = \underline{f}\colon Y_1 \to Y_2$ be a morphism in $\underline{\mathrm{mod}}$–A. Then one of the following conditions is satisfied*:

(a) $\mathrm{supp}(\widetilde{Y}_2)$ *is of the form*

$$- \cdots \to r_i \leftarrow \cdots \leftarrow r_{i+1} \to \cdots \to r_{i+2} \leftarrow \cdots \to r_j \leftarrow \cdots -$$

$\mathrm{supp}(\widetilde{Y}_1)$ *is of the form*

$$- \cdots \to x \leftarrow \cdots \leftarrow r_i \to \cdots \to r'_{i+1} \leftarrow \cdots \leftarrow r_{i+2} \cdots r'_{j-1} \leftarrow \cdots \leftarrow r_j - \cdots -$$

where $x \in (r'_{i-1} \leftarrow \cdots \leftarrow r_i)$, $x \neq r'_{i-1}$, $x \neq r_0$ and $\underline{\tilde{f}}$ is given by a composition of a projection of $\widetilde{Y}_1$ onto S_{r_i} with an injection of S_{r_i} into $\widetilde{Y}_2$.

(b) $\mathrm{supp}(\widetilde{Y}_2)$ *is of the form*

$$- \cdots \to r_i \leftarrow \cdots \leftarrow r_{i+1} \to \cdots \to r_{i+2} \leftarrow \cdots \to r_j \leftarrow \cdots -$$

$\mathrm{supp}(\widetilde{Y}_1)$ *is of the form*

$$- \cdots \to x \leftarrow \cdots \leftarrow r_i \leftarrow \cdots \leftarrow r_{i+1} \to \cdots \to r_j - \cdots -$$

where $x \in (r_i \to \cdots \to r'_{i+1})$ $x \neq r'_{i+1}$, $x \neq r_0$, and $\underline{\tilde{f}}$ is given by a composition of a projection of $\widetilde{Y}_1$ onto an indecomposable $\tilde{A}$–module $\widetilde{X}$ whose support is

$$r_i \leftarrow \cdots \leftarrow r_{i+1} \to \cdots \to r_{i+2} \leftarrow \cdots \to r_j \leftarrow \cdots \leftarrow y$$

or

$$r_i \leftarrow \cdots \leftarrow r_{i+1} \to \dots r_{i+2} \leftarrow \cdots \to r_j$$

with an injection of $\widetilde{X}$ *into* V_2.

(c) $\mathrm{supp}(\widetilde{Y}_2)$ *is of the form*

$$- \cdots \to r_i \leftarrow \cdots \leftarrow r_{i+1} \to \cdots \to r_{i+2} \leftarrow \cdots \to r_j \leftarrow \cdots -$$

$\mathrm{supp}(\widetilde{Y}_1)$ *is of the form*

$$- \cdots \leftarrow x \to \cdots \to r_i \leftarrow \cdots \leftarrow r_{i+1} \to \cdots \to r_j - \cdots -$$

where $x \in (r_{i-1} \to \cdots \to r_i)$ *and* $x = r_{i-1}$ *implies* $i = 1$, $\underline{\tilde{f}}$ *is given by a composition of a projection of* $\widetilde{Y}_1$ *onto an indecomposable* $\tilde{A}$*–module* $\widetilde{X}$ *whose support is of the form*

$$x \to \cdots \to r_i \leftarrow \cdots \leftarrow r_{i+1} \to \cdots \to r_{i+2} \leftarrow \cdots \to r_j \leftarrow \cdots \leftarrow y$$

or

$$x \to \cdots \to r_i \leftarrow \cdots \leftarrow r_{i+1} \to \cdots \to r_{i+2} \leftarrow \cdots \to r_j$$

with an injection of $\widetilde{X}$ *into* $\widetilde{Y}_2$.

(d) $\mathrm{supp}(\widetilde{Y}_2)$ *is of the form*

$$- \cdots \to r_i \leftarrow \cdots \leftarrow r_{i+1} \to \cdots \to r_{i+2} \leftarrow \cdots \to r_j \leftarrow \cdots -$$

$\mathrm{supp}(\widetilde{Y}_1)$ *is of the form*

$$- \cdots \leftarrow x \to \cdots \to r_i \to \cdots \to r'_{i+1} \leftarrow \cdots \leftarrow r_{i+2} \to \cdots \leftarrow r_j - \cdots -$$

where $x \in (r_i \leftarrow \cdots \leftarrow r_{i+1})$, $x \neq r_{i+1}$, *and* $\underline{\tilde{f}}$ *is given by a composition of a projection of* $\widetilde{Y}_1$ *onto an indecomposable* $\tilde{A}$*–module* $\widetilde{X}$ *whose support is* $x \to \cdots \to r_i$ *with an injection of* $\widetilde{X}$ *into* $\widetilde{Y}_2$.

Proof. Under the assumptions of our lemma suppose that $0 \neq F_\lambda(\underline{\tilde{f}}) = \underline{f}\colon Y_1 \to Y_2$. Thus $\underline{\tilde{f}}\colon \widetilde{Y}_1 \to \widetilde{Y}_2$ and $\mathrm{supp}(\widetilde{Y}_2) \cap \mathrm{supp}(\widetilde{Y}_1) \neq \varnothing$. Suppose that vertices of $\mathrm{supp}(\widetilde{Y}_1)$ are numbered by integers in such a way that they increase from the left hand to the right hand. Let z be the lowest vertex of $\mathrm{supp}(\widetilde{Y}_1)$ that is contained in $\mathrm{supp}(\widetilde{Y}_2)$. If the neighbourhood of z in $\mathrm{supp}(\widetilde{Y}_2)$ is of the form $\cdots \to z \to \cdots$ then $z \in (r_{i-1} \to \cdots \to r_i)$ and it is not hard to verify that (b) or (c) or (d) holds by Lemmas 1, 2. If the neighbourhood of z in $\mathrm{supp}(\widetilde{Y}_2)$ is of the form $\cdots \to z \leftarrow \cdots$, then $z = r_i$ and by Lemmas 1, 2 (a) holds. If the neighbourhood of z is of the form $\cdots \leftarrow z \to \cdots$, then there cannot be such a morphism $0 \neq \underline{f}\colon \widetilde{Y}_1 \to \widetilde{Y}_2$ that factors through an indecomposable $\tilde{A}$-module $\widetilde{X}$ with $z \in \mathrm{supp}(\widetilde{X})$. This finishes the proof of our lemma. □

If $Y_2 \in \mathcal{M}_A$ in Lemma 3, then we call the vertex x of $\mathrm{supp}(\widetilde{Y}_1)$ a *left frame* of $\underline{\tilde{f}}$ and we denote it $\mathrm{lf}(\underline{\tilde{f}})$. Similarly we define a *right frame* $\mathrm{rf}(\underline{\tilde{f}})$ of $\underline{\tilde{f}}$. A *frame* of $\underline{\tilde{f}}$ is a left or right frame.

Lemma 4. *Let A be a special biserial selfinjective K–algebra which is not a local Nakayama algebra. Let $F_\lambda(\widetilde{Y}_1) = Y_1$, $F_\lambda(\widetilde{Y}_2) = Y_2$ be two indecomposable A–modules of the first kind. Let $0 \neq F_\lambda(\underline{\tilde{g}}) = \underline{g}\colon Y_2 \to Y_1$ be a morphism in $\underline{\mathrm{mod}}$–A. Then one of the following conditions is satisfied:*

(a) $\mathrm{supp}(\widetilde{Y}_2)$ *is of the form*

$$- \cdots \leftarrow r_i \to \cdots \to r_{i+1} \leftarrow \cdots \leftarrow r_{i+2} \to \cdots \leftarrow r_j \to \cdots -$$

$\operatorname{supp}(\widetilde{Y}_1)$ *is of the form*

$$- \cdots \leftarrow x \rightarrow \cdots \rightarrow r_i \leftarrow \cdots \leftarrow r^*_{i+1} \rightarrow \cdots \rightarrow r_{i+2} \leftarrow \cdots \rightarrow r_j \leftarrow \cdots -$$

where $x \in (r^*_{i-1} \rightarrow \cdots \rightarrow r_i)$, $x \neq r^*_{i-1}$, $x \neq r_0$, *and* $\underline{\tilde{g}}$ *is given by a composition of a projection of* $\widetilde{Y}_2$ *onto* S_{r_i} *with an injection of* S_{r_i} *into* $\widetilde{Y}_1$.

(b) $\operatorname{supp}(\widetilde{Y}_2)$ *is of the form*

$$- \cdots \leftarrow r_i \rightarrow \cdots \rightarrow r_{i+1} \leftarrow \cdots \leftarrow r_{i+2} \rightarrow \cdots \leftarrow r_j \rightarrow \cdots -$$

$\operatorname{supp}(\widetilde{Y}_1)$ *is of the form*

$$- \cdots \leftarrow x \rightarrow \cdots \rightarrow r_i \rightarrow \cdots \rightarrow r_{i+1} \leftarrow \cdots \leftarrow r_{i+2} \rightarrow \cdots \leftarrow r_j - \cdots -$$

where $x \in (r_i \leftarrow \cdots \leftarrow r^*_{i+1})$, $x \neq r^*_{i+1}$, $x \neq r_0$, *and* $\underline{\tilde{g}}$ *is given by a composition of a projection of* $\widetilde{Y}_2$ *onto an indecomposable* $\tilde{A}$*–module* $\widetilde{X}$ *whose support is either*

$$r_i \rightarrow \cdots \rightarrow r_{i+1} \leftarrow \cdots \leftarrow r_{i+2} \rightarrow \cdots \leftarrow r_j \rightarrow \cdots \rightarrow y$$

or

$$r_i \rightarrow \cdots \rightarrow r_{i+1} \leftarrow \cdots \leftarrow r_{i+2} \rightarrow \cdots \leftarrow r_j$$

with an injection of $\widetilde{X}$ *into* $\widetilde{Y}_1$.

(c) $\operatorname{supp}(\widetilde{Y}_2)$ *is of the form*

$$- \cdots \leftarrow r_i \rightarrow \cdots \rightarrow r_{i+1} \leftarrow \cdots \leftarrow r_{i+2} \rightarrow \cdots \leftarrow r_j \rightarrow \cdots -$$

$\operatorname{supp}(\widetilde{Y}_1)$ *is of the form*

$$- \cdots \rightarrow x \leftarrow \cdots \leftarrow r_i \rightarrow \cdots \rightarrow r_{i+1} \leftarrow \cdots \leftarrow r_{i+2} \rightarrow \cdots \leftarrow r_j - \cdots -$$

where $x \in (r_{i-1} \leftarrow \cdots \leftarrow r_i)$ *and* $x = r_{i-1}$ *implies* $i = 1$, $\underline{\tilde{g}}$ *is given by a composition of a projection of* $\widetilde{Y}_2$ *onto an indecomposable* $\tilde{A}$*-module* $\widetilde{X}$ *whose support is either*

$$x \leftarrow \cdots \leftarrow r_i \rightarrow \cdots \rightarrow r_{i+1} \leftarrow \cdots \leftarrow r_{i+2} \rightarrow \cdots \leftarrow r_j \rightarrow \cdots \rightarrow y$$

or

$$x \leftarrow \cdots \leftarrow r_i \rightarrow \cdots \rightarrow r_{i+1} \leftarrow \cdots \leftarrow r_{i+2} \rightarrow \cdots \leftarrow r_j$$

with an injection of $\widetilde{X}$ *into* $\widetilde{Y}_1$.

(d) $\operatorname{supp}(\widetilde{Y}_2)$ *is of the form*

$$- \cdots \leftarrow r_i \rightarrow \cdots \rightarrow r_{i+1} \leftarrow \cdots \leftarrow r_{i+2} \rightarrow \cdots \leftarrow r_j \rightarrow \cdots -$$

$\operatorname{supp}(\widetilde{Y}_1)$ *is of the form*

$$- \cdots \rightarrow x \leftarrow \cdots \leftarrow r_i \leftarrow \cdots \leftarrow r^*_{i+1} \rightarrow \cdots \rightarrow r_{i+2} \leftarrow \cdots \rightarrow r_j - \cdots -$$

where $x \in (r_i \rightarrow \cdots \rightarrow r_{i+1})$, $x \neq r_{i+1}$ *and* $\underline{\tilde{g}}$ *is given by a composition of a projection of* $\widetilde{Y}_2$ *onto an indecomposable* $\tilde{A}$*–module* $\widetilde{X}$ *whose support is* $x \leftarrow \cdots \leftarrow r_i$ *with an injection of* $\widetilde{X}$ *into* $\widetilde{Y}_1$.

Proof. The proof is dual to that of Lemma 3 and we omit it. □

If $Y_2 \in \mathcal{M}_A$ in Lemma 4, then we call the vertex x of $\operatorname{supp}(\widetilde{Y}_1)$ a *left coframe* of $\underline{\tilde{g}}$ and we denote it $\operatorname{lcf}(\underline{\tilde{g}})$. Similarly we define a *right coframe* $\operatorname{rcf}(\underline{\tilde{g}})$ of $\underline{\tilde{g}}$. A *coframe* of $\underline{\tilde{g}}$ is a left or right coframe.

Lemma 5. *Let A be a special biserial selfinjective K–algebra which is not a local Nakayama algebra. If $F_\lambda(\widetilde{Y}) = Y$ is an indecomposable A–module of the first kind that is of τ–period 1 then* $\text{supp}(\widetilde{Y})$ *is of the form*

$$\xrightarrow{\lambda_{1,1}} \cdots \xrightarrow{\lambda_{1,t}} \xleftarrow{\alpha_2} \xrightarrow{\lambda_{2,1}} \cdots \xrightarrow{\lambda_{2,t}} \xleftarrow{\alpha_3} \xrightarrow{\lambda_{3,1}} \cdots \xrightarrow{\lambda_{3,t}} \xleftarrow{\alpha_4} \cdots \xrightarrow{\lambda_{m,1}} \cdots \xrightarrow{\lambda_{m,t}}$$

$m \geq 1$, *where* $F(\lambda_{i,j}) = F(\lambda_{1,j}$, $i = 1, 2, \ldots, m$, $F(\alpha_s) = F(\alpha_2)$, $s = 2, \ldots, m$ *and* $\lambda_{1,1} \cdots \lambda_{1,t}$ *is a maximal nonzero path that does not connect a top of an indecomposable projective* $\tilde{A}$*–module with its socle.*

Proof. This lemma follows immediatly from the description of indecomposable modules for special biserial algebras contained in [9, 27]. □

Lemma 6. *Let A be a selfinjective special biserial K-algebra which is not a local Nakayama algebra. If $F_\lambda(\widetilde{Y}) = Y$ is an indecomposable A-module of the first kind that is of τ–period 1 then* $\text{s-top}(Y) \neq 0$ *and* $\text{s-soc}(Y) \neq 0$.

Proof. Suppose that $F_\lambda(\widetilde{Y}) = Y$ satisfies the assumptions of the lemma. We deduce from Lemma 7 that $\text{supp}(\widetilde{Y})$ is of the following form :

$$\xrightarrow{\lambda_{1,1}} \cdots \xrightarrow{\lambda_{1,t}} \xleftarrow{\alpha_2} \xrightarrow{\lambda_{2,1}} \cdots \xrightarrow{\lambda_{2,t}} \xleftarrow{\alpha_3} \cdots \xrightarrow{\lambda_{m,1}} \cdots \xrightarrow{\lambda_{m,t}} .$$

But consider an $\tilde{A}$-module $\widetilde{Y}_1$ whose support is of the form

$$\xrightarrow{\lambda_{1,1}} \cdots \xrightarrow{\lambda_{1,t}} \xleftarrow{\alpha_2} \cdots \xrightarrow{\lambda_{m,1}} \cdots \xrightarrow{\lambda_{m,t}} \xleftarrow{\alpha_{m+1}}$$

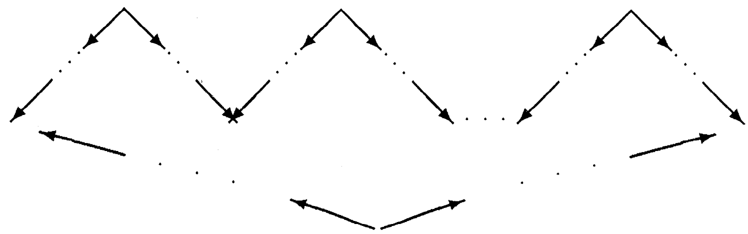
with $F(\alpha_{m+1}) = F(\alpha_2)$. Then $\text{s-top}(Y_1) \neq 0$, because it is not of τ-period 1. Consequently $\text{s-top}(Y) \neq 0$. Dually one proves that $\text{s-soc}(Y) \neq 0$. □

An A-module X is said to have a *finite nonzero s-top* (resp. *finite nonzero s-socle*) if $\text{s-top}(X)$ (resp $\text{s-soc}(X)$) is a direct sum of finitely many nonzero indecomposable A-modules.

