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# ON THE NAIMARK EXTENSION OF A COMMUTATIVE NORMALIZED POV–MEASURE ACTING ON A HILBERT SPACE OF FUNCTIONS OVER A DIFFERENTIAL MANIFOLD

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RÉSUMÉ. Nous construisons l'extension de Naimark d'une mesure à valeurs dans les opérateurs positifs, commutative et normalisée, définie dans la tribu des parties d'une variété différentielle. Voir le résumé substantiel en français à la fin de l'article.

ABSTRACT. We construct the Naimark extension of a commutative normalized positive-operator-valued measure, defined on the Borel sets of a differential manifold.

**1. Introduction.** The Naimark extension of a positive-operator-valued (POV) measure (on a Hilbert space) to a projection-valued (PV) measure (on an extended space) is a powerful mathematical tool for the quantization of a physical system living on a differential $n$–manifold $\mathcal{M}$ (see e.g. [2, 3]). The quantization is usually performed by extending an associated system of covariance to a system of imprimitivity [1, 4]. In this paper we consider a commutative POV–measure and its Naimark extension to a PV–measure, under the assumption of informational equivalence of the POV–measure and the canonical PV–measure for the manifold $\mathcal{M}$.

In Section 2 we discuss informational equivalence of POV– and PV–measures, and show that this equivalence implies that the von Neumann algebra generated by the given POV–measure is maximal Abelian. Consequently, the measure has a representation by a family of point-dependent multiplication operators. In Section 3 the Naimark extension of the POV–measure is constructed.

**2. Notation and Preliminary Results.** Let $\mathcal{X}$ be a locally compact topological space, $\mathcal{B}(\mathcal{X})$ the $\sigma$–algebra of Borel–subsets of $\mathcal{X}$; let $\mathcal{H}$ be a complex separable Hilbert space, $\mathcal{L}(\mathcal{H})$ the set of all bounded linear operators on $\mathcal{H}$ and $\mathcal{L}_+(\mathcal{H})$ its positive cone.

**Definition 2.1.** A *commutative, normalized positive-operator-valued* (POV)*–measure* for $\mathcal{X}$ on $\mathcal{H}$ is a map

$$a\colon \mathcal{B}(\mathcal{X}) \to \mathcal{L}_+(\mathcal{H}),$$

which satisfies

(1) $a(\mathcal{X}) = I$, the identity operator on $\mathcal{H}$ and $a(\varnothing) = 0$.
(2) $a(\bigcup_{i\in J} E_i) = \sum_{i\in J} a(E_i)$, for any discrete index set $J$, $E_i \cap E_j = \varnothing$ whenever $i \neq j$, and the sum being assumed to converge weakly.
(3) $a(E)a(F) = a(F)a(E)\ \forall E, F \in \mathcal{B}(\mathcal{X})$.

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If, furthermore, the POV–measure satisfies

$$a(E)a(F) = a(E \cap F) \qquad \forall E, F \in \mathcal{B}(\mathcal{X}).$$

it will be called a *normalized projection-valued* (PV)–*measure* and will generally be denoted by $P$.

**Definition 2.2.** Two POV–measures $a$ and $\hat{a}$ for $\mathcal{X}$ on $\mathcal{H}$ are said to be *informationally equivalent* if and only if for any $\rho \in \mathcal{T}(\mathcal{H})$ (the set of all trace-class operators on $\mathcal{H}$),

$$\mathrm{tr}[\rho a(E)] = 0 \qquad \forall E \in \mathcal{B}(\mathcal{X})$$

implies

$$\mathrm{tr}[\rho \hat{a}(E)] = 0 \qquad \forall E \in \mathcal{B}(\mathcal{X})$$

and vice versa.

For a physical justification of this notion see [5].

Given a POV–measure $a$ let $A(a)$ be the von Neumann algebra generated by the set $\{a(E) \mid E \in \mathcal{B}(\mathcal{X})\}$. Fix a normalized PV-measure $P$ for $\mathcal{X}$ on $\mathcal{H}$ such that $A(P)$ is maximal Abelian.

