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THE EULER-KRONECKER CONSTANT OF A CYCLOTOMIC FIELD

V. KUMAR MURTY

To Paulo Ribenboim on his 80th birthday.

RÉSUMÉ. Lorsque K est un corps de nombres, il est bien connu que la fonction zêta de Dedekind $\zeta_K(s)$ possède un pôle simple en $s = 1$, de sorte que l'on peut écrire au voisinage de $s = 1$ le développement

$$\zeta_K(s) = c_{-1}(s-1)^{-1} + c_0 + \mathbf{O}(s-1).$$

Adoptant la définition de Ihara, nous appelons $\gamma_K = c_0/c_{-1}$, la constante de Euler-Kronecker de K. Lorsque K est le corps cyclotomique $\mathbb{Q}(\zeta_m)$, Ihara a conjecturé que l'ordre de grandeur de cette constante est $\mathbf{O}(\log m)$. Nous prouvons que ceci est vrai en moyenne pour les entiers m premiers.

ABSTRACT. For a number field K, it is well-known that the Dedekind zeta function $\zeta_K(s)$ has a simple pole at $s = 1$ and so we may write the expansion near $s = 1$ as

$$\zeta_K(s) = c_{-1}(s-1)^{-1} + c_0 + \mathbf{O}(s-1).$$

Following Ihara, we call $\gamma_K = c_0/c_{-1}$ the Euler-Kronecker constant of K. When K is the cyclotomic field $\mathbb{Q}(\zeta_m)$, Ihara has conjectured that this constant is $\mathbf{O}(\log m)$. We prove that this holds on average for m prime.

1. Introduction

For a number field K, let us denote by $\zeta_K(s)$ the Dedekind zeta function of K. For $\mathrm{Re}(s) > 1$, it is defined by

$$\zeta_K(s) \;=\; \sum_{\mathfrak{a}} (N\mathfrak{a})^{-s}$$

where the sum is over integral ideals $\mathfrak{a}$ of K. It is well-known that $\zeta_K(s)$ has an analytic continuation for all s with a simple pole at $s = 1$. In particular, we may write

$$\zeta_K(s) \;=\; c_{-1}(s-1)^{-1} \;+\; c_0 \;+\; \mathbf{O}(s-1)\,.$$

Let us set

$$\gamma_K \;=\; c_0/c_{-1}\,.$$

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Following Ihara [4], we call γ_K the Euler-Kronecker constant of K. It can equivalently be defined by

$$-\frac{\zeta_K'}{\zeta_K}(s) \;=\; \frac{1}{s-1} \;-\; \gamma_K \;+\; \mathbf{O}(s-1)\,.$$

For $K = \mathbb{Q}$, it is the usual Euler constant γ given by

$$\lim_{x\to\infty}\biggl(\sum_{n\le x}\frac{1}{n} \;-\; \log x\biggr).$$

In [4], Theorem 1 and Proposition 3, Ihara proved that for $K \ne \mathbb{Q}$, the Riemann Hypothesis for Dedekind zeta functions (GRH) implies that there are absolute constants $c_1, c_2 > 0$ so that

$$-c_1 \log d_K \;\le\; \gamma_K \;\le\; c_2 \log\log d_K\,. \tag{1.1}$$

Here, d_K is the absolute value of the discriminant of $K/\mathbb{Q}$. As pointed out in [4], the estimation of γ_K from below is related to the existence or non-existence of primes of K with small norm. Tsfasman [8] showed, assuming the GRH, that as we range over number fields K with $d_K \;\to\; \infty$, we have

$$\liminf \frac{\gamma_K}{\log d_K} \;\ge\; -0.13024\ldots$$

In [5], the Euler-Kronecker constant is considered in more detail for certain families of fields. Of special interest is the Euler-Kronecker constant of a cyclotomic field. Let us set $\gamma_m \;=\; \gamma_{\mathbb{Q}(\zeta_m)}$. Then, as a consequence of the factorization

$$\zeta_{\mathbb{Q}(\zeta_m)}(s) \;=\; \prod_{\chi \mod m} L(s,\chi)$$

of the Dedekind zeta function of $\mathbb{Q}(\zeta_m)$ into Dirichlet L-functions, we have

$$\gamma_m \;=\; \gamma \;+\; \sum_{\chi\ne\chi_0}\frac{L'}{L}(1,\chi)\,.$$

Here χ_0 denotes the principal character modulo m. From (1.1), we have for $m = q$ prime,