Corollary 1. *Let A be a selfinjective special biserial K–algebra which is not a local Nakayama algebra. Every nonzero finite-dimensional A–module of the first kind has a finite nonzero s-top and a finite nonzero s-socle.*

Proof. By definition of the maximal system of orthogonal stable A-bricks and by Lemma 6 our corollary is obvious. □

6. Modules of the second kind. The aim of this section is proving that A–modules of the second kind have also nonzero finite s-tops and nonzero finite s-socles. We start with some known facts. Let $A = KQ_A/I_A$ and the bound quiver (Q_A, I_A) satisfies the required conditions for special biserial algebras. We are interested in closed walks which are assumed to have the property that their start points coincide with their end points. A closed walk w in (Q_A, I_A) will be called *primitive* [27] if it is not of the form v^n for some natural $n \geq 2$, and w is not of the form $w = w_1 u w_2$, where u is a path (resp. a formal inverse of a path) such that either u (resp. u^{-1}) lies in I_A, or $u - v$ (resp. $u^{-1} - v$) belongs to I_A for some path $v \neq \lambda u$ (resp. $v \neq \lambda u^{-1}$) in Q_A, $\lambda \in K^*$, or else u is of the forms $\alpha\alpha^{-1}$, $\alpha^{-1}\alpha$ for some arrow α in Q_A. It is well-known (see [27]) that primitive walks in (Q_A, I_A) produce A-modules of the second kind. We shall visualize primitive closed walks w as the following quivers



and we shall identify them with covering functors w from the path categories of the above quivers Q_w to KQ_A/I_A. Thus every indecomposable A-module of the second kind is (up to isomorphism) of the form $F_w(M(Q_w, m, \lambda))$, where $F_w\colon \text{mod-}KQ_w \to \text{mod-}KQ_A/I_A$ is induced by w, and $M(Q_w, m, \lambda)$ is a representation of Q_w which has K^m at each vertex, the identity map at each but one arrow and the Jordan box $J_m(\lambda)$ at the exceptional arrow (it does not matter which one) for some $\lambda \in K^*$ (see [27]). Consequently we can look at A-modules of the second kind as at KQ_w–modules of the second kind. Moreover nonzero maps between A-modules of the first kind and of the second kind are induced by nonzero functors between supports of finite-dimensional $\tilde{A}$–modules and KQ_w (in particular by nontrivial maps between their quivers).

For an A-module Z of the form $Z = F_w(M(Q_w, m, \lambda))$ consider an A–module $Z^\vee$ which is a direct sum of m copies of $F_w(L_x)$ for all sinks x in Q_w, where L_x is an injective KQ_w–module with $\text{s-soc}(L_x) \cong S_x$. Thus we have an injection i_Z from Z to $Z^\vee$. Dually consider an A–module $Z^\wedge$ which is a direct sum of m copies of $F_w(C_y)$ for all sources y in Q_w, where C_y is a projective KQ_w-module with $\text{s-top}(C_y) \cong S_y$. Consequently we have a projection π_Z from $Z^\wedge$ to Z. Let us denote by $\text{Hom}_A^{\pi_Z}(Z^\wedge, Y)$ the set of morphisms $f\colon Z^\wedge \to Y$ such that $f|_{\ker(\pi_Z)} = 0$. Thus $(\pi_Z)_* = \text{Hom}_A(\pi_Z, Y)$ establishes an isomorphism between $\text{Hom}_A^{\pi_Z}(Z^\wedge, Y)$ and $\text{Hom}_A(Z, Y)$ for every A–module Y. Dually let $\text{Hom}_A^{i_Z}(Y, Z^\vee)$ denotes the set of morphisms $g\colon Y \to Z^\vee$ such that the composition $hg = 0$, where $h\colon Z^\vee \to \text{coker}(i_Z)$. Consequently we have $\text{Hom}_A^{i_Z}(Y, Z^\vee) \cong \text{Hom}_A(Y, Z)$ established by the isomorphism $(i_Z)^* = \text{Hom}_A(Y, i_Z)$. Moreover the following lemma is true.

Lemma 7. *Let A be a selfinjective special biserial K–algebra which is not a local Nakayama algebra. For every A–module Y and for every indecomposable A–module $Z = F_w(M(Q_w, m, \lambda))$ the isomorphisms $(i_Z)^*$ and $(\pi_Z)_*$ induce the following isomorphisms:*

$$\underline{\text{Hom}}_A^{i_Z}(Y, Z^\vee) \cong \underline{\text{Hom}}_A(Y, Z), \qquad \underline{\text{Hom}}_A^{\pi_Z}(Z^\wedge, Y) \cong \underline{\text{Hom}}_A(Z, Y).$$

Proof. In order to prove that $\underline{\text{Hom}}_A^{\pi_Z}(Z^\wedge, Y) \cong \underline{\text{Hom}}_A(Z, Y)$ it is enough to show that for every $f \in \text{Hom}_A(Z, Y)$ it holds: if $\underline{f\pi_Z} = 0$ then $\underline{f} = 0$. But if $\underline{f\pi_Z} = 0$ then $f\pi_Z$ factors through a projective A–module, hence $f\pi_Z$ factors through an injective envelope $E(Z^\wedge)$ of $Z^\wedge$. But $f|_{\ker(\pi_Z)} = 0$ so $f\pi_Z$ factors through an injective envelope $E(Z)$ of Z. Therefore f factors through $E(Z)$ and $\underline{f} = 0$. Dual arguments show that $\underline{\text{Hom}}_A^{i_Z}(Y, Z^\vee) \cong \underline{\text{Hom}}_A(Y, Z)$. □

Lemma 8. *Let A be a special biserial selfinjective K–algebra which is not a local Nakayama algebra. Every finite-dimensional A–module of the second kind has a finite nonzero s-top and a finite nonzero s-socle.*

Proof. The lemma is an easy consequence of Lemma 7 and Corollary 1. □

7 Maximal modules. Let $(\widetilde{Y}) = Y$ be an indecomposable $\tilde{A}$–module and set $Y = F_\lambda(\widetilde{Y})$. Let $F_\lambda(\widetilde{M}) = M \in \mathcal{M}_A$ and $0 \neq F_\lambda(\tilde{p}) = \underline{p}\colon Y \to M$. An indecomposable A–module $X = F_\lambda(\widetilde{X})$ is said to be *produced by* $\mathrm{lf}(\underline{\tilde{p}})$ if one of the following conditions is satisfied:

(1) $\mathrm{supp}(\widetilde{M})$ is of the form

$$- \cdots \to r_i \leftarrow \cdots \leftarrow r_{i+1} \to \cdots \to r_{i+2} \leftarrow \cdots \to r_j \leftarrow \cdots -$$

$\mathrm{supp}(\widetilde{Y})$ is of the form

$$- \cdots \to x \leftarrow \cdots \leftarrow r_i \to \cdots \to r'_{i+1} \leftarrow \cdots \leftarrow r_{i+2} \to \cdots \leftarrow r_j - \cdots -$$

with $x \in (r'_{i-1} \leftarrow \cdots \leftarrow r_i)$ $x \neq r'_{i-1}$, $x \neq r_0$, and $\underline{\tilde{p}}$ is given by a composition of a projection of $\widetilde{Y}$ onto S_{r_i} with an injection of S_{r_i} into $\widetilde{M}$, $\mathrm{supp}(\widetilde{X})$ is of the form:

$$r_{-1} \to \cdots \to r'_0 \leftarrow \cdots \leftarrow r_{i-4} \to \cdots \to r'_{i-3} \leftarrow \cdots \leftarrow r_{i-2} \to \ldots$$
$$\to r'_{i-1} \leftarrow \cdots \leftarrow x \leftarrow \cdots \leftarrow r_i \to \cdots \to r'_{i+1} \leftarrow \cdots \leftarrow r_j - \cdots -$$

if r_0 is a source in $\mathrm{supp}(\widetilde{M})$ and

$$r_{0,+} \to \cdots \to r'_1 \leftarrow \cdots \leftarrow r_{i-2} \to \cdots \to r'_{i-1} \leftarrow \cdots \leftarrow x \leftarrow \cdots \leftarrow r_i \to \ldots$$
$$\to r'_{i+1} \leftarrow \cdots \leftarrow r_j - \cdots -$$

if r_0 is a sink in $\mathrm{supp}(\widetilde{M})$.

(2) $\mathrm{supp}(\widetilde{M})$ is of the form

$$- \cdots \to r_i \leftarrow \cdots \leftarrow r_{i+1} \to \cdots \to r_{i+2} \leftarrow \cdots \to r_j \leftarrow \cdots -$$

$\mathrm{supp}(\widetilde{Y})$ is of the form

$$- \cdots \to x \leftarrow \cdots \leftarrow r_i \leftarrow \cdots \leftarrow r_{i+1} \to \cdots \to r_{i+2} \leftarrow \cdots \to r_j - \cdots -$$

with $x \in (r_i \to \cdots \to r'_{i+1})$, $x \neq r'_{i+1}$, $x \neq r_0$ and $\underline{\tilde{p}}$ is given by a composition of a projection of $\widetilde{Y}$ onto an indecomposable $\tilde{A}$-module $\widetilde{Y}_1$ whose support is

$$r_i \leftarrow \cdots \leftarrow r_{i+1} \to \cdots \to r_{i+2} \leftarrow \cdots \to r_j \leftarrow \cdots \leftarrow y$$

or

$$r_i \leftarrow \cdots \leftarrow r_{i+1} \to \cdots \to r_{i+2} \leftarrow \cdots \to r_j$$

with an injection of $\widetilde{Y}_1$ into $\widetilde{M}$, and $\mathrm{supp}(\widetilde{X})$ is of the form

$$- \cdots \to x \leftarrow \cdots \leftarrow r_i \to \cdots \to r'_{i-1} \leftarrow \cdots \leftarrow r_{i-2} \to \cdots \to r'_0 \leftarrow \cdots \leftarrow r_{-1}$$

if r_0 is a source in $\mathrm{supp}(\widetilde{M})$ and

$$- \cdots \to x \leftarrow \cdots \leftarrow r_i \to \cdots \to r'_{i-1} \leftarrow \cdots \leftarrow r_{i-2} \to \cdots \to r'_1 \leftarrow \cdots \leftarrow r_{0,+}$$

if r_0 is a sink in $\mathrm{supp}(\widetilde{M})$

(3) $\mathrm{supp}(\widetilde{M})$ is of the form

$$-\cdots\to r_i \leftarrow\cdots\leftarrow r_{i+1}\to\cdots\to r_{i+2}\leftarrow\cdots\to r_j\leftarrow\cdots-$$

$\mathrm{supp}(\widetilde{Y})$ is of the form

$$-\cdots\leftarrow x\to\cdots\to r_i\leftarrow\cdots\leftarrow r_{i+1}\to\cdots\to r_{i+2}\leftarrow\cdots\to r_j-\cdots-$$

with $x\in(r_{i-1}\to\cdots\to r_i)$ and $x=r_{i-1}$ implies $i=1$, $\underline{\tilde{p}}$ is given by a composition of a projection of $\widetilde{Y}$ onto an indecomposable $\tilde{A}$-module $\widetilde{Y}_1$ whose support is

$$x\to\cdots\to r_i\leftarrow\cdots\leftarrow r_{i+1}\to\cdots\to r_{i+2}\leftarrow\cdots\to r_j\leftarrow\cdots\leftarrow y$$

or

$$x\to\cdots\to r_i\leftarrow\cdots\leftarrow r_{i+1}\to\cdots\to r_{i+2}\leftarrow\dots r_j$$

with an injection of $\widetilde{Y}_1$ into $\widetilde{M}$, and $\mathrm{supp}(\widetilde{X})$ is of the form

$$-\cdots\leftarrow x\to\cdots\to r_i\to\cdots\to r'_{i-1}\leftarrow\cdots\leftarrow r_{i-2}\to\cdots\to r'_0\leftarrow\cdots\leftarrow r_{-1}$$

if r_0 is a source in $\mathrm{supp}(\widetilde{M})$ and $x\neq r_0$, or

$$-\cdots\leftarrow x\to\cdots\to r_i\to\cdots\to r'_{i-1}\leftarrow\cdots\leftarrow r_{i-2}\to\cdots\to r'_1\leftarrow\cdots\leftarrow r_{0,+}$$

if r_0 is a sink in $\mathrm{supp}(\widetilde{M})$ and $x\neq r_0$, or else $-\cdots-x_-$ if $x=r_0$.

(4) $\mathrm{supp}(\widetilde{M})$ is of the form

$$-\cdots\to r_i\leftarrow\cdots\leftarrow r_{i+1}\to\cdots\to r_{i+2}\leftarrow\cdots\to r_j\leftarrow\cdots-$$

$\mathrm{supp}(\widetilde{Y})$ is of the form

$$-\cdots\leftarrow x\to\cdots\to r_i\to\cdots\to r'_{i+1}\leftarrow\cdots\leftarrow r_{i+2}\to\cdots\leftarrow r_j-\cdots-$$

with $x\in(r_i\leftarrow\cdots\leftarrow r_{i+1})$, $x\neq r_{i+1}$, and $\underline{\tilde{p}}$ is given by a composition of a projection of $\widetilde{Y}$ onto an indecomposable $\tilde{A}$-module $\widetilde{Y}_1$ whose support is $x\to\cdots\to r_i$ with an injection of $\widetilde{Y}_1$ into $\widetilde{M}$, $\mathrm{supp}(\widetilde{X})$ is of the form

$$-\cdots\leftarrow x\to\cdots\to r_i\to\cdots\to r'_{i+1}\leftarrow\cdots\leftarrow r_{i+2}\to\cdots\to r'_{t+1}\leftarrow\cdots\leftarrow r_{t+2}$$

if r_{t+1} is a source in $\mathrm{supp}(\widetilde{M})$,or

$$-\cdots\leftarrow x\to\cdots\to r_i\to\cdots\to r'_{i+1}\leftarrow\cdots\leftarrow r_{i+2}\to\cdots\to r'_t\leftarrow\cdots\leftarrow r_{t+1,-}$$

if r_{t+1} is a sink in $\mathrm{supp}(\widetilde{M})$.

Symmetrically we define a *module produced by* $\mathrm{rf}(\underline{\tilde{p}})$.

Lemma 9. *Let A be a selfinjective special biserial K–algebra which is not a local Nakayama algebra. Let $F_\lambda(\widetilde{M}) = M \in \mathcal{M}_A$. Let $F_\lambda(\widetilde{Y}) = Y$ be an indecomposable A-module of the first kind. Suppose that* s-top$(Y) \cong M$ *and* $0 \neq F_\lambda(\underline{\tilde{p}}) = \underline{p}\colon Y \to M$. *Let $X_1 = F_\lambda(\widetilde{X}_1)$ be produced by* $\mathrm{lf}(\underline{\tilde{p}})$ *and let $X_2 = F_\lambda(\widetilde{X}_2)$ be produced by* $\mathrm{rf}(\underline{\tilde{p}})$. *Then the following implications hold:*

(1) *If $X_1 = 0$, then* s-top(X_2) *is indecomposable and for every* $0 \neq F_\lambda(\underline{\tilde{q}}) = \underline{q}\colon X_2 \to$ s-top(X_2) *one of the modules produced by* $\mathrm{lf}(\underline{\tilde{q}})$ *and by* $\mathrm{rf}(\underline{\tilde{q}})$ *is zero.*

(2) *If $X_2 = 0$, then* s-top(X_1) *is indecomposable and for every* $0 \neq F_\lambda(\underline{\tilde{q}}) = \underline{q}\colon X_1 \to$ s-top(X_1) *one of the modules produced by* $\mathrm{lf}(\underline{\tilde{q}})$ *and by* $\mathrm{rf}(\underline{\tilde{q}})$ *is zero.*

Proof. Under the assumptions and notations of the lemma suppose that $X_1 = 0$. Moreover assume that $\mathrm{supp}(\widetilde{M})$ is of the form

$$r_0 - \cdots \to r_i \leftarrow \cdots \leftarrow r_{i+1} \to \cdots \to r_{i+2} \leftarrow \cdots \to r_j \leftarrow \cdots - r_{t+1}.$$

A handy analysis shows that if $X_1 = 0$ then $\mathrm{supp}(\widetilde{Y})$ must be one of the following forms:

$$r_0 - \cdots \to r_i \leftarrow \cdots \leftarrow r_{i+1} \to \cdots \to r_{i+2} \leftarrow \cdots \to r_j \to \cdots \to \mathrm{rf}(\underline{\tilde{p}}) \leftarrow \cdots - \qquad \text{(i)}$$

$$r_0 - \cdots \to r_i \leftarrow \cdots \leftarrow r_{i+1} \to \cdots \to r_{i+2} \leftarrow \cdots \to r_j \leftarrow \cdots \leftarrow rf(\underline{\tilde{p}}) \to \cdots - \qquad \text{(ii)}$$

by definition of the modules produced by frames.