**Lemma 2.3.** *Let $a$ be a normalized commutative POV–measure for $\mathcal{X}$ on $\mathcal{H}$, which is informationally equivalent to $P$. Then $A(a)$ is maximal Abelian.*

*Proof.* Let $\mathcal{T}^*(a)$ (respectively $\mathcal{T}^*(P)$) be the vector space generated by the set of all linear functionals on $\mathcal{T}(\mathcal{H})$ defined by the set of operators $\{a(E) \mid E \in \mathcal{B}(\mathcal{X})\}$ (respectively by the set of operators $\{P(E) \mid E \in \mathcal{B}(\mathcal{X})\}$). Then clearly $\mathcal{T}^*(a) \subset A(a)$, while $\mathcal{T}^*(P) = A(P) = A(P)' = A(P)''$. Let $\rho \in \mathcal{T}(\mathcal{H})$ be such that

$$\langle \rho, f \rangle = 0 \qquad \forall f \in \mathcal{T}^*(a)$$

where $\langle\ \rangle$ denotes the natural pairing between $\mathcal{T}(\mathcal{H})$ and its dual. Then we have in particular that

$$\langle \rho, a(E) \rangle = 0 \qquad \forall E \in \mathcal{B}(\mathcal{X}),$$

and hence by informational equivalence of $a$ and $P$,

$$\langle \rho, P(E) \rangle = 0 \qquad \forall E \in \mathcal{B}(\mathcal{X}).$$

Thus,

$$\langle \rho, g \rangle = 0 \qquad \forall g \in \mathcal{T}^*(P),$$

so that

$$\mathcal{T}^*(P) \subset \overline{\mathcal{T}^*(a)}$$

(in the weak *–topology) and therefore

$$A(P) \subset A(a).$$

Finally, since $A(a)$ is Abelian and $A(P)$ is maximal Abelian, the same is true of $A(a)$. □

An immediate consequence of this lemma is the existence of a $\sigma$–finite measure $\mu$ on $\mathcal{X}$ such that $\mathcal{H}$ may be realized as $L^2(\mathcal{X}, \mu)$ (see [8]) and the commutative POV–measure $a$ is given by

$$(a(E)\psi)(x) = \nu_x(E)\psi(x) \qquad \forall E \in \mathcal{B}(\mathcal{X})\ \forall \psi \in L^2(\mathcal{X}, \mu)\ \forall x \in \mathcal{X},$$

where $\nu_x \colon \mathcal{B}(\mathcal{X}) \to [0,1]$ is a probability measure on $\mathcal{X}$ for any $x \in \mathcal{X}$ (see [1], in particular Lemma 6, Theorem 6 and Remark 3.1). This gives

**Proposition 2.4.** *Let $a\colon \mathcal{B}(\mathcal{X}) \to \mathcal{L}_+(\mathcal{H})$ be a normalized commutative POV–measure which is informationally equivalent to the PV–measure $P$ whose associated von Neumann algebra $A(P)$ is maximal Abelian. Then there exists a $\sigma$–finite Borel measure $\mu$ on $\mathcal{X}$ such that*

(1) $\mathcal{H} \cong L^2\mathcal{X}, \mu)$; *and*
(2) *the POV–measure $a$ acts on $L^2(\mathcal{X}, \mu)$ as*

$$\big(a(E)\psi\big)(x) = \nu_x(E)\psi(x) \qquad \forall E \in \mathcal{B}(\mathcal{X})\,\forall\psi \in L^2(\mathcal{X}, \mu)\,\forall x \in \mathcal{X},$$

*with $\nu_x$ a probability measure on $\mathcal{X}$.*

**3. Naimark Extension for a Commutative POV–Measure on a Manifold.** We assume in this section that the topological space $\mathcal{X}$ is endowed with the structure of an $n$–dimensional differential manifold, and we will denote it from now on by $\mathcal{M}$.

**Definition 3.1.** A Borel measure $\nu$ on $\mathcal{M}$ is said to be *smooth*, if for each local chart $(U, \gamma, V)$ of $\mathcal{M}$ there exists $r \in C^\infty(V, \mathbb{R}^+)$ such that

$$\nu \circ \gamma^{-1} = r \cdot \lambda^n \text{ on } V \tag{3.1}$$

where $\lambda^n$ denotes the Lebesgue measure on $\mathbb{R}^n$.

Such a measure always exists on $\mathcal{M}$ (see e.g. [2, 3]) and is $\sigma$–finite.