$$-c_1 q\log q \;\le\; \gamma_q \;\le\; c_2\log q\,.$$

In [6], Ihara, K. Murty and M. Shimura proved that under the GRH, we have

$$\gamma_q \;\ll\; (\log q)^2.$$

In other words, there is an absolute constant $c_3 > 0$ so that

$$|\gamma_q| \;\le\; c_3(\log q)^2. \tag{1.2}$$

Unconditionally, they proved that

$$\gamma_q \;\ll\; q^{\epsilon}.$$

Badzyan [1] has shown that under the GRH, (1.2) can be improved to

$$\gamma_q \;\ll\; (\log q)(\log\log q)\,.$$

Ihara [5], Conjecture 1, has conjectured that there are positive constants $0 < c_4, c_5 \le 2$ such that for any m sufficiently large (not necessarily prime), and any $\epsilon > 0$ we have

$$(c_4 - \epsilon)\log m \;<\; \gamma_m \;<\; (c_5 + \epsilon)\log m\,.$$

In this note, we prove that for m prime, the upper bound of Ihara's conjecture holds unconditionally on average. More precisely, we prove the following.

Theorem 1.1. *We have*

$$\sum_{\frac{1}{2}Q<q\le Q} |\gamma_q| \ \ll\ \pi^*(Q)(\log Q)$$

where $\pi^*(Q)$ denotes the number of primes in the interval $(\frac{1}{2}Q, Q]$ and the sum is over primes q in this interval.

The proof closely follows the methods of [6]. Throughout this paper, we shall use q to denote a prime number.

2. The sum $\Phi_\chi(x)$ and its average

As in [6], we consider for $x > 1$ the sum

$$\Phi_\chi(x) \ =\ \frac{1}{x-1}\int_1^x \left(\sum_{n\le t} \frac{\Lambda(n)}{n}\chi(n)\right) dt\,.$$

We have

$$\sum_{\chi\ne\chi_0} \Phi_\chi(x) \ =\ \frac{1}{x-1}\int_1^x \left(\phi(q) \sum_{\substack{n\le t\\ n\equiv 1 \ (\mathrm{mod}\ q)}} \frac{\Lambda(n)}{n} \ -\ \sum_{\substack{n\le t\\ (n,q)=1}} \frac{\Lambda(n)}{n}\right) dt\,. \tag{2.1}$$

As usual, let us set

$$\psi(x,q,a) \ =\ \sum_{\substack{n\le x\\ n\equiv a \ (\mathrm{mod}\ q)}} \Lambda(n)$$

and

$$\psi(x) \ =\ \sum_{n\le x} \Lambda(n)\,.$$

By partial summation, we have

$$\sum_{\substack{n\le t\\ n\equiv 1 \ (\mathrm{mod}\ q)}} \frac{\Lambda(n)}{n} \ =\ \frac{1}{t}\psi(t,q,1) + \int_1^t \frac{\psi(u,q,1)}{u^2}\, du\,. \tag{2.2}$$

On the other hand, we have for q prime

$$\sum_{\substack{n\le t\\ (n,q)=1}} \frac{\Lambda(n)}{n} \ =\ \sum_{n\le t} \frac{\Lambda(n)}{n} \ +\ \mathbf{O}\Big(\frac{\log q}{q}\Big). \tag{2.3}$$

Moreover,

$$\sum_{n\le t} \frac{\Lambda(n)}{n} \ =\ \frac{\psi(t)}{t} \ +\ \int_1^t \frac{\psi(u)}{u^2}\, du\,. \tag{2.4}$$

Proposition 2.1. *We have for $x > 1$ and $Q \geq 2$,*

$$\sum_{\frac{1}{2}Q<q\leq Q} \left| \sum_{\chi\neq\chi_0} \Phi_\chi(x) \right| \ll \pi^*(Q)(\log Q)$$

where $\pi^*(Q)$ denotes the number of primes in the interval $(\frac{1}{2}Q, Q]$.

Proof. The contribution of the error term of (2.3) is

$$\sum_{\frac{1}{2}Q<q\leq Q} \frac{\log q}{q} \ll 1 .$$

Now inserting (2.2) and (2.4) into (2.1) and rearranging, we get

$$(2.5) \quad \sum_{\frac{1}{2}Q<q\leq Q} \left| \sum_{\chi\neq\chi_0} \Phi_\chi(x) \right| \ll \frac{x}{x-1} \int_1^x \frac{\sum_{\frac{1}{2}Q<q\leq Q} |\phi(q)\psi(u,q,1) - \psi(u)|}{u^2}\, du + \mathbf{O}(1) .$$