Thus in case (i) $\mathrm{supp}(\widetilde{X}_2)$ is one of the following forms:

$$r_{-1} \to \cdots \to r_0' \leftarrow \cdots \leftarrow r_1 \to \cdots \to r_{i-1}' \leftarrow \cdots \leftarrow r_i \to \dots$$
$$\leftarrow r_{j-2} \to \cdots \to r_{j-1}' \leftarrow \cdots \leftarrow \mathrm{rf}(\underline{\tilde{p}}) \leftarrow \cdots - \qquad (\mathrm{i}_1)$$

if r_0 is a source in $\mathrm{supp}(\widetilde{M})$ and

$$(r_0)_+ \to \cdots \to r_1' \leftarrow \cdots \leftarrow r_2 \to \cdots \leftarrow r_i \to \dots$$
$$\leftarrow r_{j-2} \to \cdots \to r_{j-1}' \leftarrow \cdots \leftarrow \mathrm{rf}(\underline{\tilde{p}}) \leftarrow \cdots - \qquad (\mathrm{i}_2)$$

if r_0 is a sink in $\mathrm{supp}(\widetilde{M})$. Moreover in case (ii) $\mathrm{supp}(\widetilde{X}_2)$ is one of the following forms:

$$r_{t+2} \to \cdots \to r_{t+1}' \leftarrow \cdots \leftarrow r_{j+2} \to \cdots \to r_{j+1}' \leftarrow \dots r_j \leftarrow \mathrm{rf}(\underline{\tilde{p}}) \leftarrow \cdots - \qquad (\mathrm{ii}_1)$$

if r_{t+1} is a source in $\mathrm{supp}(\widetilde{M})$ and

$$(r_{t+1})_- \to \cdots \to r_t' \leftarrow \cdots \leftarrow r_{t-1} \to \cdots \leftarrow r_{j+2} \to \dots$$
$$\to r_{j+1}' \leftarrow \cdots \leftarrow r_j \leftarrow \cdots \leftarrow \mathrm{rf}(\underline{\tilde{p}}) \leftarrow \cdots - \qquad (\mathrm{ii}_2)$$

if r_{t+1} is a sink in $\mathrm{supp}(\widetilde{M})$ and

$$\mathrm{rf}(\underline{\tilde{p}}) \to \cdots - \qquad (\mathrm{ii}_3)$$

if $\mathrm{rf}(\underline{\tilde{p}}) = r_{t+1}$.

It is easy to deduce from Lemma 3 and the orthogonality of elements in $\mathcal{M}_A$ that if $F_\lambda(\widetilde{M_1}) = M_1 \in \mathcal{M}_A$ and M_1 is a direct summand in s-top(X_2) then $\mathrm{supp}(\widetilde{M_1})$ is of the form $r_{-1} - \cdots -$ in Case (i_1), $(r_0)_+ \to \cdots -$ in Case (i_2), $r_{t+2} - \cdots -$ in Case (ii_1), $(r_{t+1})_- - \cdots -$ in

Case (ii_2), $\mathrm{rf}(\tilde{p})$ — $\cdots$ — in Case (ii_3). Hence $\text{s-top}(X_2)$ is indecomposable and implication (1) easily follows.

A similar analysis shows implication (2) which finishes the proof. □

Let Y be an indecomposable nonprojective A-module. Let $0 \neq \underline{p}\colon Y \to M$ with $M \in \mathcal{M}_A$. An A-module X without projective direct summands is said to be *$\underline{p}$–maximal* for Y if the following condition holds: if $X \neq 0$ then there is $0 \neq \underline{f}\colon X \to Y$ such that

(1) $\underline{pf} = 0$
(2) If Z is an A–module such that there is $0 \neq \underline{g}\colon Z \to Y$ with $\underline{pg} = 0$, then there is $\underline{h}\colon Z \to X$ such that $\underline{g} = \underline{f}\underline{h}$.

We have the following description of $\underline{p}$-maximal modules for indecomposable A–modules of the first kind.

Proposition 1. *Let A be a special biserial selfinjective K–algebra which is not a local Nakayama algebra. Let $F_\lambda(\widetilde{M}) = M \in \mathcal{M}_A$. Let $F_\lambda(\widetilde{Y}) = Y$ be an indecomposable A–module of the first kind. Let $0 \neq F_\lambda(\underline{\tilde{p}}) = \underline{p}\colon Y \to M$. If X is a $\underline{p}$–maximal module for Y with $0 \neq \underline{f}: X \to Y$ then $X \cong X_1 \oplus X_2$ and the following conditions are satisfied:*

(a) *$F_\lambda(\widetilde{X}_1) = X_1$ is produced by $\mathrm{lf}(\tilde{p})$, $F_\lambda(\widetilde{X}_2) = X_2$ is produced by $\mathrm{rf}(\tilde{p})$.*
(b) *If $0 \neq \underline{q}\colon Y \to M'$ with $M' \in \mathcal{M}_A$ and $\underline{q} \neq \lambda\underline{p}$ for any $\lambda \in K^*$ then $\underline{qf} \neq 0$, and for $\underline{f} = (\underline{f}_1, \underline{f}_2)$ it holds either $\underline{qf}_1 \neq 0$ and $\underline{qf}_2 = 0$ or $\underline{qf}_1 = 0$ and $\underline{qf}_2 \neq 0$.*
(c) *If $M' \in \mathcal{M}_A$ and there is $0 \neq \underline{q}\colon M' \to X$ then $\underline{fq} \neq 0$.*
(d) *If $M' \in \mathcal{M}_A$ and there is $0 \neq \underline{q}\colon M' \to Y$ then there is $0 \neq \underline{g}\colon M' \to X$ such that for $0 \neq \underline{f} = (\underline{f}_1, \underline{f}_2)$ either $\underline{f}_1\underline{g} = \underline{q}$ and $\underline{f}_2\underline{g} = 0$ or $\underline{f}_2\underline{g} = \underline{q}$ and $\underline{f}_1\underline{g} = 0$.*
(e) *If there is $M' \in \mathcal{M}_A$ such that M' is a direct summand in $\text{s-top}(X)$ and $0 \neq \underline{q}\colon X \to M'$ does not belong to $\underline{\mathrm{Hom}}_A(Y, \text{s-top}(Y))$, then there is an indecomposable direct summand L in $\text{s-rad}(N)$ with N being s-projective whose s-top is M such that $M' = \text{s-top}(L)$. Moreover there are at most two such modules M', M'' and one of them is a direct summand in $\text{s-top}(X_1)$ and the other one is a direct summand in $\text{s-top}(X_2)$.*
(f) *If X_i, $i = 1, 2$, does not have a direct summand M' in its s-top such that there is $0 \neq \underline{q}\colon X \to M'$ with $\underline{q} \notin \underline{\mathrm{Hom}}_A(Y, \text{s-top}(Y))$, then one of the direct summands in $\text{s-top}(\text{s-rad}(N))$, say M_i'', has the property that if $M_i'' \cong \text{s-top}(L_i'')$, L_i'' is an indecomposable direct summand in $\text{s-rad}(N)$, and N is s-projective with $\text{s-top}(N) \cong M$, then there is $0 \neq \underline{t}_i\colon M'' \to X_i$.*
(g) *Let N be s-projective with $\text{s-top}(N) = M$ and let L be an indecomposable direct summand in $\text{s-rad}(N)$. Let $\underline{\alpha}_{N,L}\colon L \to N$ be a coset of an irreducible map $\alpha_{N,L}\colon L \to N$. Then there is $0 \neq \underline{g}\colon N \to Y$ and there is $\lambda_{N,L}(Y) \in K$ such that $\underline{f} \circ (\lambda_{N,L}(Y) \cdot \underline{\alpha}_{N,L})$ is a morphism from L to X_i, $i = 1, 2$, where $\text{s-top}(L)$ is either a direct summand in $\text{s-top}(X)$, or a direct summand in $\text{s-soc}(X)$.*

Proof. Under the assumptions and the notations of the proposition suppose that $\mathrm{supp}(\widetilde{M})$ is of the following form

$$- \cdots \to r_i \leftarrow \cdots \leftarrow r_{i+1} \to \cdots \to r_{i+2} \leftarrow \cdots \to r_j \leftarrow \cdots - .$$

We shall consider two typical cases of $\mathrm{supp}(\widetilde{Y})$.

1. Suppose that $\mathrm{supp}(\widetilde{Y})$ is of the form

$$- \cdots \to x \leftarrow \cdots \leftarrow r_i \to \cdots \to r'_{i+1} \leftarrow \cdots \leftarrow r_j \to \cdots \to y \leftarrow \cdots -$$

with $x \in (r'_{i-1} \leftarrow \cdots \leftarrow r_i)$, $x \neq r'_{i-1}$, $y \in (r_j \rightarrow \cdots \rightarrow r'_{j+1})$, $y \neq r'_{j+1}$. Let $F_\lambda(\widetilde{X}_1) = X_1$, $F_\lambda(\widetilde{X}_2) = X_2$ be the modules produced by $\mathrm{lf}(\underline{\tilde{p}})$ and by $\mathrm{rf}(\underline{\tilde{p}})$ respectively. Thus by definition $\mathrm{supp}(\widetilde{X}_2)$ is of the form

$$\overset{\kappa_0}{—} \cdots \overset{\kappa_l}{—}\ x \leftarrow \cdots \leftarrow r_i \rightarrow \cdots \rightarrow r'_{i+1} \leftarrow \cdots \leftarrow r_{i+2} \rightarrow \cdots \leftarrow r_j \rightarrow \cdots \rightarrow y \leftarrow \cdots —$$

and $\mathrm{supp}(\widetilde{X}_1)$ is of the form

$$— \cdots \rightarrow x \leftarrow \cdots \leftarrow r_i \rightarrow \cdots \rightarrow r'_{i+1} \leftarrow \cdots \rightarrow r'_{j-1} \leftarrow \cdots \leftarrow r_j \rightarrow \cdots \rightarrow y \overset{\rho_0}{—} \cdots \overset{\rho_s}{—}$$

where

$$\overset{\kappa_0}{—} \cdots \overset{\kappa_l}{—} = \begin{cases} r_{-1} \rightarrow \cdots \rightarrow r'_0 \leftarrow \cdots \leftarrow r_1 \rightarrow \cdots \leftarrow r_{i-2} \rightarrow \cdots \rightarrow r'_{i-1} \leftarrow \cdots \leftarrow x \\ \qquad \text{if } r_0 \text{ is a source in } \mathrm{supp}(\widetilde{M}) \\ r_{0,+} \rightarrow \cdots \rightarrow r'_1 \leftarrow \cdots \rightarrow r'_{i-1} \leftarrow \cdots \leftarrow x \\ \qquad \text{if } r_0 \text{ is a sink in } \mathrm{supp}(\widetilde{M}) \end{cases}$$

$$\overset{\rho_0}{—} \cdots \overset{\rho_s}{—} = \begin{cases} y \rightarrow \cdots \rightarrow r'_{j+1} \leftarrow \cdots \leftarrow r_t \rightarrow \cdots \rightarrow r'_{t+1} \leftarrow \cdots \leftarrow r_{t+2} \\ \qquad \text{if } r_{t+1} \text{ is a source in } \mathrm{supp}(\widetilde{M}) \\ y \rightarrow \cdots \rightarrow r'_{j+1} \leftarrow \cdots \leftarrow r_{t-1} \rightarrow \cdots \rightarrow r'_t \leftarrow \cdots \leftarrow r_{t+1,-} \\ \qquad \text{if } r_{t+1} \text{ is a sink in } \mathrm{supp}(\widetilde{M}) \end{cases}$$

It is easy to see that there is $0 \neq \underline{f}\colon X_1 \oplus X_2 \rightarrow Y$ which has the property $\underline{pf} = 0$ by Lemmas 1, 2, and $\underline{f} = (\underline{f}_1, \underline{f}_2)$. If Z is a nonzero A–module of the first kind that is indecomposable and there is $0 \neq \underline{g}\colon Z \rightarrow Y$ then $Z = F_\lambda(\tilde{Z})$ and $\underline{g} = F_\lambda(\underline{\tilde{g}})$. If $\underline{pg} = 0$ and $\mathrm{supp}(\widetilde{Z})$ is disjoint with $r_i \rightarrow \cdots \rightarrow r_{i+1} \leftarrow \cdots \rightarrow r_j$ then obviously $\underline{g}$ factors through $\underline{f}$. If $\mathrm{supp}(\widetilde{Z})$ is not disjoint with $r_i \rightarrow \cdots \rightarrow r_{i+1} \leftarrow \cdots \rightarrow r_j$ then let r_{i_0} be the lowest sink of $\mathrm{supp}(\widetilde{M})$ that is contained in $\mathrm{supp}(\widetilde{Z})$ and let r_{i_1} be the highest sink of $\mathrm{supp}(\widetilde{M})$ that is contained in $\mathrm{supp}(\widetilde{Z})$. Thus $\mathrm{supp}(\widetilde{Z})$ must be of the form

$$— \cdots \leftarrow r_{i_0} \rightarrow \cdots \rightarrow r'_{i_0+1} \leftarrow \cdots \leftarrow r_{i_0+2} \rightarrow \cdots \leftarrow r_{i_1} \rightarrow \cdots —$$

and an easy verification shows that there exists $\underline{h}\colon Z \rightarrow X_1 \oplus X_2$ which has the required properties. Consequently (a) is proved in this case, because for A–modules of the second kind we apply Lemma 7.

In order to prove (b) let us observe that if $0 \neq \underline{q}\colon Y \rightarrow M'$ with $M' = F_\lambda(\tilde{M'}) \in \mathcal{M}_A$ and $\underline{q} = F_\lambda(\underline{\tilde{q}})$ then for $\underline{f}\colon X \rightarrow Y$ it holds $\underline{qf} \neq 0$ for $F_\lambda(\underline{\tilde{q}}) = \underline{q}$ with $\mathrm{lf}(\underline{\tilde{q}}) \geq x$, or $\mathrm{rf}(\underline{\tilde{q}}) \leq y$. Moreover if $\mathrm{lf}(\underline{\tilde{q}}) \geq x$, $\mathrm{rf}(\underline{\tilde{q}}) \neq y$ then $\underline{qf}_2 \neq 0$ and $\underline{qf}_1 = 0$, and if $\mathrm{rf}(\underline{\tilde{q}}) \leq y$, $\mathrm{lf}(\underline{\tilde{q}}) \neq x$ then $\underline{qf}_1 \neq 0$ and $\underline{qf}_2 = 0$. We should only consider the case $\mathrm{lf}(\underline{\tilde{q}}) < x$ and $\mathrm{rf}(\underline{\tilde{q}}) > y$. But if such an M' exists then $M' \cong F_\lambda(\tilde{M'})$ and by Lemma 3 $\mathrm{supp}(\tilde{M'})$ is of the form

$$— \cdots \rightarrow l_{i_0} \leftarrow \cdots \leftarrow l_{i_0+1} \rightarrow \cdots \rightarrow l_{i_0+2} \leftarrow \cdots \leftarrow l_{i_1} \rightarrow \\ \cdots \rightarrow r_{i_1} \leftarrow \cdots \rightarrow r_j \leftarrow \cdots \leftarrow l_{j_1} \rightarrow \cdots —$$

or

$$— \cdots \rightarrow x \leftarrow \cdots \leftarrow r_i \rightarrow \cdots \rightarrow r'_{i+1} \leftarrow \cdots \leftarrow r_{i+2} \rightarrow \cdots \leftarrow r_j \rightarrow \cdots \rightarrow y \leftarrow \cdots — .$$

In the first case $\underline{\mathrm{Hom}}_A(M, M') \neq 0$ and in the other one $\underline{\mathrm{Hom}}_A(M', M) \neq 0$ which contradicts to the fact that M', $M \in \mathcal{M}_A$. If $\mathrm{lf}(\underline{\tilde{q}}) = x$, $\mathrm{rf}(\underline{\tilde{q}}) = y$ then it is easily seen that (b) holds too. Consequently (b) is proved in this case.

In order to prove (c) suppose that $F_\lambda(\tilde{M'}) = M' \in \mathcal{M}_A$ and there is $0 \neq F_\lambda(\underline{\tilde{q}}) = \underline{q}\colon M' \to X$. We may assume that $0 \neq \underline{q}\colon M' \to X_2$. If $\mathrm{lf}(\underline{\tilde{q}}) \geq x$ then it is obvious that $\underline{fq} \neq 0$. We should only check that if $\mathrm{lf}(\underline{\tilde{q}}) < x$ then (c) also holds. But consider a module $T = F_\lambda(\tilde{T})$ for which $\mathrm{supp}(\tilde{T})$ is of the form

$$\overset{\kappa_0}{\text{---}} \cdots \overset{\kappa_l}{\text{---}}\ x \leftarrow \cdots \leftarrow r_i \rightarrow \cdots \rightarrow r'_{i+1} \leftarrow \cdots \leftarrow r_{i+2} \rightarrow$$
$$\cdots \leftarrow r_j \rightarrow \cdots \rightarrow y \rightarrow \cdots \rightarrow r'_{j+1} \leftarrow \cdots \leftarrow r_{t+1} \rightarrow \cdots \rightarrow v$$

where S_v is a direct summand in $P_{r_{t+1}}/\,\text{s-soc}(P_{r_{t+1}})$ if r_{t+1} is a sink in $\mathrm{supp}(\widetilde{M})$, or

$$\overset{\kappa_0}{\text{---}} \cdots \overset{\kappa_l}{\text{---}}\ x \leftarrow \cdots \leftarrow r_i \rightarrow \cdots \rightarrow r'_{i+1} \leftarrow \cdots \leftarrow r_{i+2} \rightarrow$$
$$\cdots \leftarrow r_j \rightarrow \cdots \rightarrow y \rightarrow \cdots \rightarrow r'_{j+1} \leftarrow \cdots \leftarrow r_t \rightarrow \cdots \rightarrow r'^{-}_{t+1}$$

if r_{t+1} is a source in $\mathrm{supp}(\widetilde{M})$. By [21, Proposition 2] and by (a) and Lemma 9 $\text{s-soc}(T)$ is indecomposable and it holds $\mathrm{supp}(\tilde{T})$ is of the form

$$\leftarrow \cdots \leftarrow w \rightarrow \cdots \rightarrow r'_{i_0} \leftarrow \cdots \rightarrow r'_{i-1} \leftarrow \cdots \leftarrow x \leftarrow \cdots \leftarrow r_i \rightarrow$$
$$\cdots \rightarrow r'_{i+1} \leftarrow \cdots \leftarrow r_{i+2} \rightarrow \cdots \leftarrow r_j \rightarrow \cdots \rightarrow y \rightarrow \cdots \rightarrow r'_{j+1} \leftarrow \cdots - z$$

with $z = v$ or $z = r_{t+1,-}$, where $w \in (r_{i_0-1} \rightarrow \cdots \rightarrow r'_{i_0})$, $w \neq r_{i_0-1}$. Consequently $\text{s-soc}(T)$ is the only M' such that there is $0 \neq \underline{q}\colon M' \to X_2$ with $\mathrm{lf}(\underline{\tilde{q}}) < x$, and the composition $\underline{fq} \neq 0$. In the same manner one proves (c) if we replace X_2 by X_1. Moreover the above M' satisfies also (d) by [21]. Applying [21, Proposition 2] one proves (e), (f) dually to (c), (d).