Let $\nu$ be a smooth Borel measure on $\mathcal{M}$ and set $H = L^2(\mathcal{M}, \nu)$. Consider a commutative normalized POV–measure on $\mathcal{M}$,

$$a\colon \mathcal{B}(\mathcal{M}) \to \mathcal{L}_+\big(L^2(\mathcal{M}, \nu)\big)$$

which is informationally equivalent to the canonical PV-measure $P$ for $\mathcal{M}$ on $L^2(\mathcal{M}, \nu)$,

$$(P(E)\psi)(m) = \chi_E(m)\psi(m) \qquad \forall E \in \mathcal{B}(\mathcal{M})\,\forall\psi \in L^2(\mathcal{M}, \nu)\,\forall m \in \mathcal{M}.$$

Here, $\chi_E$ denotes the characteristic function of $E \subset \mathcal{M}$. Since the von Neumann algebra $A(P)$ is maximal Abelian, we conclude with Lemma 2.3 and Proposition 2.4 that (up to isomorphism) $a$ is given as

$$(a(E)\psi)(m) = \mu_m(E)\psi(m) \qquad \forall E \in \mathcal{B}(\mathcal{M})\,\forall\psi \in L^2(\mathcal{M}, \nu)\,\forall m \in \mathcal{M},$$

with $\mu_m$ a probability measure on $\mathcal{M}$ for each $m \in \mathcal{M}$.

Let $\{(U_i, \gamma_i, V_i) \mid U_i \subset \mathcal{M}, V_i \subset \mathbb{R}^n, i \in \mathbb{N}\}$ be a maximal atlas for $\mathcal{M}$ and $\{r_i \mid i \in \mathbb{N}\}$ the corresponding family of positive $C^\infty$–maps with property (3.1). We construct a cover of $\mathcal{M}$ by disjoint Borel sets as follows:

$$\widehat{U}_1 = U_1, \qquad \widehat{U}_k = U_k \Big\backslash \bigcup_{i=1}^{k-1} \widehat{U}_i \quad \forall k \geq 2.$$

Let $g_k$ denote the restriction of the diffeomorphism $\gamma_k$ to $\widehat{U}_k$ and set $\widehat{V}_k = g_k(\widehat{U}_k)$, for all $k \in \mathbb{N}$. Then the restriction $\kappa_k$ of $r_k$ to $\widehat{V}_k$ satisfies

$$\nu \circ g_k^{-1} = \kappa_k \cdot \lambda^n \text{ on } \widehat{V}_k$$

for any $k$ and, without loss of generality, we may assume that $\operatorname{supp}\kappa_k \subset \widehat{V}_k$ while viewing $\kappa_k$ as a function from $\mathbb{R}^n$ to $\mathbb{R}^+$. For each $k$, the diffeomorphism $g_k$ induces an isometric isomorphism of $L^2$-spaces:

$$\begin{aligned}\Psi_k\colon L^2(\widehat{U}_k,\nu_k) &\to L^2(\widehat{V}_k,\kappa_k\cdot\lambda^n),\\ f &\mapsto \Psi_k(f)=h, \quad h(x)=f\big(g_k^{-1}(x)\big).\end{aligned}$$

Thus, with $\nu_k = \nu\big|_{\mathcal{B}(\mathcal{M})\cap\widehat{U}_k}$, we may write $L^2(\mathcal{M},\nu)$ as a countable direct sum of Hilbert spaces,

$$\begin{aligned}L^2(\mathcal{M},\nu) &= \bigoplus_{k=1}^{\infty} L^2(\widehat{U}_k,\nu_k)\\ &\cong \bigoplus_{k=1}^{\infty} L^2(\widehat{V}_k,\kappa_k\cdot\lambda^n)\\ &\cong \bigoplus_{k=1}^{\infty} L^2(\mathbb{R}^n,\kappa_k\cdot\lambda^n).\end{aligned} \tag{3.2}$$

The POV–measure $a$, restricted to $L^2(\widehat{U}_k,\nu_k)$, is commutative for each k. Let $\Gamma_k$ denote the isometric isomorphism between $L^2(\widehat{U}_k,\nu_k)$ and $L^2(\mathbb{R}^n,\kappa_k\cdot\lambda^n)$ induced by $g_k$. Then the family $\{\Gamma_k a(E)\Gamma_k^{-1} \mid E\in\mathcal{B}(\mathcal{M})\}$ is a commutative normalized POV–measure for $\mathcal{M}$ on $L^2(\mathbb{R}^n,\kappa_k\cdot\lambda^n)$, with

$$(\Gamma_k a(E)\Gamma_k^{-1}\psi)(x) = \mu_{g_k^{-1}(x)}(E)\psi(x), \tag{3.3}$$

$\forall\psi\in L^2(\mathbb{R}^n,\kappa_k\cdot\lambda^n)$, $\forall E\in\mathcal{B}(\mathcal{M})$ and $\forall x\in\mathbb{R}^n$.