(g) is obvious by the shapes of $\mathrm{supp}(\widetilde{M})$, $\mathrm{supp}(\widetilde{Y})$, $\mathrm{supp}(\widetilde{X}_1)$, $\mathrm{supp}(\widetilde{X}_2)$ and [21, Lemma 14].

2. Suppose that $\mathrm{supp}(\widetilde{Y})$ is of the form

$$- \cdots \leftarrow x \rightarrow \cdots \rightarrow r_i \leftarrow \cdots \leftarrow r_{i+1} \rightarrow \cdots \rightarrow r_{i+2} \leftarrow$$
$$\cdots \leftarrow r_{j-1} \rightarrow \cdots \rightarrow r_j \leftarrow \cdots \leftarrow y \rightarrow \cdots -$$

with $x \in (r_{i-1} \rightarrow \cdots \rightarrow r_i)$, $x = r_{i-1}$ implies $i = 1$, $y \in (r_j \leftarrow \cdots \leftarrow r_{j+1})$, $y = r_{j+1}$ implies $j = t$. Let $F_\lambda(\widetilde{X}_1) = X_1$, $F_\lambda(\widetilde{X}_2) = X_2$ be the modules produced by $\mathrm{lf}(\underline{\tilde{p}})$ and $\mathrm{rf}(\underline{\tilde{p}})$ respectively. Thus by definition $\mathrm{supp}(\widetilde{X}_1)$ is of the form

$$- \cdots \leftarrow x \rightarrow \cdots \rightarrow r_i \overset{\rho_0}{\text{---}} \cdots \overset{\rho_s}{\text{---}}$$

and $\mathrm{supp}(\widetilde{X}_2)$ is of the form

$$\overset{\kappa_0}{\text{---}} \cdots \overset{\kappa_l}{\text{---}}\ r_j \leftarrow \cdots \leftarrow y \rightarrow \cdots -$$

where

$$\frac{\rho_0}{} \cdots \frac{\rho_s}{} = \begin{cases} r_{-1} \to \cdots \to r'_0 \leftarrow \cdots \leftarrow r_1 \to \cdots \to r'_{i-1} \leftarrow \cdots \leftarrow r_i \\ \qquad \text{if } r_0 \text{ is a source in } \operatorname{supp}(\widetilde{M}) \\ r_{0,+} \to \ldots \to r'_1 \leftarrow \cdots \to r'_{i-1} \leftarrow \cdots \leftarrow r_i \\ \qquad \text{if } r_0 \text{ is a sink in } \operatorname{supp}(\widetilde{M}) \end{cases}$$

$$\frac{\kappa_0}{} \cdots \frac{\kappa_l}{} = \begin{cases} r_j \to \cdots \to r'_{j+1} \leftarrow \cdots \to r'_{t+1} \leftarrow \cdots \leftarrow r_{t+2} \\ \qquad \text{if } r_{t+1} \text{ is a source in } \operatorname{supp}(\widetilde{M}) \\ r_j \to \cdots \to r'_{j+1} \leftarrow \cdots \to r'_t \leftarrow \cdots \leftarrow r_{t+1,-} \\ \qquad \text{if } r_{t+1} \text{ is a sink in } \operatorname{supp}(\widetilde{M}) \end{cases}$$

It is easily seen that there is $0 \neq \underline{f} = (\underline{f}_1, \underline{f}_2)\colon X_1 \oplus X_2 \to Y$ which has the property that $\underline{pf} = 0$ by Lemmas 1, 2. If Z is a nonzero A–module of the first kind that is indecomposable and there is $0 \neq \underline{g}\colon Z \to Y$ then $Z = F_\lambda(\tilde{Z})$ and $\underline{g} = F_\lambda(\tilde{g})$. If $\underline{pg} = 0$ then $\operatorname{supp}(\tilde{Z})$ cannot be contained in $r_i \to \cdots \to r'_{i+1} \leftarrow \cdots \to r'_{j-1} \leftarrow \cdots \leftarrow r_j$, otherwise $\underline{\operatorname{Hom}}_A(Z, Y) = 0$ or $\underline{\operatorname{Hom}}_A(Z, M) \neq 0$. Now we can follow the arguments used in 1. and (a) –(g) hold.

1. and 2. are typical cases of $\operatorname{supp}(\widetilde{Y})$, and in each another case one proceeds similarly to 1., 2. We leave the details to the reader. □

8 s-radicals. The aim of this section is a generalization of the notion of an s-radical that was introduced for s-projective modules only.

Let Y be a nonprojective A–module. An A–module X without projective direct summands is said to be an *s-radical* of Y, and is denoted by $\text{s-rad}(Y)$, if there is $0 \neq \underline{f}\colon X \to Y$ such that the following conditions are satisfied:

(1) If $0 \neq \underline{p}\colon Y \to \text{s-top}(Y)$ then $\underline{pf} = 0$.
(2) If Z is such an A–module that there is $0 \neq \underline{g} : Z \to Y$ with $\underline{pg} = 0$ for any $0 \neq \underline{p} : Y \to \text{s-top}(Y)$ then there exists $0 \neq \underline{h} : Z \to X$ such that $\underline{g} = \underline{fh}$.

Remark 1. The s-radical of an s-projective A–module defined in Section 4 shares the above properties.

Proposition 2. *Let A be a selfinjective special biserial K–algebra which is not a local Nakayama algebra. Every nonprojective A–module Y of the first kind has its s-radical whose s-socle is contained in* $\text{s-soc}(Y)$. *Moreover,* $\text{s-rad}(Y)$ *is an A-module of the first kind.*

Proof. It is obvious that we need only to show the proposition for indecomposable A–modules of the first kind. Let Y be such an A-module. We fix a K–basis $\{\underline{p}_1, \ldots, \underline{p}_s\}$ of $\underline{\operatorname{Hom}}_A(Y, \text{s-top}(Y))$ in such a way that each $\underline{p}_i$ is in $\underline{\operatorname{Hom}}_A(Y, M)$, $M \in \mathcal{M}_A$. Thus, taking the $\underline{p}_1$–maximal module Y_1 for Y we have that $\{\underline{p}_2, \ldots, \underline{p}_s\}$ is a K–basis of $\underline{\operatorname{Hom}}_A(Y_1, \text{s-top}(Y))$ by Proposition 1. Consequently we can take Y_2 to be the $\underline{p}_2$–maximal module for Y_1. Continuing this procedure successivly we obtain a module Y_s that is $\text{s-rad}(Y)$ and our proposition follows by Proposition 1. □

Lemma 10. *Let $A = KQ_A/I_A$ be a selfinjective special biserial algebra that is not a local Nakayama algebra. There are only finitely many nonisomorphic indecomposable A–modules of the first kind with a fixed finite s-top.*

Proof. We shall prove our lemma in two steps. Let Y be an indecomposable A–module of the first kind with $F_\lambda(\widetilde{Y}) = Y$. Let $\text{s-top}(Y) = M \in \mathcal{M}_A$ be indecomposable with $F_\lambda(\widetilde{M}) = M$. Let

$0 \neq F_\lambda(\underline{\tilde{p}}) = \underline{p}\colon Y \to M$. Then, by Lemma 3, $\mathrm{supp}(\widetilde{M})$ is of the form

$$- \cdots \to r_i \leftarrow \cdots \leftarrow r_{i+1} \to \cdots \to r_{i+2} \leftarrow \cdots \to r_j \leftarrow \cdots -$$

and $\mathrm{supp}(\widetilde{Y})$ is one of the following forms:

(i) $- \cdots \to \mathrm{lf}(\underline{\tilde{p}}) \leftarrow \cdots \leftarrow r_i \to \cdots \to r'_{i+1} \leftarrow \cdots \leftarrow r_{i+2} \to \cdots \leftarrow r_j - \cdots -$

(ii) $- \cdots \to \mathrm{lf}(\underline{\tilde{p}}) \leftarrow \cdots \leftarrow r_i \leftarrow \cdots \leftarrow r_{i+1} \to \cdots \to r_{i+2} \leftarrow \cdots \to r_j - \cdots -$

(iii) $- \cdots \leftarrow \mathrm{lf}(\underline{\tilde{p}}) \to \cdots \to r_i \leftarrow \cdots \leftarrow r_{i+1} \to \cdots \to r_{i+2} \leftarrow \cdots \to r_j - \cdots -$

(iv) $- \cdots \leftarrow \mathrm{lf}(\underline{\tilde{p}}) \to \cdots \to r_i \to \cdots \to r'_{i+1} \leftarrow \cdots \leftarrow r_{i+2} \to \cdots \leftarrow r_j - \cdots -$.

In each of the above cases, if $(- \cdots \to \mathrm{lf}(\underline{\tilde{p}})) \neq \mathrm{lf}(\underline{\tilde{p}})$ or $(- \cdots \leftarrow \mathrm{lf}(\underline{\tilde{p}})) \neq \mathrm{lf}(\underline{\tilde{p}})$, then the indecomposable A–module $Y_1 = F_\lambda(\widetilde{Y}_1)$ with $\mathrm{supp}(\widetilde{Y}_1)$ of the form $- \cdots \to lf(\underline{\tilde{p}})$ or $- \cdots \leftarrow lf(\underline{\tilde{p}})$ respectively has also its s-top which is not given by $\lambda \cdot \underline{p}$ for any $\lambda \in K^*$ by the properties of $\mathrm{lf}(\underline{\tilde{p}})$. An easy verification shows that $\text{s-top}(Y_1) \subset \text{s-top}(Y)$, hence $\text{s-top}(Y)$ is not indecomposable. We can do the same with $\mathrm{rf}(\underline{\tilde{p}})$ and we obtain that $\mathrm{supp}(\widetilde{Y})$ starts at $\mathrm{lf}(\underline{\tilde{p}})$ and ends at $\mathrm{rf}(\underline{\tilde{p}})$. Hence the number of isoclasses of indecomposable A–modules Y of the first kind with $\text{s-top}(Y) \cong M$ is bounded by the maximal number of relation-free walks between vertices of $\mathrm{supp}(\widetilde{M}) \cup \mathrm{supp}(\widetilde{N})$, where $N = F_\lambda(\widetilde{N})$ is the s-projective A-module whose s-top is M. Consequently this number is finite and the required condition holds.

Let Y be an indecomposable A-module of the first kind with $\dim_K \underline{\mathrm{Hom}}_A(Y, \text{s-top}(Y)) \geq 2$. Let $Y = F_\lambda(\widetilde{Y})$ and let the vertices of $\mathrm{supp}(\widetilde{Y})$ be numbered increasingly from the left to the right.

Let $\underline{p} = F_\lambda(\underline{\tilde{p}}) : Y \to M \in \mathcal{M}_A$ be an element of a fixed K–basis of $\underline{\mathrm{Hom}}_A(Y, \text{s-top}(Y))$ such that $\mathrm{lf}(\underline{\tilde{p}})$ is minimal in the family of all left frames of the fixed K–basis. Thus in the same way as above we can show that $\mathrm{supp}(\widetilde{Y})$ starts at $\mathrm{lf}(\underline{\tilde{p}})$ and ends at $\mathrm{rf}(\underline{\tilde{q}})$ for some $\underline{q} = F_\lambda(\underline{\tilde{q}})$ belonging to the fixed K-basis. Furthermore in the same manner one can prove that if $\mathrm{rf}(\underline{\tilde{p}}_1) < \mathrm{lf}(\underline{\tilde{p}}_2)$ then there is $\underline{p}_3 = F_\lambda(\underline{\tilde{p}}_3)$ with $\mathrm{lf}(\underline{\tilde{p}}_3) \leq \mathrm{rf}(\underline{\tilde{p}}_1)$ and $\mathrm{rf}(\underline{\tilde{p}}_3) > \mathrm{rf}(\underline{\tilde{p}}_1)$. Consequently the number of isoclasses of Y with a fixed s-top is bounded by the number of composed walks of the form as in the first part of the proof. This number is also finite and our lemma is proved. □

Lemma 11. *Let $A = KQ_A/I_A$ be a selfinjective special biserial K–algebra that is not a local Nakayama algebra. There are only finitely many nonisomorphic indecomposable A–modules of the first kind with a fixed finite s-socle.*

Proof. The proof is dual to that of Lemma 10. □

Now we can define inductively $\text{s-rad}^{n+1}(Y) = \text{s-rad}(\text{s-rad}^n(Y))$ for every natural number n, where $\text{s-rad}^0(Y) = Y$.

Proposition 3. *Let $A = KQ_A/I_A$ be a selfinjective special biserial K–algebra that is not a local Nakayama algebra. For every finite-dimensional A–module Y of the first kind there exists a natural number n_Y with* $\text{s-rad}^{n_Y}(Y) = 0$.

Proof. Let $A \cong KQ_A/I_A$ be a selfinjective special biserial K-algebra that is not a local Nakayama algebra. If Y is a finite-dimensional of the first kind then by Proposition 2 $\text{s-rad}(Y)$ is an A-module of the first kind whose s-socle is contained in $\text{s-soc}(Y)$. If $\text{s-rad}^n(Y) \neq 0$ for every natural n then by definition we have an infinite sequence of nonzero maps

$$\cdots \to \text{s-rad}^n(Y) \xrightarrow{f_n} \text{s-rad}^{n-1}(Y) \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_2} \text{s-rad}(Y) \xrightarrow{f_1} Y$$

such that for each indecomposable direct summand M in s-soc(s-rad$^n(Y)$) and every nonzero map $\underline{q}\colon M \to$ s-soc(s-rad$^n(Y)$) it holds $\underline{f}_1\underline{f}_2\cdots\underline{f}_n\underline{q} \neq 0$. Moreover, by Lemma 11, there is only finitely many such modules, hence $\underline{f}_m\cdots\underline{f}_r$ is an isomorphism for some natural $r > m$. Therefore $\underline{f}_m$ is an isomorphism for some natural m which contradicts to the definition of s-radicals. Consequently there is a natural number n_Y with s-rad$^{n_Y}(Y) = 0$. $\square$

9. s-supports of A-modules of the first kind. Let Y be an indecomposable A-module of the first kind. For each s-projective A–module N with respect to $\mathcal{M}_A$ and for each indecomposable direct summand L in s-rad(N) we fix a coset $\underline{\alpha}_{N,L}$ of an irreducible map $\alpha_{N,L}\colon L \to N$. Thus an *s-support* of Y, that will be denoted by s-supp$_{\mathcal{M}_A}(Y)$, is the path category of the following relation-free quiver $Q_{\mathcal{M}_A}(Y)$: vertices of $Q_{\mathcal{M}_A}(Y)$ are indecomposable direct summands in s-top(s-rad$^n(Y)$) for all $n = 0, 1, 2, \ldots$, where we do not identify isomorphic direct summands. If M_1, M_2 are direct summands in s-top(s-rad$^{n_1}(Y)$) and s-top(s-rad$^{n_2}(Y)$) respectively for some $n_1, n_2 = 0, 1, 2, \ldots$ then there is an arrow $M_1 \xrightarrow{\underline{\alpha}_{N_1,L_1}} M_2$ in $Q_{\mathcal{M}_A}(Y)$ iff $n_2 = n_1 + 1$ and there is a coset $\underline{\alpha}_{N_1,L_1}$ such that s-top$(N_1) \cong M_1$ and s-top$(L_1) \cong M_2$.

Lemma 12. *Let A be a special biserial selfinjective K-algebra which is a local Nakayama algebra. Let Y be an indecomposable A-module of the first kind. Then* s-supp$_{\mathcal{M}_A}(Y)$ *is a path category of a finite connected quiver $Q_{\mathcal{M}_A}(Y)$ of Dynkin type $\mathbf{A}_n$ and the following conditions hold:*

(a) *The sources in $Q_{\mathcal{M}_A}(Y)$ correspond to the indecomposable direct summands in* s-top(Y).

(b) *The sinks in $Q_{\mathcal{M}_A}(Y)$ correspond to the indecomposable direct summands in* s-soc(Y).

(1) *(c) If Y is s-projective then $Q_{\mathcal{M}_A}(Y)$ is one of the forms*

$$\xleftarrow{\underline{\alpha}_{N_{2s+1},L_{2s+1,2s+1}}} \cdots \xleftarrow{\underline{\alpha}_{N_3,L_{3,3}}} \xleftarrow{\underline{\alpha}_{N_1,L_{1,1}}} \xleftarrow{\underline{\alpha}_{Y,L_1}} \xrightarrow{\underline{\alpha}_{Y,L_2}} \xrightarrow{\underline{\alpha}_{N_2,L_{2,2}}} \cdots \xrightarrow{\underline{\alpha}_{N_{2t},L_{2t,2t}}} \xrightarrow{\underline{\alpha}_{N,L}} \xrightarrow{\underline{\alpha}_{N_1,L_1}} \cdots \xrightarrow{\underline{\alpha}_{N_s,L_s}}.$$

(d) *If $Q = \leftarrow \cdots \xleftarrow{\underline{\alpha}_{N,L_1}} \xrightarrow{\underline{\alpha}_{N,L_2}} \cdots \rightarrow$ is a subquiver in $Q_{\mathcal{M}_A}(Y)$ then Q is a subquiver of $Q_{\mathcal{M}_A}(N)$.*

(e) *If $Q = \xrightarrow{\underline{\alpha}_{N,L}} \cdots \rightarrow$ is a subquiver in $Q_{\mathcal{M}_A}(Y)$ then Q is a subquiver in $Q_{\mathcal{M}_A}(N)$.*

Proof. Let Y be an indecomposable A–module of the first kind. By Corollary 1, Proposition 3 and by the above construction of $Q_{\mathcal{M}_A}(Y)$ we infer that $Q_{\mathcal{M}_A}(Y)$ is finite. Inductively on the number of vertices in $Q_{\mathcal{M}_A}(Y)$ we shall prove the remained part of our lemma. If $Q_{\mathcal{M}_A}(Y)$ has only one vertex then the required conditions are obvious, since $Y \in \mathcal{M}_A$. Suppose that our assertions hold for all A–modules X whose quivers $Q_{\mathcal{M}_A}(X)$ have less vertices than n, and let Y be such a module that $Q_{\mathcal{M}_A}(Y)$ has n vertices. Thus s-rad(Y) is a direct sum of indecomposable A–modules of the first kind and each indecomposable direct summand Y_i in s-rad(Y) has the property $Q_{\mathcal{M}_A}(Y_i)$ has less vertices than n. By the inductive assumption $Q_{\mathcal{M}_A}(Y_i)$ is connected of type $\mathbf{A}_n$ and (a)–(e) hold. But by the construction of $Q_{\mathcal{M}_A}(Y)$ and by the construction of s-rad(Y) in the proof of Proposition 2 we infer that $Q_{\mathcal{M}_A}(Y)$ is of type $\mathbf{A}_n$ in view of Proposition 1 and (a), (b) hold. Since by [21, Proposition 2] each indecomposable direct summand in s-rad(Y) has an indecomposable s-top and an indecomposable s-socle for s-projective Y by Proposition 2 and Lemma 9, hence (c) holds. In order to prove (d) observe that by the definition of an s-projective module we have a nonzero map $\underline{l}\colon N \to Y$ and by Proposition 1, s-rad$(N) = L_1 \oplus L_2$, $L_1, L_2 \neq 0$. Let Q be of the form

$$\leftarrow \cdots \xleftarrow{\underline{\alpha}_{N',L'_1}} \xleftarrow{\underline{\alpha}_{N_{2m+1},L_{2m+1,2m+1}}} \cdots \xleftarrow{\underline{\alpha}_{N_1,L_{1,1}}} \xleftarrow{\underline{\alpha}_{N,L_1}} \xrightarrow{\underline{\alpha}_{N,L_2}} \xrightarrow{\underline{\alpha}_{N_2,L_{2,2}}} \cdots \xrightarrow{\underline{\alpha}_{N_{2r},L_{2r,2r}}} \xrightarrow{\underline{\alpha}_{N'',L''_1}} \cdots \rightarrow.$$

Suppose that $\underline{\alpha}_{N',L'_1} \neq \underline{\alpha}_{N_{2m+3},L_{2m+3,2m+3}}$. But in this case s-top$(L_{2m+1,2m+1})$ is contained in s-soc(Y) since we can consider an A-module R that has the following property: for every

$0 \neq \underline{h}\colon N \to Y$ with $\underline{h}\big|_{\text{s-rad}(L_{2m+1,2m+1}} = 0$, $\underline{h}$ factors through R. It is easy to verify that such an R exists (by a dual version of Proposition 1) and $\text{s-top}(L_{2m+1,2m+1})$ is a direct summand in $\text{s-soc}(Y)$, so $\underline{\alpha}_{N',L'_1}$ does not exist in Q. In the same manner we prove that $\underline{\alpha}_{N'',L''_1}$ does not exist in Q and (d) is proved. Similarly we prove (e) and our lemma is proved. □

Corollary 2. *Let A be a selfinjective special biserial K–algebra which is not a local Nakayama algebra. Let Y be an indecomposable A–module of the first kind. Let X be a $\underline{p}$–maximal module for Y with $\underline{p}\colon Y \to M$. Then $Q_{\mathcal{M}_A}(X)$ is a subquiver of $Q_{\mathcal{M}_A}(Y)$ and $Q_{\mathcal{M}_A}(Y)\backslash Q_{\mathcal{M}_A}(X) = \{M\}$.*

Proof. The corollary is an obvious consequence of the constructions of $Q_{\mathcal{M}_A}(Y)$ and $\text{s-rad}(Y)$. □

10 τ–shifts of the s-projective modules. We starts this section with a lemma that will be of great importance in our further considerations.

Lemma 13. *Let A be a selfinjective special biserial K–algebra which is a local Nakayama algebra. Let N be an s-projective A-module whose s-top is M. Then $\text{s-soc}(\tau(N))$ is indecomposable and $\text{s-top}(\tau(N)) \cong \text{s-top}(\text{s-rad}(N))$.*

Moreover if

$$0 \to \tau(N) \xrightarrow{\binom{g_1}{g_2}} L_1 \oplus L_2 \xrightarrow{(f_1,f_2)} N \to 0$$

is an Auslander–Reiten sequence in mod–A then there is $\lambda \in K^*$ such that $\underline{f}_1\underline{g}_1 = \lambda \underline{f}_2\underline{g}_2$ with $\underline{f}_1\underline{g}_1 \neq 0$.

Proof. Under the notations of the lemma let $F_\lambda(\tilde{N}) = N$, $F_\lambda(\widetilde{M}) = M$. Suppose that $\text{supp}(\widetilde{M})$ is of the form

$$r_0 \leftarrow \cdots \leftarrow r_1 \to \cdots \to r_2 \leftarrow \cdots \leftarrow r_3 \to \cdots \leftarrow r_t \to \cdots \to r_{t+1}$$

$t \geq 1$. Then by [21, Lemma 12] we obtain that $\text{supp}(\tilde{N})$ is of the form

$$(r'_0)^- \leftarrow \cdots \leftarrow r_0 \to \cdots \to r'_1 \leftarrow \cdots \leftarrow r_2 \to \ldots$$
$$\to r'_t \leftarrow \cdots \leftarrow r_{t+1} \to \cdots \to (r'_{t+1})^+.$$

One can deduce from [27] that $\text{supp}(\tau(\tilde{N}))$ is of the form

$$(r_0)_+ \to \cdots \to r'_1 \leftarrow \cdots \leftarrow r_2 \to \cdots \to r'_3 \leftarrow \cdots \to r'_t \leftarrow \cdots \leftarrow (r_{t+1})_-.$$

By Proposition 2 and Lemma 9 we know that $\text{s-soc}(\tilde{N})$ is a direct sum of at most two indecomposable $\tilde{A}$–modules and $\text{s-soc}(\tilde{N}) = \text{s-soc}(\text{s-rad}(\tilde{N}))$. Moreover, if $M' \in \mathcal{M}_A$ with $F_\lambda(\widetilde{M'}) = M'$ and there is $0 \neq \underline{\tilde{q}}_1 : \widetilde{M'} \longrightarrow \tilde{L}'_1$ and there is $0 \neq \underline{\tilde{q}}_2 : \widetilde{M'} \to \tilde{L}_2$ with $\text{s-rad}(\tilde{N}) = \tilde{L}_1 \oplus \tilde{L}_2$, where $\underline{\tilde{q}}_1$, $\underline{\tilde{q}}_2$ factor through $\tau(\tilde{N})$, then by Lemmas 1, 2, 3 $\lambda \cdot \underline{\alpha}_{\tilde{N},\tilde{L}_1}\underline{\tilde{q}}_1 = \underline{\alpha}_{\tilde{N},\tilde{L}_2}\underline{\tilde{q}}_2$ for some $\lambda \in K^*$. Therefore one of $\tilde{L}_1$, $\tilde{L}_2$ has the property that its s-socle decomposes into two direct summands which contradicts to the fact that $\text{s-soc}(\tilde{L}_i)$ is indecomposable. Consequently if $M' \in \mathcal{M}_A$ and $0 \neq \underline{\tilde{q}} : \widetilde{M'} \to \tau(\tilde{N})$ then $\underline{\tilde{q}}$ cannot be prolongated to a nonzero morphism from $\widetilde{M'}$ to $\tilde{N}$. In fact there is only one $\tilde{A}$–module with this property and its support is of the form

$$r'_0 \leftarrow \cdots \leftarrow r'_1 \to \cdots \to r'_2 \leftarrow \cdots \leftarrow r'_3 \to \cdots \leftarrow r'_t \to \cdots \to r'_{t+1}.$$

This shows that $\text{s-soc}(\tau(N))$ is indecomposable in the considered case.

Suppose that $\mathrm{supp}(\widetilde{M})$ is of the form

$$r_0 \to \cdots \to r_1 \leftarrow \cdots \leftarrow r_2 \to \cdots \to r_3 \leftarrow \cdots \to r_t \leftarrow \cdots \leftarrow r_{t+1}$$

$t \geq 1$. Thus $\mathrm{supp}(\tilde{N})$ is of the form

$$(r'_0)^+ \leftarrow \cdots \leftarrow r_1 \to \cdots \to r'_2 \leftarrow \cdots \leftarrow r_3 \to \cdots \leftarrow r_t \to \cdots \to (r'_{t+1})^-$$

and $\mathrm{supp}(\tau(\tilde{N}))$ is of the following form

$$(r_0)_- \to \cdots \to r'_0 \leftarrow \cdots \leftarrow r_1 \to \cdots \to r'_2 \leftarrow \cdots \leftarrow r_t \to \cdots \to r'_{t+1} \leftarrow \cdots \leftarrow (r_{t+1})_+$$

by [27]. Similar arguments as above show that $\mathrm{supp}(\text{s-soc}(\tau(\tilde{N})))$ is of the form

$$((r'_0)_-)_+ \leftarrow \cdots \leftarrow r'_0 \to \cdots \to r'_1 \leftarrow \cdots \to r'_t \leftarrow \cdots \leftarrow r'_{t+1} \to \cdots \to ((r_{t+1})'_+)_-$$

and the required assertion holds in this case.

Suppose that $\mathrm{supp}(\widetilde{M})$ is of the form

$$r_0 \to \cdots \to r_1 \leftarrow \cdots \leftarrow r_2 \to \cdots \leftarrow r_t \to \cdots \to r_{t+1}.$$

Then $\mathrm{supp}(\tilde{N})$ is of the form

$$(r'_0)^+ \leftarrow \cdots \leftarrow r_1 \to \cdots \to r'_2 \leftarrow \cdots \to r'_t \leftarrow \cdots \leftarrow r_{t+1} \to \cdots \to (r'_{t+1})^+$$

and $\mathrm{supp}(\tau(\tilde{N}))$ is of the following form

$$(r_0)_- \to \cdots \to r'_0 \leftarrow \cdots \leftarrow r_1 \to \cdots \to r'_2 \leftarrow \cdots \to r'_t \leftarrow \cdots \leftarrow (r_{t+1})_-$$

by [27]. Similarly we obtain that $\mathrm{supp}(\text{s-soc}(\tau(\tilde{N})))$ is of the form

$$((r_0)'_-)_+ \leftarrow \cdots \leftarrow r'_0 \to \cdots \to r'_1 \leftarrow \cdots \leftarrow r'_2 \to \cdots \leftarrow r'_t \to \cdots \to r'_{t+1}.$$

Consequently, $\text{s-soc}(\tau(\tilde{N}))$ is also indecomposable in this case.

In order to finish the proof, it is enough to observe that every nonzero map starting at $\tau(\widetilde{N})$ must factor through a linear combination of the irreducible maps from $\tau(\widetilde{N})$ to $\tilde{L}_1$ and to $\tilde{L}_2$. Consequently $\text{s-top}(\tau(\tilde{N}))$ coincides with $\text{s-top}(\text{s-rad}(\tilde{N}))$. The last sentence in the lemma is obvious what finishes the proof. □

Corollary 3. *Let A be a selfinjective special biserial K–algebra which is not a local Nakayama algebra. Let N be an s-projective A–module whose s-support is the path category of the quiver*

$$\xleftarrow{\alpha_{N_{2s+1},L_{2s+1,2s+1}}} \cdots \xleftarrow{\alpha_{N_1,L_{1,1}}} \xleftarrow{\alpha_{N,L_1}} \xrightarrow{\alpha_{N,L_2}} \xrightarrow{\alpha_{N_2,L_{2,2}}} \cdots \xrightarrow{\alpha_{N_{2t},L_{2t,2t}}}$$

(respectively $\xrightarrow{\alpha_{N,L}} \xrightarrow{\alpha_{N_1,L_1}} \cdots \xrightarrow{\alpha_{N_r,L_r}}$) then $Q_{\mathcal{M}_A}(\tau(N))$ is of the form

$$\xrightarrow{\alpha_{N_1,L_{1,1}}} \cdots \xrightarrow{\alpha_{N_{2s+1},L_{2s+1,2s+1}}} \xrightarrow{\alpha_{N_{2s+3},L_{2s+3,2s+3}}} \xleftarrow{\alpha_{N_{2t+2},L_{2t+2,2t+2}}} \xleftarrow{\alpha_{N_{2t},L_{2t,2t}}} \cdots \xleftarrow{\alpha_{N_2,L_{2,2}}}$$

(resp. $\xrightarrow{\alpha_{N_1,L_1}} \cdots \xrightarrow{\alpha_{N_r,L_r}} \xrightarrow{\alpha_{N_{r+1},L_{r+1}}}$).

Proof. The corollary is an easy consequence of Lemmas 12, 13, of the construction of s-supports and of the construction of s-radicals. All details are left to the reader. □

11. s-supports of A-modules of the second kind. Throughout this section let Z be an indecomposable A–module of the second kind, where A is a selfinjective special biserial algebra. Now we are going to interpret Lemma 7 in terms of s-projective A–modules with respect to $\mathcal{M}_A$. In order to do it we need the following lemma.