We use at this point a theorem due to Naimark (see for example, [6]), according to which the POV–measure in (3.3) can be extended to a PV–measure $P$ on an enlarged Hilbert space $\widetilde{\mathcal{H}}$, i.e. one which contains $L^2(\mathbb{R}^n,\kappa_k\cdot\lambda^n)$ as an isometrically embedded subspace. The POV–measure can be recovered as the restriction of $P$ to this subspace. More precisely,

**Lemma 3.2.** *Let $\mathcal{X}$ be a locally compact topolocical space, $\lambda$ a Borel measure on $\mathbb{R}^n$, and let $a\colon\mathcal{B}(\mathcal{X})\to\mathcal{L}_+\big(L^2(\mathbb{R}^n,\lambda)\big)$ be a commutative normalized POV–measure, given as*

$$(a(E)\psi)(x) = \rho_x(E)\psi(x) \qquad \forall E\in\mathcal{B}(\mathcal{X})\;\forall\psi\in L^2(\mathbb{R}^n,\lambda),$$

*where $\forall x\in\mathbb{R}^n$, $\rho_x$ is a probability measure on $\mathcal{X}$. Then there exists a PV–measure $P\colon\mathcal{B}(\mathcal{X})\to\mathcal{L}_+(\mathcal{H})$, where*

$$\mathcal{H} = \int_{\mathbb{R}^n}^{\oplus} L^2(\mathcal{X},\rho_y)\lambda(dy),$$

*and a projector $\mathbf{P}\colon\mathcal{H}\to L^2(\mathbb{R}^n,\lambda)$ such that*

$$a(E) = \mathbf{P}P(E)\mathbf{P}, \qquad \forall E\in\mathcal{B}(\mathcal{X}).$$

*Proof.* Following [1, 4, 6] we construct the Hilbert space

$$\mathcal{K} = \overline{B\otimes L^2(\mathbb{R}^n,\lambda)/\mathcal{N}}^{\langle\ \rangle},$$

where $B$ denotes the $C^*$–algebra of all bounded measurable functions from $\mathcal{X}$ to $\mathbb{C}$, with inner product

$$\langle f \otimes \varphi, g \otimes \psi \rangle = \int_{\mathbb{R}^n} \int_{\mathcal{X}} \overline{f(x)} g(x) \rho_y(dx) \overline{\varphi(y)} \psi(y) \lambda(dy),$$

and

$$\mathcal{N} = \{\xi \in B \otimes L^2(\mathbb{R}^n, \lambda) \mid \langle \xi, \xi \rangle = 0\}.$$

Define a representation $\beta \colon B \to \mathcal{L}(\mathcal{K})$ by

$$\beta(f)(g \otimes \psi) = (f \cdot g) \otimes \psi,$$

$\forall f \in B$ and $\forall g \otimes \psi \in B \otimes L^2(\mathbb{R}^n, \lambda)$, and we embed $L^2(\mathbb{R}^n, \lambda)$ into $\mathcal{K}$ by

$$\gamma \colon L^2(\mathbb{R}^n, \lambda) \to \mathcal{K},$$
$$\gamma(\psi) = \mathbf{1} \otimes \psi, \qquad \forall \psi \in L^2(\mathbb{R}^n, \lambda),$$

where $\mathbf{1}(x) = 1$, $\forall x \in \mathcal{X}$. Since $\mathbf{1} \otimes L^2(\mathbb{R}^n, \lambda)$ is a proper subspace of $\mathcal{K}$ and since each $\rho_x$ is a probability measure on $\mathcal{X}$, the map $\gamma$ is an isometric isomorphism. Moreover, for each $f \in B$, the equality,

$$(\gamma^* \beta(f) \gamma \psi, \varphi) = \langle f \otimes \psi, \mathbf{1} \otimes \varphi \rangle \tag{3.4}$$

holds. If we define $P \colon \mathcal{B}(\mathcal{X}) \to \mathcal{L}_+(\mathcal{K})$ by $P(E) = \beta(\chi_E)$, then $P$ is a PV-measure for $\mathcal{X}$ on $\mathcal{K}$. Combining (3.4) with the specific form of the POV-measure $a$, we get

$$\gamma^* P(E) \gamma = a(E) \qquad \forall E \in \mathcal{B}(\mathcal{X}).$$

Identifying $L^2(\mathbb{R}^n, \lambda)$ with its image in $\mathcal{K}$ and denoting by $\mathbf{P}$ the projection operator from $\mathcal{K}$ onto $L^2(\mathbb{R}^n, \lambda)$, we have

$$\mathbf{P}^* P(E) \mathbf{P} = a(E) \qquad \forall E \in \mathcal{B}(\mathcal{X}). \tag{3.5}$$