Lemma 14. *If* $M \in \mathcal{M}_A$ *and* $0 \to Z \xrightarrow{w} Y \xrightarrow{r} Z \to 0$ *is an Auslander–Reiten sequence in mod–A then there is the following short exact sequence* $0 \to \underline{\mathrm{Hom}}_A(Z, M) \xrightarrow{\underline{r}^*} \underline{\mathrm{Hom}}_A(Y, M) \xrightarrow{\underline{w}^*} \underline{\mathrm{Hom}}_A(Z, M) \to 0$ *of K–spaces.*

Proof. Suppose That $M \in \mathcal{M}_A$ and $0 \to Z \xrightarrow{w} Y \xrightarrow{r} Z \to 0$ is an Auslander–Reiten sequence in mod–A. If $\underline{p} \in \underline{\mathrm{Hom}}_A(Z, M)$ is nonzero then $p \neq 0$ is not a splitable monomorphism. Hence there is a nonzero map $t : Y \to M$ such that $p = tw$ Moreover $\underline{t} \neq 0$, because $\underline{p} = \underline{tw}$. Consequently $\underline{w}^* : \underline{\mathrm{Hom}}_A(Y, M) \to \underline{\mathrm{Hom}}_A(Z, M)$ is an epimorphism of K–spaces. Suppose now that $\underline{pr} = 0$. Then there is a factorization of pr through the injective envelope $E(Y)$ of Y and we have the following commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{r} & Z \\ {\scriptstyle l}\downarrow & & \downarrow{\scriptstyle p} \\ E(Y) & \xrightarrow{s} & M. \end{array}$$

But it is easily seen that $E(Y) \cong E(Z) \oplus E(Z)$ and $l = (l_1, l_2)$. Furthermore there is $q : Z \hookrightarrow E(Z)$ with $l_2 = qr$. If $s = \binom{s_1}{s_2}$ then we have $sl = \binom{s_1}{s_2}(l_1, l_2) = s_1 l_1 + s_2 l_2 = pr$. Consequently $sl = s_1 l_1 + s_2 l_2 = pr$. But $l_1 w \neq 0$, hence $s_1 l_1 = 0$ and $s_2 qr = pr$. But r is an epimorphism, so $s_2 q = p$ which gives a contradiction to the assumption that $\underline{p} \neq 0$. Therefore $\underline{\mathrm{Hom}}_A(\underline{r}, M) = \underline{r}^*$ is a monomorphism. Of course $\underline{w}^* \underline{r}^* = 0$ what shows that we should check for $0 \neq \underline{l} \colon Y \to M$ whether $\underline{lw} = 0$ implies that there is $0 \neq \underline{p} : Z \to M$ such that $\underline{pr} = \underline{l}$. In order to check the last implication observe that $\underline{lw} = 0$ implies that lw factors through $E(Z)$ e.g. we have the following commutative diagram

$$\begin{array}{ccc} Z & \xrightarrow{w} & Y \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle l} \\ E(Z) & \xrightarrow{s} & M. \end{array}$$

Moreover there is $t \colon Y \to E(Z)$ such that $i = tw$, hence $lw = stw$. Now we are able to define a homomorphism $p \colon Z \to M$ by the formula $p(r(y)) = l(y) - st(y)$. It is easy to check that p does not depend on the choice of representatives of $r(y)$ and $\underline{l} = \underline{pr}$. Consequently our lemma is proved. □

Corollary 4. *Let Z be an indecomposable A–module of the second kind that is of the form* $F_w(M(Q_w, m, \lambda))$. *Then* $\text{s-top}(Z) \cong [\text{s-top}(F_w(M(Q_w, 1, \lambda)))]^m$.

Proof. The corollary is an easy consequence of Lemma 14. It can be proved inductively on m. We leave the detailes to the reader. □

An indecomposable A–module Y of the first kind is said to be *s-local* if its s-top is indecomposable. A family $\{V_i\}_{i=1,\dots,l}$ of s-local A-modules is said to be *primitive* if the following conditions are satisfied:

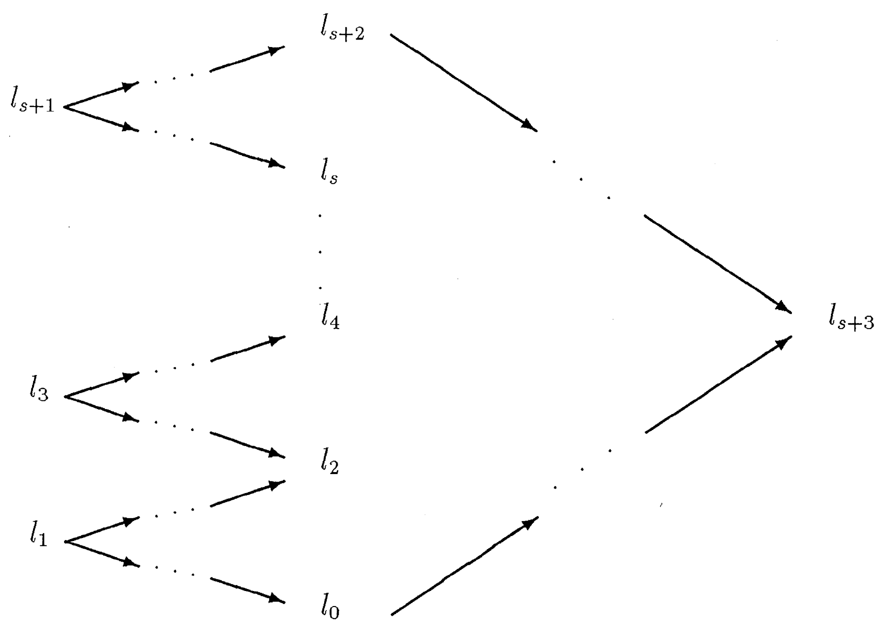
(i) if $l = 1$ then $\text{s-soc}(V_1) = M \oplus M$, $M \in \mathcal{M}_A$
(ii) if $l > 1$ then $\text{s-soc}(V_i) = M_{i_1} \oplus M_{i_2}$ with $M_{i_1}, M_{i_2} \in \mathcal{M}_A$ and $M_{i_2} \cong M_{(i+1)_1}$ for $i = 1, 2, \dots, l-1$ and $M_{l_2} \cong M_{1_1}$.

Proposition 4. *Let Z be an indecomposable A–module of the second kind. Then there exists a primitive family $\{V_i\}_{i=1,\dots,l}$ of s-local A–modules and there is a natural number r such that the following conditions are satisfied:*

(a) $[\text{s-top}(\bigoplus_{i=1}^{l} V_i)]^r \cong \text{s-top}(Z)$.

(b) *There exists a map $0 \neq \underline{q}\colon (\bigoplus_{i=1}^{l} M_{i_1})^r \to (\bigoplus_{i=1}^{l} V_i)^r$ such that for every A-module Y it holds $\underline{\text{Hom}}_A^{\pi_Z}(Z^\wedge, Y) \cong \underline{\text{Hom}}_A^{q}((\bigoplus_{i=1}^{l} V_i)^r, Y)$, where $\underline{\text{Hom}}_A^{q}((\bigoplus_{i=1}^{l} V_i)^r, Y)$ is a subspace in $\underline{\text{Hom}}_A((\bigoplus_{i=1}^{l} V_i)^r, Y)$ consisting of the morphisms $\underline{f}$ that satisfy $\underline{f}\underline{q} = 0$.*

Proof. Let Z be an indecomposable A–module of the second kind. We begin our proof with the case $Z \cong F_w(M(Q_w, 1, \lambda))$. Let $M_1 = F_\lambda(\widetilde{M_1}) \in \mathcal{M}_A$ be an indecomposable direct summand in s-top(Z). Let N_1 be s-projective with s-top$(N_1) \cong M_1$. Thus by definition there is a morphism $0 \neq \underline{f}\colon N_1 \to Z$ which satisfies $\underline{p}\underline{f} \neq 0$ for every $0 \neq \underline{p}\colon Z \to M_1$. Let V_1 be an s-local A–module such that $\underline{f}$ factors through V_1 and for $\underline{f} = \underline{f}_1\underline{f}_2$ with $\underline{f}_1\colon V_1 \to Z$ it holds $\underline{f}_1\underline{g}_1 \neq 0$, where $0 \neq \underline{g}_1\colon$ s-soc$(V_1) \to V_1$. First we should show that s-soc(V_1) decomposes into a direct sum of two indecomposable A-modules. Suppose that Q_w is of the form



Since $F_w(P_{l_i}) = L_{l_i}$ is a submodule in Z, hence there is some l_{i_0}, say l_1, such that for $0 \neq \underline{p}\colon Z \to M_1$ it holds $\underline{p}|_{L_{l_1}} \neq 0$ or $\underline{p}|_{L_{l_1}} = 0$ and $p|_{L_{l_1}} \neq 0$. Therefore in the first case there is $\underline{f}'_1\colon V_1 \to L_{l_1}$ with $\underline{p}|_{L_{l_1}}\underline{f}'_1 \neq 0$. It is easy to verify by construction that a $\underline{p}|_{L_{l_1}}$–maximal A–module for L_{l_1} is a direct sum of exactly two indecomposable A–modules, hence s-top(s-rad(V_1)) decomposes into two indecomposable direct summands by Lemma 11 and Lemma 12 implies that s-soc(V_1) decomposes into a direct sum of exactly two indecomposable A–modules. In the second case L_{l_1} is a submodule of s-rad(N) and s-soc(L_{l_1}) decomposes, hence s-soc(V_1) decomposes. Suppose that s-soc$(V_1) \cong M' \oplus M''$ with $M', M'' \in \mathcal{M}_A$. If M_2 is another indecomposable direct summand in s-top(Z) in the sense that there is $0 \neq \underline{p}_1\colon Z \to M_2$ with $\underline{p} \neq \lambda\underline{p}_1$ for every $\lambda \in K^*$ then we construct in the above way V_2. Continuing this procedure we obtain a family of s-local A–modules $\{V_i\}_{i=1,\dots,l}$. Applying a usual duality D we can show the same for $D(Z)$ what shows that $\{V_i\}_{i=1,\dots,l}$ is a primitive family of s-local A-modules.

In order to finish the proof of the considered case observe that $\text{s-top}(\bigoplus_{i=1}^{l} V_i) = \text{s-top}(Z)$ by the construction of the family $\{V_i\}_{i=1,\dots,l}$ what shows (a) in this case. Now we should indicate a morphism $0 \neq \underline{q}\colon \bigoplus_{i=1}^{l} M_{i_1} \to \bigoplus_{i=1}^{l} V_i$. But $\underline{q}$ acts in such a manner that for each $i = 1,\dots,l$ the following formula is true $\underline{f_{i-1}q_{i_2}}(m) \neq \underline{f_i q_{i_1}}(m)$ for every element m of M_{i_1}, where $0 \neq \underline{q_{i_1}}\colon M_{i_1} \to V_i$ and $0 \neq \underline{q_{i_2}}\colon M_{i_1} = M_{(i-1)_2} \to V_{i-1}$. Then (b) holds in this case for $\underline{\text{Hom}}_A^q((\bigoplus_{i=1}^{l} V_i), Y) \cong \underline{\text{Hom}}_A(Z,Y)$. Indeed, the morphism

$$\begin{pmatrix} \underline{f_1} \\ \vdots \\ \underline{f_l} \end{pmatrix} : \bigoplus_{i=1}^{l} V_i \to Z$$

yields a needed isomorphism. Consequently the case $Z \cong F_w(M(Q_w, 1, \lambda))$ is proved.

The general case $Z \cong F_w(M(Q_w, m, \lambda))$ is obtained by applying Lemma 14, Corollary 4 and the above analysis. All details in this case are left to the reader. □

Lemma 15. *If $M \in \mathcal{M}_A$ and $0 \to Z \xrightarrow{w} Y \xrightarrow{p} Z \to 0$ is an Auslander–Reiten sequence in mod–A then there is the following short exact sequence $0 \to \underline{\text{Hom}}_A(M,Z) \xrightarrow{w_*} \underline{\text{Hom}}_A(M,Y) \xrightarrow{p_*} \underline{\text{Hom}}_A(M,Z) \to 0$ of K–spaces.*

Proof. The proof is dual to the proof of Lemma 14. □

Corollary 5. *Let Z be an indecomposable A–module of the second kind which is of the form $F_w(M(Q_w, m, \lambda))$. Then $\text{s-soc}(Z) \cong [\text{s-soc}(F_w(M(Q_w, 1, \lambda)))]^m$.*

Proof. The corollary is easy proved inductively on m by using of Lemma 15. □

An indecomposable A-module Y of the first kind is said to be *s-colocal* if its s-socle is indecomposable. A family $\{U_i\}_{i=1,\dots,l}$ of s-colocal A-modules is said to be *primitive* if the following conditions are satisfied:

(i) if $l = 1$ then $\text{s-top}(U_1) = M \oplus M$, $M \in \mathcal{M}_A$

(ii) if $l > 1$ then $\text{s-top}(U_i) \cong M_{i_1} \oplus M_{i_2}$ with $M_{i_1}, M_{i_2} \in \mathcal{M}_A$ and $M_{i_2} \cong M_{(i+1)_1}$ for $i = 1, 2, \dots, l-1$ and $M_{l_2} \cong M_{1_1}$.

Proposition 5. *Let Z be an indecomposable A–module of the second kind. Then there exists a primitive family $\{U_i\}_{i=1,\dots,l}$ of s-colocal A–modules and there is a natural number r such that the following conditions are satisfied:*

(a) $[\text{s-soc}(\bigoplus_{i=1}^{l} U_i)]^r = \text{s-soc}(Z)$.

(b) *There exists a map $0 \neq \underline{p}\colon (\bigoplus_{i=1}^{l} U_i)^r \to (\bigoplus_{i=1}^{l} M_{i_1})^r$ such that for any A–module Y it holds $\underline{\text{Hom}}_A^{iz}(Y, Z^\vee) \cong \underline{\text{Hom}}_A^{p}(Y, (\bigoplus_{i=1}^{l} U_i)^r)$, where $\underline{\text{Hom}}_A^{p}(Y, (\bigoplus_{i=1}^{l} U_i)^r)$ is a subspace of $\underline{\text{Hom}}_A(Y, (\bigoplus_{i=1}^{l} U_i)^r)$ consisting of the morphisms $\underline{f}$ that staisfy $\underline{pf} = 0$.*

Proof. By applying the usual duality D to Proposition 4 one obtains the proposition at once. □

Now we are able to define s-supports for indecomposable A–modules of the second kind. Let Z be an indecomposable A-module of the second kind. Then *s-support* of Z, that will be denoted also by $\text{s-supp}_{\mathcal{M}_A}(Z)$, is a path category of the following relation-free quiver $Q_{\mathcal{M}_A}(Z)$. If $Z \cong F_w(M(Q_w, m, \lambda))$ then we put $Q_{\mathcal{M}_A}(Z) = Q_{\mathcal{M}_A}(F_w(M(Q_w, 1, \lambda)))$.

Moreover, $Q_{\mathcal{M}_A}(F_w(M(Q_w, 1, \lambda)))$ is defined as follows: if $\{V_i\}_{i=1,\dots,l}$ is a primitive family of s-local A-modules from Proposition 4 then $Q_{\mathcal{M}_A}(F_w(M(Q_w, 1, \lambda)))$ is obtained from $Q_{\mathcal{M}_A}(V_1) \cup \cdots \cup Q_{\mathcal{M}_A}(V_l)$ by the identifications of the following sinks M_{i_2} with $M_{(i+1)_1}$ for all $i = 1, \dots, l-1$ and M_{l_2} with M_{1_1}.

Lemma 16. *Let Z be an indecomposable A–module of the second kind. Then* s-supp$_{\mathcal{M}_A}(Z)$ *is a path category of a finite connected quiver $Q_{\mathcal{M}_A}(Z)$ of extended Dynkin type $\tilde{\mathbf{A}}_n$ and the following conditions hold:*

(a) *The sources in $Q_{\mathcal{M}_A}(Z)$ correspond to the indecomposable direct summands in* s-top(Z).
(b) *The sinks in $Q_{\mathcal{M}_A}(Z)$ correspond to the indecomposable direct summands in* s-soc(Z).

Proof. The lemma is an obvious consequence of Proposition 4, Lemma 12 and the definition of $Q_{\mathcal{M}_A}(Z)$ for indecomposable A-modules of the second kind. □

12. Algebras produced by maximal systems of orthogonal stable A–bricks. Let $\mathcal{M}_A$ be a fixed maximal system of orthogonal stable A-bricks, where A is a special biserial selfinjective algebra which is not a local Nakayama algebra. We start this section with defining a quiver $Q_{\mathcal{M}_A}$ produced by $\mathcal{M}_A$. The vertices of $Q_{\mathcal{M}_A}$ are the elements of $\mathcal{M}_A$. For any $M_1, M_2 \in \mathcal{M}_A$ there is an arrow $\underline{\alpha}_{N_1,L_1}$ from M_1 to M_2 iff there is a coset $\underline{\alpha}_{N_1,L_1}$ such that s-top$(N_1) \cong M_1$ and s-top$(L_1) \cong M_2$. Moreover, different cosets of the form $\underline{\alpha}_{N,L}$, $\underline{\alpha'}_{N,L}$ produce different arrows in $Q_{\mathcal{M}_A}$ iff $\lambda\underline{\alpha'}_{N,L} \neq \underline{\alpha}_{N,L}$ for all $\lambda \in K^*$.

Now we can define a two-sided ideal in the path category $KQ_{\mathcal{M}_A}$ of $\mathcal{M}_A$ to be an ideal $I_{\mathcal{M}_A}$ generated by the differences

$$\underline{\alpha}_{N,L_1}\underline{\alpha}_{N_1,L_{1,1}} \cdots \underline{\alpha}_{N_{2s+1},L_{2s+1,2s+1}}\underline{\alpha}_{N_{2s+3},L_{2s+3,2s+3}} - \underline{\alpha}_{N,L_2}\underline{\alpha}_{N_2,L_{2,2}} \cdots \underline{\alpha}_{N_{2t},L_{2t,2t}}\underline{\alpha}_{N_{2t+2},L_{2t+2,2t+2}}$$

for N with s-rad$(N) = L_1 \oplus L_2$, $L_1, L_2 \neq 0$, and by the paths that are not subpaths of the following paths $\underline{\alpha}_{N,L_1}\underline{\alpha}_{N_1,L_{1,1}} \cdots \underline{\alpha}_{N_{2s+1},L_{2s+1,2s+1}}\underline{\alpha}_{N_{2s+3},L_{2s+3,2s+3}}$, $\underline{\alpha}_{N,L_2}\underline{\alpha}_{N_2,L_{2,2}} \cdots \underline{\alpha}_{N_{2t},L_{2t,2t}}$ $\underline{\alpha}_{N_{2t+2},L_{2t+2,2t+2}}$ for N with s-rad$(N) = L_1 \oplus L_2$, $L_1, L_2 \neq 0$; $\underline{\alpha}_{N,L}\underline{\alpha}_{N_1,L_1} \cdots \underline{\alpha}_{N_r,L_r}\underline{\alpha}_{N_{r+1},L_{r+1}}$ for N with s-rad$(N) = L$ indecomposable. We shall denote the algebra $KQ_{\mathcal{M}_A}/I_{\mathcal{M}_A}$ by $\Lambda_{\mathcal{M}_A}$. The algebra $\Lambda_{\mathcal{M}_A}$ is called *$\mathcal{M}_A$-algebra.*

Lemma 17. *Let $\mathcal{M}_A$ be a maximal system of orthogonal stable A–bricks. Then an $\mathcal{M}_A$–algebra $\Lambda_{\mathcal{M}_A}$ is finite-dimensional selfinjective special biserial connected.*

Proof. Obvious by the construction of $\Lambda_{\mathcal{M}_A}$ and by Lemma 12, Corollary 3. □

Let Y be an indecomposable A–module. Consider the following morphism of quivers $l_Y\colon Q_{\mathcal{M}_A}(Y) \to Q_{\mathcal{M}_A}$ that acts as follows: for each $M \in \mathcal{M}_A$ we put $l_Y(M) = M$, and for each arrow $\underline{\alpha}_{N,L}$ in $Q_{\mathcal{M}_A}(Y)$ we put $l_Y(\underline{\alpha}_{N,L}) = \underline{\alpha}_{N,L}$. It is easy to observe that l_Y induces a K–linear functor of locally bounded K–categories $l_Y\colon$ s-supp$_{\mathcal{M}_A}(Y) \to \Lambda_{\mathcal{M}_A}$.

Lemma 18. *For every indecomposable A–module Y the functor $l_Y\colon$ s-supp$_{\mathcal{M}_A}(Y) \to \Lambda_{\mathcal{M}_A}$ is a covering functor.*

Proof. An easy verification shows the lemma. □

13. Specified quivers and stable morphisms. A quiver Q is said to be *specified* if the arrows in Q have their names. It may happen that different arrows in a specified quiver have the same names. A subquiver Q' in a specified quiver Q is said to be a *specified subquiver* if Q' is a specified quiver and the names of arrows in Q' coincide to their names in Q.

Let Y_1, Y_2 be two indecomposable A–modules of the first kind. A specified quiver Q (connected or not) is said to be an *essential subquiver* of $Q_{\mathcal{M}_A}(Y_1)$ with respect to $Q_{\mathcal{M}_A}(Y_2)$ if it is a specified subquiver of $Q_{\mathcal{M}_A}(Y_1)$ and it is a specified subquiver of $Q_{\mathcal{M}_A}(Y_2)$ such that if x is a source in Q then all paths in $Q_{\mathcal{M}_A}(Y_1)$ starting at x are contained in Q and if y is a sink in Q then all paths in $Q_{\mathcal{M}_A}(Y_2)$ ending at y are contained in Q.

Lemma 19. *Let A be a selfinjective special biserial K–algebra which is not a local Nakayama algebra. Let Y_1, Y_2 be two indecomposable A–modules of the first kind that are not projective. If $0 \neq \underline{f}\colon Y_1 \to Y_2$ then there exists a uniquely determined essential specified subquiver Q of $Q_{\mathcal{M}_A}(Y_2)$ with respect to $Q_{\mathcal{M}_A}(Y_1)$ and there exists a uniquely determined by f family $\{\underline{f}_M\}_{M \in Q_0}$ of morphisms $\underline{f}_M\colon M \to M$ such that the following conditions are satisfied:*

(a) *For each arrow $\underline{\alpha}_{N,L}\colon M_1 \to M_2$ in Q it holds $\lambda_{N,L}(Y_2) \cdot \underline{f}_{M_1} = \underline{f}_{M_2} \cdot \lambda_{N,L}(Y_1)$.*

(b) *If M' is a source in $Q_{\mathcal{M}_A}(Y_2)$ such that $\underline{\alpha}_{N,L} \in Q_1$ is contained in a path starting at M' with an arrow $\underline{\alpha}_{N',L'_1}$ then the following conditions are satisfied:*

(b1) If $\text{s-rad}(N')$ is indecomposable then there is not a path in $Q_{\mathcal{M}_A}(Y_1)$ that contains $\underline{\alpha}_{N,L}$ and passes through M'' with $M'' = \text{s-soc}(\tau(N'))$, where N' is s-projective with $\text{s-top}(N') = M'$.

(b2) If $\text{s-rad}(N')$ is decomposable and $\text{s-rad}(N') = L'_1 \oplus L'_2$ then in case that there is a path v in $Q_{\mathcal{M}_A}(Y_1)$ which contains $\underline{\alpha}_{N,L}$ and passes through M'' with $M'' = \text{s-soc}(\tau(N'))$ it holds M'' is a sink in $Q_{\mathcal{M}_A}(Y_1)$ and there is another path w in $Q_{\mathcal{M}_A}(Y_1)$ connecting Q with M'' for which there is a path in $Q_{\mathcal{M}_A}(Y_2)$ starting at M' with the arrow $\underline{\alpha}_{N',L'_2}$ and ending at a vertex that belongs to w. Moreover in this case if $\underline{f}$ is such that $\underline{f}_M \neq 0$ only for M lying on the intersection of Q with a path $\underline{\alpha}_{N',L'_1} z$ connecting M' with M'' and $\underline{h}\colon Y_1 \to Y_2$ is such that $\underline{h}_M \neq 0$ only for M lying on the intersection of Q with a path $\underline{\alpha}_{N',L'_2} z_1$ connecting M' with M'' then $\lambda \underline{f} = \underline{h}$ for some $\lambda \in K^*$.

Moreover every essential specified subquiver Q of $Q_{\mathcal{M}_A}(Y_2)$ with respect to $Q_{\mathcal{M}_A}(Y_1)$ and every family $\{\underline{f}_M\}_{M \in Q_0}$ of morphisms $\underline{f}_M\colon M \to M$ satisfying (a) *and* (b) *determines uniquely a nonzero morphism $\underline{f}\colon Y_1 \to Y_2$.*

Proof. We shall prove our lemma by induction on the number m of vertices in $Q_{\mathcal{M}_A}(Y_2)$. If $m = 1$ then $Y_2 \in \mathcal{M}_A$ and the required conditions hold obviously. Suppose now that the lemma holds for all $0 \neq \underline{f}\colon Y_1 \to Y_2$ with the property that $Q_{\mathcal{M}_A}(Y_2)$ has m_0 vertices or less than m_0 vertices. Consider $0 \neq \underline{f}\colon Y_1 \to Y_2$ such that $Q_{\mathcal{M}_A}(Y_2)$ has $m_0 + 1$ vertices. Suppose that there exists $M \in \mathcal{M}_A$ and $0 \neq \underline{p}: Y_2 \to M$ with $\underline{pf} = 0$. Thus $\underline{f}$ factors through a $\underline{p}$-maximal A–module X_2 for Y_2, hence $\underline{f} = \underline{f'f''}$ with $0 \neq \underline{f''}\colon Y_1 \to X_2$, $0 \neq \underline{f'}: X_2 \to Y_2$. By Corollary 2 we obtain that $Q_{\mathcal{M}_A}(X_2)$ has m_0 vertices. Therefore the lemma holds for $\underline{f''}$ by inductive assumption. Let Q be the uniquely determined essential specified subquiver of $Q_{\mathcal{M}_A}(X_2)$ with respect to $Q_{\mathcal{M}_A}(Y_1)$ for which there exists a uniquely determined by f'' family $\{\underline{f''}_M\}_{M \in Q}$ of morphisms satisfying (a) and (b). Since Q is also an essential specified subquiver of $Q_{\mathcal{M}_A}(Y_2)$ with respect to $Q_{\mathcal{M}_A}(Y_1)$ and $\underline{pf} = 0$, hence Q and $\{\underline{f''}_M\}$ are uniquely determined by f, and (a) holds obviously. In order to prove (b) in this case suppose that $\underline{\alpha}_{N',L'} \in Q_1$ is contained in a path starting at M with an arrow $\underline{\alpha}_{N,L_1}$ and $\text{s-rad}(N)$ is indecomposable, where N is s-projective with $\text{s-top}(N) \cong M$. Moreover suppose that there is a path v in $Q_{\mathcal{M}_A}(Y_1)$ that contains $\underline{\alpha}_{N',L'}$ and passes through M'' with $M'' \cong \text{s-soc}(\tau(N))$. Then it is easily seen that $\underline{f}$ factors through $\tau(N)$ and consequently $\underline{f} = 0$. Now suppose that $\underline{\alpha}_{N',L'} \in Q_1$ is contained in a path starting at M with an arrow $\underline{\alpha}_{N,L_1}$ and $\text{s-rad}(N) = L_1 \oplus L_2$, $L_1, L_2 \neq 0$, where N is s-projective with $\text{s-top}(N) \cong M$. Moreover, suppose that there is a path v in $Q_{\mathcal{M}_A}(Y_1)$ which contains $\underline{\alpha}_{N',L'}$ and passes through M'' with

$M'' \cong$ s-soc$(\tau(N))$. If M'' is not a sink in $Q_{\mathcal{M}_A}(Y_1)$ then we get a contradiction to Lemma 12 and Corollary 3. Consequently $\underline{f}$ factors through $\tau(N)$ and the required assertion is an easy consequence of Lemma 13. Therefore (b) holds in the considered case.

In order to finish the proof we should consider the case that for each $M \in \mathcal{M}_A$ with $0 \neq \underline{p}\colon Y_2 \to M$ it holds $\underline{pf} \neq 0$. But in this case it is easy to verify that $Q = Q_{\mathcal{M}_A}(Y_2)$ is an essential specified subquiver in $Q_{\mathcal{M}_A}(Y_2)$ with respect to $Q_{\mathcal{M}_A}(Y_1)$, moreover $\underline{f}$ induces a nonzero morphism $\underline{f}_1\colon X_1 \to X_2$, where X_2 is a $\underline{p}$–maximal A–module for Y_2 and X_1 is a $\underline{pf}$–maximal A-module for X_1 for some $0 \neq \underline{p}\colon Y_2 \to M \in \mathcal{M}_A$. Of course $Q_{\mathcal{M}_A}(X_2)$ has m_0 vertices and $Q_{\mathcal{M}_A}(X_2)$ is an essential specified subquiver in $Q_{\mathcal{M}_A}(X_2)$ with respect to $Q_{\mathcal{M}_A}(X_1)$. Consequently the lemma holds for $\underline{f}_1$ by inductive assumption. By Proposition 1(g) we obtain a uniquely determined by f family $\{\underline{f}_M\}_{M\in Q_{\mathcal{M}_A}(Y_2)}$ of morphisms satisfying (a) from a uniquely determined by f_1 family $\{\underline{f_1}_M\}_{M\in Q_{\mathcal{M}_A}(X_2)}$ of morphisms satisfying (a) and (b). Repeatting our arguments from the first part of the proof we obtain that the lemma holds also for $\underline{f}$. Therefore our lemma is proved. □

Remark 2. The above lemma shows that in terms of s-supports of A–modules of the first kind there are the same laws for morphisms as in Lemmas 1, 2 in terms of ordinary supports.

14. Supports of indecomposable $\Lambda_{\mathcal{M}_A}$-modules. Throughout we can fix a Galois covering $F : \tilde{\Lambda}_{\mathcal{M}_A} \to \Lambda_{\mathcal{M}_A}$ with $\tilde{\Lambda}_{\mathcal{M}_A}$ simply connected. Then $\tilde{\Lambda}_{\mathcal{M}_A} = K\tilde{Q}_{\mathcal{M}_A}/\tilde{I}_{\mathcal{M}_A}$ and every arrow β in $\tilde{Q}_{\mathcal{M}_A}$ with $F(\beta) = \underline{\alpha}_{N,L}$ will be named also by $\underline{\alpha}_{N,L}$. Thus for every indecomposable A–module Y of the first kind its specified quiver $Q_{\mathcal{M}_A}(Y)$ can be considered as a specified subquiver of $(\tilde{Q}_{\mathcal{M}_A}, \tilde{I}_{\mathcal{M}_A})$. Furthermore every covering functor $l_Y\colon$ s-supp$_{\mathcal{M}_A}(Y) \to \Lambda_{\mathcal{M}_A}$ can be considered as $F|_{\text{s-supp}_{\mathcal{M}_A}(Y)}$. The first question we should answer is whether there is an indecomposable A–module Y of the first kind whose s-support s-supp$_{\mathcal{M}_A}(Y)$ coincides with supp(T) for any indecomposable $\tilde{\Lambda}_{\mathcal{M}_A}$–module T. The following proposition answers this question in affirmative.

Proposition 6. *For a special biserial selfinjective K–algebra A which is not a local Nakayama algebra let T be an indecomposable $\tilde{\Lambda}_{\mathcal{M}_A}$–module. Then there exists an indecomposable A–module Y of the first kind such that* s-supp$_{\mathcal{M}_A}(Y) =$ supp(T).

Proof. Let T be an indecomposable $\tilde{\Lambda}_{\mathcal{M}_A}$–module whose support is a path category of a quiver Q of Dynkin type $\mathbf{A}_n$. We shall prove by induction on the number m of vertices in Q that there is an indecomposable A–module Y of the first kind such that $Q_{\mathcal{M}_A}(Y) = Q$. If $m = 1$ then the required assertion is obvious. Assume that if Q has m_0 vertices then there is an indecomposable A–module Y_0 of the first kind with $Q_{\mathcal{M}_A}(Y_0) = Q$. Suppose now that Q has $m_0 + 1$ vertices. Let Q be of the form $M_x \to M_y - \cdots -$. Thus by the inductive assumption there is an indecomposable A–module Y' of the first kind such that $Q_{\mathcal{M}_A}(Y') = Q'$, where Q' is of the form $M_y - \cdots -$. Consider the case y is a source in Q'. In this case $Q_{\mathcal{M}_A}(Y')$ is of the form $M_y \to \cdots -$ and consequently Y' is an indecomposable A–module of the first kind such that for $0 \neq \underline{p}\colon Y' \to M_y$ there is an indecomposable $\underline{p}$–maximal A–module X' for Y'. If $F_\lambda(\widetilde{Y'}) = Y'$, $F_\lambda(\widetilde{M_y}) = M_y$ and $\underline{p} = F_\lambda(\underline{\tilde{p}})$ then by Proposition 1(a) a simple analisys shows that we have one of the following possibilities: supp$(\widetilde{M_y})$ is of the form

$$r_0 \to \cdots \to r_1 \leftarrow \cdots \leftarrow r_2 \to \cdots \to r_j \leftarrow \cdots -,$$

supp$(\tilde{Y'})$ is of the form

$$r_0 \to \cdots \to r_1 \leftarrow \cdots \leftarrow r_2 \to \cdots \to r_j - \cdots - \mathrm{rf}(\underline{\tilde{p}}) - \cdots -,$$

or $\mathrm{supp}(\widetilde{M_y})$ is of the form

$$r_0 \leftarrow \cdots \leftarrow r_1 \rightarrow \cdots \rightarrow r_2 \leftarrow \cdots \rightarrow r_j \leftarrow \cdots -$$

and $\mathrm{supp}(\tilde{Y'})$ is of the form

$$r_0 \leftarrow \cdots \leftarrow r_1 \rightarrow \cdots \rightarrow r_2 \leftarrow \cdots \rightarrow r_j - \cdots - \mathrm{rf}(\underline{\tilde{p}}) - \cdots - .$$

By the construction of $\Lambda_{\mathcal{M}_A}$ we infer that if N_x is the s-projective A–module with $\text{s-top}(N_x) \cong M_x$ then there exists an indecomposable direct summand L_x in $\text{s-rad}(N_x)$ such that $\text{s-top}(L_x) \cong M_y$. If $F_\lambda(\tilde{M_x}) = M_x$ then we have one of the following possibilities: $\mathrm{supp}(\tilde{M_x})$ is of the form

(i) $r_1^* \rightarrow \cdots \rightarrow r_2 \leftarrow \cdots \leftarrow r_3^* \rightarrow \cdots \rightarrow r_s \leftarrow \cdots -$
(ii) $r_0^- \leftarrow \cdots \leftarrow r_1^* \rightarrow \cdots \rightarrow r_2 \leftarrow \cdots \rightarrow r_s \leftarrow \cdots -$
(iii) $r_{-1}^+ \rightarrow \cdots \rightarrow r_0^- \leftarrow \cdots \leftarrow r_0^* \rightarrow \cdots \rightarrow r_1 \leftarrow \cdots \rightarrow r_s \leftarrow \cdots -$
(iv) $r_0^- \leftarrow \cdots \leftarrow r_0^* \rightarrow \cdots \rightarrow r_1 \leftarrow \cdots \leftarrow r_2^* \rightarrow \cdots \rightarrow r_s \leftarrow \cdots -$,

where r_s is a vertex in $\mathrm{supp}(\widetilde{M_x}) \cap \mathrm{supp}(\widetilde{M_y})$ with maximal s. In each case if $\mathrm{rf}(\underline{\tilde{p}})$ is a source in $\mathrm{supp}(\tilde{Y'})$ and $j < s$ then the composition $M_x \rightarrow M_y \rightarrow$ lies in $\tilde{I}_{\mathcal{M}_A}$ which contradicts to our assumptions. Consequently $s < j$ and $\widetilde{Y}$ with $\mathrm{supp}(\widetilde{Y})$ of the form

$$- \cdots \leftarrow \mathrm{rf}(\underline{\tilde{p}}) \rightarrow \cdots \rightarrow r_j \leftarrow \cdots \leftarrow r_{j-1} \rightarrow \cdots \leftarrow r_s \leftarrow \cdots -$$

or

$$- \cdots \leftarrow \mathrm{rf}(\underline{\tilde{p}}) \rightarrow \cdots \rightarrow r_j \leftarrow \cdots \leftarrow r_{j-1} \rightarrow \cdots \leftarrow (r_{s+1}^*)^- \leftarrow r_{s+1}^* \rightarrow \cdots \rightarrow r_s \leftarrow \cdots - l$$

satisfies the required condition, where l is equal to either r_1^*, or to r_0^-, or to r_{-1}^* or else to r_0^- in case (i), (ii), (iii), (iv) respectively. If $\mathrm{rf}(\underline{\tilde{p}})$ is a sink in $\mathrm{supp}(\widetilde{Y'})$ then always $M_x \rightarrow M_y \rightarrow$ lies in $\tilde{I}_{\mathcal{M}_A}$ which contradicts to our assumptions.