Finally, observing that the functions $f \in B$ are square integrable with respect to $\rho_y$, $\forall y \in \mathbb{R}^n$ (which follows from the square integrability of any $f$ with respect to $(a(E)\psi, \psi)$, $\forall \psi \in L^2(\mathbb{R}^n, \lambda)$), $\mathcal{K}$ is isometrically isomorphic to

$$\widetilde{\mathcal{H}} = \int_{\mathbb{R}^n}^{\oplus} L^2(\mathcal{X}, \rho_y) \lambda(dy). \quad \square \tag{3.6}$$

**Corollary 3.3.** *If for all $y \in \mathbb{R}^n$, $L^2(\mathcal{X}, \rho_y) = L^2(\mathcal{X}, \rho)$ for some fixed probability measure $\rho$ on $\mathcal{B}(\mathcal{X})$, then $\widetilde{\mathcal{H}} = L^2(\mathcal{X} \times \mathbb{R}^n, \rho \cdot \lambda)$.*

Applying Lemma 3.2 to the POV–measure $a$ as defined in (3.3), we find as an extended Hilbert space $\widetilde{\mathcal{H}}_k$ for $L^2(\mathbb{R}^n, \kappa_k \cdot \lambda^n J)$,

$$\widetilde{\mathcal{H}}_k = \int_{\mathbb{R}^n}^{\oplus} L^2(\mathcal{M}, \mu_{g_k^{-1}(x)}) \kappa_k(x) \lambda^n(dx) = \int_{\widehat{\mathcal{V}}_k}^{\oplus} L^2(\mathcal{M}, \mu_{g_k^{-1}(x)}) \kappa_k(x) \lambda^n(dx),$$

and by (3.5) the corresponding PV–measure is given as a family of operators of multiplication by the characteristic functions. Carrying out this extension for each $k \in \mathbb{N}$, and writing the POV–measure $a$ in the form

$$a(E) = \sum_{k=1}^{\infty} \Gamma_k \mathbf{P}(\widehat{U}_k) a(E) \mathbf{P}(\widehat{U}_k) \Gamma_k^{-1},$$

one finds for the enlarged Hilbert space,

$$\widetilde{\mathcal{H}} = \bigoplus_{k=1}^{\infty} \widetilde{\mathcal{H}}_k = \bigoplus_{k=1}^{\infty} \int_{\mathbb{R}^n}^{\oplus} L^2(\mathcal{M}, \mu_{g_k^{-1}(x)}) \kappa_k(x) \lambda^n(dx).$$

Thus we have shown

**Proposition 3.4.** *Let $a\colon \mathcal{B}(\mathcal{M}) \to \mathcal{L}_+\big(L^2(\mathcal{M},\nu)\big)$ be a commutative, normalized POV–measure for $\mathcal{M}$ on $L^2(\mathcal{M},\nu)$ which is informationally equivalent to the canonical PV–measure for $\mathcal{M}$ on $L^2(\mathcal{M},\nu)$. Then*

(1) *For all $E \in \mathcal{B}(\mathcal{M})$ and for all $\psi \in L^2(\mathcal{M},\nu)$,*

$$(a(E)\psi)(m) = \mu_m(E)\psi(m)$$

*with $\mu_m$ a probability measure on $\mathcal{M}$ for each $m$.*

(2) *There exists a PV–measure $\widetilde{P}\colon \mathcal{B}(\mathcal{M}) \to \mathcal{L}_+(\widetilde{\mathcal{H}})$, where*

$$\widetilde{\mathcal{H}} = \bigoplus_{k=1}^{\infty} \widetilde{\mathcal{H}}_k = \bigoplus_{k=1}^{\infty} \int_{\mathbb{R}^n}^{\oplus} L^2\big(\mathcal{M},\mu_{g_k^{-1}(x)}\big)\kappa_k(x)\lambda^n(dx),$$

*and a projection operator $\mathbf{P}\colon \widetilde{\mathcal{H}} \to \bigoplus_{k=1}^{\infty} L^2(\mathbb{R}^n,\kappa_k\cdot\lambda^n) \subset \widetilde{\mathcal{H}}$, such that*

$$\mathbf{P}\widetilde{P}(E)\mathbf{P} = \sum_{k=1}^{\infty} \Gamma_k \mathbf{P}(\widehat{U}_k)a(E)\mathbf{P}(\widehat{U}_k)\Gamma_k^{-1},$$

*for all $E \in \mathcal{B}(\mathcal{M})$.*

Proposition 3.4 can be further simplified, if Corollary 3.3 is applicable to $H_k$ for each $k$ — i.e. if there is a measure $\mu_k$ such that $\forall x \in \mathbb{R}^n$,

$$L^2\big(\mathcal{M},\mu_{g_k^{-1}(x)}\big) = L^2(\mathcal{M},\mu_k),$$

for then,

$$\begin{aligned}\widetilde{\mathcal{H}} &= \bigoplus_{k=1}^{\infty} L^2(\mathcal{M}\times\mathbb{R}^n,\mu_k\cdot\kappa_k\lambda^n)\\ &= \bigoplus_{k=1}^{\infty} L^2(\mathcal{M}\times\mathbb{R}^n,\Lambda_k).\end{aligned} \tag{3.7}$$

Here, for each $k$, $\Lambda_k$ denotes the measure on $\mathcal{M}\times\mathbb{R}^n$ with

$$\Lambda_k(E\times\mathbb{R}^n) = \mu_k(E), \qquad \forall E \in \mathcal{B}(\mathcal{M}), \tag{3.8a}$$

and

$$\Lambda_k(\mathcal{M}\times F) = \int_{\mathbb{R}^n} \chi_F(x)\kappa_k(x)\lambda^n(dx), \qquad \forall F \in \mathcal{B}(\mathbb{R}^n). \tag{3.8b}$$

If, furthermore, the measures $\Lambda_k$ have mutually disjoint supports, then (3.7) can be written as

$$\widetilde{\mathcal{H}} = L^2(\mathcal{M}\times\mathbb{R}^n,\Lambda), \tag{3.9}$$

where

$$\Lambda = \sum_{k=1}^{\infty} \chi_{\operatorname{supp}\Lambda_k}\cdot\Lambda_k.$$

The projection $\mathbf{P}\colon \widetilde{\mathcal{H}}_k \to \bigoplus_{k=1}^{\infty} L^2(\mathbb{R}^n,\kappa_k\cdot\lambda^n)$ is then given as

$$\mathbf{P} = \sum_{k=1}^{\infty} \mathbf{P}_k,$$

where $\mathbf{P}_k\colon \widetilde{\mathcal{H}}_k \to L^2(\mathbb{R}^n,\kappa_k\cdot\lambda^n)$ is the projection operator with (3.5).

Conversely, if we start with a projection valued measure $\widetilde{P}$ for $\mathcal{M}$ on $L^2(\mathcal{M}\times\mathbb{R}^n,\Lambda)$, which acts as multiplication by $\chi_E$, construction gives:

**Proposition 3.5.** *Suppose that the Borel measure $\Lambda\colon \mathcal{B}(\mathcal{M}\times\mathbb{R}^n)\to\mathbb{R}^+$ can be decomposed into $\mu\cdot\alpha$ in such a way that $\mu$ is a probability measure on $\mathcal{M}$ and that $\alpha$ is a $\sigma$–finite measure on $\mathbb{R}^n$, which is absolutely continuous with respect to the Lebesgue measure $\lambda^n$, i.e. $\alpha=\kappa\cdot\lambda^n$ where $\kappa$ is a version of the Radon–Nikodym derivative of $\alpha$ with respect to $\lambda^n$. Suppose also that there exists a disjoint cover $\{(U_i,g_i,V_i)\mid U_i\subset\mathcal{M}, V_i\subset\mathbb{R}^n, i\in\mathbb{N}\}$ of $\mathcal{M}$ by Borel sets, constituting a maximal atlas for $\mathcal{M}$, such that*

$$\kappa_k=\kappa\big|_{V_k}\in C^\infty(V_k,\mathbb{R}^+)$$

*for each $k$. Then there is a projection $\mathbf{P}\colon L^2(\mathcal{M}\times\mathbb{R}^n,\Lambda)\to\bigoplus_{k=1}^\infty L^2(\mathbb{R}^n,\kappa_k\cdot\lambda^n)$, such that the family $\{\mathbf{P}\widetilde{P}(E)\mathbf{P}\mid E\in\mathcal{B}(\mathcal{M})\}$ is a commutative, normalized POV–measure on $\bigoplus_{k=1}^\infty L^2(\mathbb{R}^n,\kappa_k\cdot\lambda^n)$.*