Now consider the case y is a sink in Q'. In this case $Q_{\mathcal{M}_A}(Y')$ is of the form $M_y \leftarrow \cdots -$, and consequently Y' is an indecomposable A–module of the first kind such that for $0 \neq D(\underline{p})\colon D(Y') \rightarrow D(M_y)$ there is an indecomposable $\underline{p}$–maximal A–module X' for $D(Y')$, where D is the usual duality. Then $\mathrm{supp}(\widetilde{M_y})$ is as above and $\mathrm{supp}(\widetilde{Y'})$ is of the form

$$r_0 \rightarrow \cdots \rightarrow r_1 \leftarrow \cdots \leftarrow r_2 \rightarrow \cdots \leftarrow r_j - \cdots - \mathrm{rcf}(\underline{\tilde{p}}) - \cdots -$$

or

$$r_0 \leftarrow \cdots \leftarrow r_1 \rightarrow \cdots \rightarrow r_2 \leftarrow \cdots \leftarrow r_j - \cdots - \mathrm{rcf}(\underline{\tilde{p}}) - \cdots -$$

respectively. Moreover $\mathrm{supp}(\widetilde{M_x})$ is one of the above forms (i)–(iv). Furthermore if $\mathrm{rcf}(\underline{\tilde{p}})$ is a source in $\mathrm{supp}(\widetilde{Y'})$ and $s < j$, then $\widetilde{Y}$ with $\mathrm{supp}(\widetilde{Y})$ of the form

$$- \cdots \leftarrow \mathrm{rcf}(\underline{\tilde{p}}) \rightarrow \cdots \rightarrow r_j \rightarrow \cdots \rightarrow r_{j-1} \leftarrow \cdots \leftarrow r_s \leftarrow \cdots -$$

or

$$- \cdots \leftarrow \mathrm{rcf}(\underline{\tilde{p}}) \rightarrow \cdots \rightarrow r_j \rightarrow \cdots \rightarrow r_{j-1} \leftarrow \cdots \leftarrow (r_{s+1}^*)^- \leftarrow r_{s+1}^* \rightarrow \cdots \leftarrow r_s \rightarrow \cdots - l$$

satisfies the required condition, where l is as above. If $\mathrm{rcf}(\underline{\tilde{p}})$ is a source and $j < s$ then $\widetilde{Y}$ with $\mathrm{supp}(\widetilde{Y})$ of the form

$$-\cdots \leftarrow \mathrm{rcf}(\underline{\tilde{p}}) \rightarrow \cdots \rightarrow r_j \leftarrow \cdots \leftarrow r^*_{j+1} \rightarrow \cdots \rightarrow r_s \leftarrow \cdots -$$

satisfies the required condition. If $\mathrm{rcf}(\underline{\tilde{p}})$ is a sink then $\widetilde{Y}$ with $\mathrm{supp}(\widetilde{Y})$ of the form

$$-\cdots \rightarrow \mathrm{rcf}(\underline{\tilde{p}}) \leftarrow \cdots \leftarrow r_j \leftarrow \cdots \leftarrow r^*_{j+1} \rightarrow \cdots \rightarrow r_s \leftarrow \cdots - \quad \text{for } j < s$$
$$-\cdots \rightarrow \mathrm{rcf}(\underline{\tilde{p}}) \leftarrow \cdots \leftarrow r_j \rightarrow \cdots \rightarrow r_{j-1} \leftarrow \cdots \leftarrow r_s \leftarrow \cdots - \quad \text{for } j > s$$

if such a module exists or

$$-\cdots \rightarrow \mathrm{rcf}(\underline{\tilde{p}}) \leftarrow \cdots \leftarrow r_j \rightarrow \cdots \rightarrow r_{j-1} \leftarrow \cdots \leftarrow (r^*_{s+1})^- \leftarrow r^*_{s+1} \rightarrow \cdots \rightarrow r_s \leftarrow \cdots -,$$

where l is as above.

If $Q_{\mathcal{M}_A}(Y') = Q'$ and Q' has no sources of exactly one arrow then we use duality D and apply the above arguments, what finishes the proof. □

Keeping the notations of Section 10 we have the following proposition.

Proposition 7.

(1) *For every primitive family* $\{V_i\}_{i=1,\dots,l}$ *of s-local A–modules there exists an indecomposable A–module Z of the second kind such that* $\text{s-top}(\bigoplus_{i=1}^{l} V_i) = \text{s-top}(Z)$ *and there exists a map* $0 \neq \underline{q}\colon \bigoplus_{i=1}^{l} M_{i_1} \to \bigoplus_{i=1}^{l} V_i$ *such that for every A–module Y it holds* $\underline{\mathrm{Hom}}_A^{\pi z}(Z^\wedge, Y) \cong \underline{\mathrm{Hom}}_A^{q}(\bigoplus_{i=1}^{l} V_i, Y)$.

(2) *For every primitive family* $\{U_i\}_{i=1,\dots,l}$ *of s-colocal A–modules there exists an indecomposable A–module Z of the second kind such that* $\text{s-soc}(\bigoplus_{i=1}^{l} U_i) = \text{s-soc}(Z)$ *and there exists a map* $0 \neq \underline{p}\colon \bigoplus_{i=1}^{l} U_i \to \bigoplus_{i=1}^{l} M_{i_1}$ *such that for every A–module Y it holds* $\underline{\mathrm{Hom}}_A^{iz}(Y, Z^\vee) \cong \underline{\mathrm{Hom}}_A^{p}(Y, \bigoplus_{i=1}^{l} U_i)$.

Proof. Simple analysis as in the proof of Proposition 6 shows that there exists a quiver Q_w of type $\tilde{\mathbf{A}}_n$ with a covering functor $F_w\colon KQ_w \to A$ such that $F_w(M(Q_w, 1, \lambda))$ satisfies the required conditions for some $\lambda \in K^*$. □

15. Main results.

The main aim of this section is a proof of the main results. Before we shall start the proofs we study sincere representations of s-supports of indecomposable A–modules. Let Y be an indecomposable A–module of the first kind. A *sincere representation* of $\text{s-supp}_{\mathcal{M}_A}(Y)$ corresponding to Y is the indecomposable representation $V(Y)$ of $Q_{\mathcal{M}_A}(Y)$ in which K stands at each vertex and there is given a multiplication by $\lambda_{N,L}(Y) \in K^*$ on the arrow $\underline{\alpha}_{N,L}$. Let Y be an indecomposable A-module of the second kind that is of the form $Y \cong F_w(M(Q_w, m, \lambda))$. A *sincere representation* of $\text{s-supp}_{\mathcal{M}_A}(Y)$ corresponding to Y is the representation $V(Y)$ of $Q_{\mathcal{M}_A}(Y)$ obtained in the following way: if $\{V_i\}_{i=1,\dots,l}$ is a family of s-local A–modules produced by Y as in Proposition 4 then we consider a family of local $\text{s-supp}_{\mathcal{M}_A}(Y)$–modules $\{L_i\}_{i=1,\dots,l}$ corresponding to V_i, $i = 1, \dots, l$, as sincere representations of subcategories $\text{s-supp}_{\mathcal{M}_A}(V_i)$ of the category $\text{s-supp}_{\mathcal{M}_A}(Y)$. Moreover let S_i simple $\text{s-supp}_{\mathcal{M}_A}(Y)$–representations corresponding to the sinks in $Q_{\mathcal{M}_A}(Y)$. Let $i\colon (\bigoplus_{i=1}^{l} S_{i_1})^r \to (\bigoplus_{i=1}^{l} L_i)^r$ be an injection induced by $0 \neq \underline{q}\colon (\bigoplus_{i=1}^{l} M_{i_1})^r \to (\bigoplus_{i=1}^{l} V_i)^r$ as in Proposition 4. In view of Lemma 19 i is really an injection, and we define $V(Y)$ to be a $\mathrm{coker}(i)$. It is easy to verify that in the case considered case $V(Y) \cong M(Q_{\mathcal{M}_A}(Y), m, \lambda)$.

Theorem 1. *Let A be a special biserial selfinjective K–algebra which is not a local Nakayama algebra. Then there is a stable equivalence $\Phi\colon \underline{mod}\text{–}A \longrightarrow \underline{mod}\text{–}\Lambda_{\mathcal{M}_A}$ for every maximal system of orthogonal stable A–bricks $\mathcal{M}_A$.*

Proof. In order to prove the theorem we should construct a functor $\Phi\colon \underline{\mathrm{mod}}\text{–}A \longrightarrow \underline{\mathrm{mod}}\text{–}\Lambda_{\mathcal{M}_A}$ that is dense full and faithful. For every indecomposable A–module Y we put $\Phi(Y) = G_\lambda(V(Y))$ in case Y is of the first kind. If Y is of the second kind then we have a covering functor $l_Y\colon \text{s-supp}_{\mathcal{M}_A}(Y) \longrightarrow \Lambda_{\mathcal{M}_A}$ by Lemma 18. Thus we define $\Phi(Y) = l_Y(V(Y))$. If $0 \neq \underline{f}\colon Y_1 \longrightarrow Y_2$ is a nonzero morphism between two indecomposable A–modules of the first kind then there exists a uniquely determined essential specified subquiver of $Q_{\mathcal{M}_A}(Y_2)$ with respect to $Q_{\mathcal{M}_A}(Y_1)$ and there exists a uniquely determined by f family $\{\underline{f}_M\}_{M\in Q_0}$ of morphisms $\underline{f}_M\colon M \longrightarrow M$ such that the conditions of Lemma 19 are satisfied. Consequently we obtain a morphism $0 \neq V(\underline{f})\colon V(Y_1) \longrightarrow V(Y_2)$ and we put $\Phi(\underline{f}) = G_\lambda(V(\underline{f}))$. By Lemma 9 and Propositions 4, 5 we can define Φ for morphisms between arbitrary indecomposable A–modules in an obvious way. Furthermore we enlarge Φ additively to the whole category $\underline{\mathrm{mod}}\text{–}A$. An easy verification shows that Φ is dense by Propositions 6, 7, Φ is full and faithful by Lemma 19 and by Propositions 4, 5. This finishes the proof of our theorem. $\square$

Theorem 2. *Let $\Phi\colon \underline{mod}\text{–}B \longrightarrow \underline{mod}\text{–}C$ be a stable equivalence for a selfinjective special biserial algebra B whose bound quiver (Q_B, I_B) does not contain double arrows and double loops and that is not a local Nakayama algebra. If $\mathcal{M}_C = \{\Phi(S_i)\}_{i=1,\dots,n}$, where $\{S_i\}_{i=1,\dots,n}$ is a set of representatives of the isoclasses of the simple B–modules, then the following conditions are satisfied:*

(1) $B \cong \Lambda_{\mathcal{M}_C}$.
(2) *Φ is induced by a stable equivalence $\Phi_1 : \underline{mod}\text{–}_1B \longrightarrow \underline{mod}\text{–}_1C$.*

Proof. Let $\Phi : \underline{\mathrm{mod}}\text{–}B \longrightarrow \underline{\mathrm{mod}}\text{–}C$ be a stable equivalence and let B be a selfinjective special biserial algebra that is not a local Nakayama algebra. Let $\mathcal{M}_C = \{\Phi(S_i)\}_{i=1,\dots,n}$, where $\{S_i\}_{i=1,\dots,n}$ is a set of representatives of the isoclasses of the simple B–modules. Then C is a selfinjective special biserial algebra that is not a local Nakayama algebra and C, B have the same number of isoclasses of the simple modules (see [21]). Thus $\mathcal{M}_C$ is a maximal system of orthogonal stable C–bricks. It is obvious that for each s-projective C–module N with respect to $\mathcal{M}_C$ its s-support coincides to an ordinary support of some $P/\,\text{s-soc}(P)$ with P indecomposable projective B-module. Moreover, $\text{s-supp}_{\mathcal{M}_C}(\tau(N))$ coincides to $\text{supp}(\text{s-rad}(P))$. Therefore by Theorem 1 we have that there is a stable equivalence $\Psi\colon \underline{\mathrm{mod}}\text{–}B \longrightarrow \underline{\mathrm{mod}}\text{–}\Lambda_{\mathcal{M}_C}$ such that $\Psi(P/\,\text{s-soc}(P)) \cong Q/\,\text{s-soc}(Q)$, $\Psi(\text{s-rad}(P)) \cong \text{s-rad}(Q)$ for each indecomposable projective B–module, where Q is an indecomposable projective $\Lambda_{\mathcal{M}_C}$–module. Moreover Ψ preserves simples. If $B \cong KQ_B/I_B$ is a special presentation then $Q_B = Q_{\mathcal{M}_C}$. If α is the only arrow between x and y then the indecomposable C–module whose support is this arrow is preserved obviously by Ψ. The only confusions are connected with double arrows, but this case is excluded by the assumption. Consequently (1) is proved. Hence (2) is obvious by Lemma 12. $\square$

Résumé substantiel en français. On note K un corps algbriquement clos; toutes les algèbres considérées sont des K-algèbres de dimension finie, basiques et connexes. Une algèbre A est dite *spéciale bissérielle* si elle est isomorphe à KQ_a/I_a, le carquois avec relations (Q_A, I_A) satisfaisant aux conditions suivantes :

(i) Tout sommet de Q_A est la source d'au plus deux flèches, et le but d'au plus deux flèches.
(ii) Pour toute flèche α de Q_A, il existe au plus une flèche β et au plus une flèche γ telles que $\alpha\beta \notin I_A$, $\gamma\alpha \notin I_A$.

Un objet indécomposable M de la catégorie stable $\underline{\text{mod}}$–A est appelé un *A–bloc stable* si l'anneau $\text{End}_A(M)$ de ses endomorphismes est isomorphe à K. On dit qu'une famille $\{M_j\}_{j\in J}$ de A–blocs stables est un *système maximal de A–blocs stables orthogonaux* si les conditions suivantes sont satisfaites:

(1) Pour tout $j \in J$, le module M_j n'est pas isomorphe à son translaté d'Auslander–Reiten τM_j.
(2) Pour i, j distincts dans J, on a $\text{Hom}(M_i, M_j)$
(3) Quel que soit le A–module indécomposable N, qui n'est ni projectif, ni isomorphe à τN, il existe j_0 et j_1 dans J avec $\text{Hom}(M_{j_0}, N) \neq 0$ et $\text{Hom}(N, M_{j_1}) \neq 0$.

Soit M_A un système maximal de A–blocs stables orthogonaux; on suppose que l'algèbre est auto-injectives, spéciale et bissérielle, mais que ce n'est pas une algèbre locale de Nakayama. Ces données permettent de construire une K–algèbre Λ_{M_A} qui estauto-injective, spéciale et bissérielle. Voici les résultats principaux de ce travail.

Théorème 1. *Les catégories $\underline{\text{mod}}$–A et $\underline{\text{mod}}$–Λ_{M_A} sont stablement équivalentes.*

Théorème 2. *Soit B une algèbre auto-injective, spéciale et bissérielle. On suppose que le carquois avec relations (Q_A, I_A) qui lui est associé ne possède pas d'arêtes doubles et de boucles doubles; on suppose aussi que l'algèbre B n'est pas une algègre locale de Nakayama. Soit Φ: $\underline{\text{mod}}$–$B \to \underline{\text{mod}}$–$C$ une équivalence stable; on note $\{S_i\}_{i=1,\ldots,n}$ un système de représentants des classes d'isomorphisme de B–modules simples et l'on pose $M_C = \{\Phi(S_i)\}_{i=1,\ldots,n}$. On a les propriétés suivantes :*

(1) *B est isomorphe à Λ_{M_C}*
(2) *Φ est induit par une équivalence stable Φ_1 de $\underline{\text{mod}}$–$_1B$ avec $\underline{\text{mod}}$–$_1C$.*

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