*Furthermore, if $\Gamma_k\colon L^2(\mathbb{R}^n,\kappa_k\cdot\lambda^n)\to L^2(U_k,\nu_k)$ denotes the isometry induced by $g_k$, with $\nu_k=(\kappa_k\cdot\lambda^n)\circ g_k$, then*

$$a(E)=\sum_{k=1}^\infty\Gamma_k\mathbf{P}\widetilde{P}(E)\mathbf{P}\Gamma_k^{-1}$$

*is a commutative, normalized POV–measure for $\mathcal{M}$ on $L^2(\mathcal{M},\nu)$.*

*Proof.* Observe that $L^2(\mathcal{M}\times\mathbb{R}^n,\Lambda)=L^2(\mathcal{M},\mu)\otimes L^2(\mathbb{R}^n,\alpha)$. Let $x\to\mu_x$ be a measurable map from $\mathbb{R}^n$ into the set of probability measures on $\mathcal{M}$ where each $\mu_x$ is equivalent to $\mu$. Define $\mathbf{P}\colon L^2(\mathcal{M}\times\mathbb{R}^n,\mu\cdot\alpha)\to\mathbf{1}\otimes L^2(\mathbb{R}^n,\alpha)$ by

$$\begin{aligned}(\mathbf{P}(fJ\otimes\psi))(m,x)&=\int_{\mathcal{M}}f(m')\mu(dm')\psi(x)\\&=\int_{\mathcal{M}}f(m')\mu(dm')(\mathbf{1}\otimes\psi)(m,x),\end{aligned}\tag{3.10}$$

and let $\kappa$ be a version of the Radon–Nikodym derivative of $\alpha$ with respect to $\lambda^n$. With our atlas of $\mathcal{M}$,

$$L^2(\mathbb{R}^n,\alpha)=\bigoplus_{k=1}^\infty L^2(\mathbb{R}^n,\kappa_k\cdot\lambda^n),$$

and

$$\psi=\sum_{k=1}^\infty\psi_k,$$

where $\psi_k\in L^2(\mathbb{R}^n,\kappa_k\cdot\lambda^n)$. Thus $\forall x\in V_k$ (3.10) reads

$$\begin{aligned}\int_{\mathcal{M}}f(m')\mu_x(dm')(\mathbf{1}\otimes\psi)(m,x)&=\sum_{k=1}^\infty\int_{\mathcal{M}}f(m')\mu_x(dm')(\mathbf{1}\otimes\psi_k)(m,x)\\&=\int_{\mathcal{M}}f(m')\mu_x(dm')\psi_k(x)\end{aligned}$$

For $\widetilde{P}(E)$ we then have

$$(\mathbf{P}\widetilde{P}(E)\mathbf{P}(fJ\otimes\psi))(m,x)=\int_{\mathcal{M}}f(m')\mu_x(dm')\mu_x(E)\psi(x)\overset{\text{def}}{=}(a(E)\psi)(x).\tag{3.11}$$

For $x \in V_k$ there exists, however, an $m \in U_k$ such that $x = g_k(m)$, and

$$\psi_k(x) = (\psi_k \circ g_k)(m) \stackrel{\text{def}}{=} \tilde{\psi}_k(m).$$

Hence, setting

$$(\Gamma_k \psi_k)(m) \stackrel{\text{def}}{=} (\psi_k \circ g_k)(m),$$

one finds that (3.11) is equivalent to

$$(\hat{a}(E)\tilde{\psi}_k t)(m) \stackrel{\text{def}}{=} (\Gamma_k^{-1} a(E)\Gamma_k \tilde{\psi}_k)(m) = \mu_{g_k(m)}(E)\tilde{\psi}_k(m),$$

which is a normalized, commutative POV–measure on $L^2(U_k, \nu_k)$. Thus for arbitrary $\tilde{\psi} \in L^2(\mathcal{M}, \nu) = \bigoplus_{k=1}^{\infty} L^2(U_k, \nu_k)$,

$$(\hat{a}(E)\tilde{\psi})(m) \stackrel{\text{def}}{=} \mu_{g_k(m)}(E)\tilde{\psi}(m), \quad m \in U_k,$$

yields a commutative, normalized POV–measure on $L^2(\mathcal{M}, \nu)$. □

**Résumé substantiel en français.** Soient $\mathcal{X}$ un espace topologique localement compact et $\mathcal{B}(\mathcal{X})$ l'algèbre $\sigma$ des sous ensembles de Borel de $\mathcal{X}$. Soient $\mathcal{H}$ un espace de Hilbert complexe et séparable, $\mathcal{L}(\mathcal{H})$ l'ensemble de tous les opérateurs linéaires bornés sur $\mathcal{H}$ et $\mathcal{L}_+(\mathcal{H})$ son cône positif. Une mesure normalisée à valeur dans les opérateurs positifs (VOP) de $\mathcal{X}$ sur $\mathcal{H}$ est une mesure qui prend ses valeurs dans $\mathcal{L}_+(\mathcal{H})$ de sorte que $a(\mathcal{X}) = I$. Si les opérateurs en question commuttent deux à deux, la mesure à VOP est dite commutative. Si tous les opérateurs sont des projecteurs orthogonaux, la mesure est alors appelée une mesure à valeur dans les projecteurs (VP). Notons $A(a)$ l'algèbre de von Neumann engendrée par la mesure à VOP $a$. Deux mesures à VOP sont dites informationnellement équivalente, ssi $\forall \rho \in \mathcal{T}(\mathcal{H})$ (l'ensemble de tous les opérateurs de la classe des opérateurs à trace finie sur $\mathcal{H}$), $\mathrm{tr}[\rho a(E)] = 0 \; \forall E \in \mathcal{B}(\mathcal{X})$ implique $\mathrm{tr}[\rho \hat{a}(E)] = 0 \; \forall E \in \mathcal{B}(\mathcal{X})$ et vice versa. Tout au long de cet article nous supposons que la mesure à VOP considérée est informationnellement équivalente à une mesure $P$ à valeur dans les projecteurs, pour laquelle $A(P)$ est abélienne maximale. Par conséquent, $A(a)$ est abélienne maximale (lemme 2.3); en outre, la mesure $a$ peut être réalisée comme une famille d'opérateurs de multiplication sur $L^2(\mathcal{X}, \mu)$, où $\mu$ est une mesure $\sigma$-finie sur $\mathcal{X}$ (proposition 2.4). Supposons maintenant que $\mathcal{X}$ a en plus la structure d'une variété différentielle $\mathcal{M}$ de dimension $n$ et que $a$ est une mesure commutative à VOP pour $\mathcal{M}$ sur $L^2(\mathcal{M}, \nu)$, $\nu$ étant une mesure de Borel continue sur $\mathcal{M}$ (voir définition 3.1). En supposant l'équivalence informationnelle avec la mesure à valeurs dans les projecteurs canonique pour $\mathcal{M}$ sur $L^2(\mathcal{M}, \nu)$, $a$ peut encore être réalisée comme une famille d'opérateurs de multiplication sur $L^2(\mathcal{M}, \nu)$. En utilisant un atlas maximal pour $\mathcal{M}$, cet espace $L^2$ peut être identifié avec une somme directe de sous espaces deux à deux disjoints. Chacun de ces sous espaces est isométriquement isomorphe à un espace $L^2$ sur $\mathbb{R}^n$. De plus la mesure à VOP considérée donne, par restriction à chaque sous espace, une mesure commutative à VOP sur ce dernier et donc sur $L^2(\mathbb{R}^n, \lambda_k)$ pour chaque $k$ (cf. formules (3.2) et (3.3)). Ainsi, il suffit donc de réaliser l'extension de Naimark pour une mesure à VOP, commutative et normalisée, pour $\mathcal{X}$ sur $L^2(\mathbb{R}^n, \lambda)$, où $\lambda$ est une mesure sur $\mathbb{R}^n$ absolument continue par rapport à la mesure de Lebesgue sur $\mathbb{R}^n$. Ceci est concrétisé dans le lemme 3.2 : nous montrons qu'il existe un espace de Hilbert étendu $\widetilde{\mathcal{H}}$ contenant $L^2(\mathbb{R}^n, \lambda)$ comme sous-espace propre, et une mesure à VOP sur $\widetilde{\mathcal{H}}$ coïncidant avec $a$ par restriction à $L^2(\mathbb{R}^n, \lambda)$. La proposition 3.4 donne le résultat pour la mesure à VOP sur $L^2(\mathcal{M}, \nu)$. L'espace de Hilbert étendu apparaît sous la forme d'une somme directe d'intégrales directes d'espaces $L^2$. Nous considérons également, de manière brève, des simplifications possibles et nous concluons en montrant comment on peut construire une mesure à VOP sur $L^2(\mathcal{M}, \nu)$, à partir d'une mesure à valeur de projecteur sur $L^2(\mathcal{M} \times \mathbb{R}^n, \Lambda)$, où $\Lambda$ est une mesure produit.

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