We have

$$(2.6) \quad \sum_{\frac{1}{2}Q<q\leq Q} \phi(q)\psi(u,q,1) = \sum_{n\leq u} \Lambda(n) \sum_{\substack{q|(n-1) \\ \frac{1}{2}Q<q\leq Q}} \phi(q) .$$

Suppose that $u \leq Q^3$. Then in the inner sum, there can be at most four terms. Hence, the contribution of prime powers to this sum is easily seen to be

$$\ll Q \sum_{p\leq\sqrt{u}} (\log p) \frac{\log u}{\log p} \ll Q\sqrt{u} .$$

Hence, the contribution to (2.5) is

$$\ll Q \int_1^{Q^3} \frac{du}{u^{3/2}} \ll \pi^*(Q) \log Q .$$

Similarly,

$$\sum_{\frac{1}{2}Q<q\leq Q} \psi(u) \ll \pi^*(Q)u .$$

Hence, the contribution of $u \leq Q^3$ to (2.5) is

$$\ll \pi^*(Q)(\log Q) .$$

The contribution of primes to (2.6) is

$$\sum_{p\leq u} (\log p) \sum_{\substack{p-1=tq \\ \frac{1}{2}Q<q\leq Q}} \phi(q) = \sum_{t\leq 2u/Q} \sum_{\substack{p\leq u, p-1=tq \\ \frac{1}{2}Q<q\leq Q}} \phi(q) \log p .$$

This is estimated by observing that ([2], Chapter 6, Exercise 13) given t, the number of prime pairs $p, q \leq u$ with $p - 1 = tq$ is

$$\ll \frac{u}{\phi(t) \log^2(u/t)} .$$

We continue to assume that $u \leq Q^3$ and distinguish two cases. Suppose first that $u < Qt$. Then, the contribution to (2.5) is

$$\ll \sum_{t \leq 2Q^2} \int_{Qt/2}^{Qt} \frac{Q}{\phi(t)} \frac{1}{(\log Q)^2} \frac{\log u}{u}\, du$$

and this is

$$\ll \frac{Q}{(\log Q)^2} \sum_{t \leq 2Q^2} \frac{1}{\phi(t)} \big((\log Qt)^2 - (\log Qt/2)^2 \big). \tag{2.7}$$

Using the elementary fact (see, for example, [2], Chapter 8, Exercise 15) that for $y > 2$, we have

$$\sum_{t \leq y} \frac{1}{\phi(t)} \ll \log y, \tag{2.8}$$

and the observation that $(\log Qt)^2 - (\log Qt/2)^2 = \mathbf{O}(\log Qt)$, we deduce that (2.7) is

$$\ll Q \ll \pi^*(Q) \log Q\,.$$

Now for $u > Qt$, we proceed as follows. Given $p - 1 = tq$, we have $p \leq tQ + 1$. Hence, the number of prime pairs $p, q \leq tQ + 1$ with $p - 1 = tq$ is

$$\ll \frac{Qt}{\phi(t)(\log Q)^2}\,.$$

Hence, the contribution to (2.5) is

$$\int_1^{Q^3} \Bigg(\sum_{t \leq u/Q} \sum_{\substack{p \leq tQ+1,\, \frac{1}{2}Q < q \leq Q \\ p-1=tq}} \phi(q) \log p \Bigg) \frac{du}{u^2}$$

which is

$$\ll \sum_{t \leq Q^2} Q (\log Qt) \frac{Qt}{\phi(t)(\log Q)^2} \int_{Qt}^{\infty} \frac{du}{u^2}\,.$$

This is

$$\ll \sum_{t \leq Q^2} \frac{Q \log Qt}{(\log Q)^2} \frac{1}{\phi(t)}\,.$$

Again using (2.8), it follows that the above quantity is

$$\ll Q \ll \pi^*(Q) \log Q\,.$$

For $u > Q^3$, we use the Bombieri-Vinogradov theorem (see [2], §9) to deduce that

$$\sum_{\frac{1}{2}Q < q \leq Q} |\phi(q)\psi(u, q, 1) - \psi(u)| = \sum_{\frac{1}{2}Q < q \leq Q} \phi(q) \left| \psi(u, q, 1) - \frac{\psi(u)}{\phi(q)} \right|$$

is

$$\ll Qu/(\log u)^3.$$

Hence, the contribution of the range $Q^3 \leq u \leq x$ to the integral in (2.5) is

$$\ll Q \int_{Q^3}^{\infty} \frac{du}{u(\log u)^3} \ll \frac{Q}{(\log Q)^2} \ll \frac{\pi^*(Q)}{\log Q}\,.$$

This proves the result. □

3. Application of zero density estimates

In this section, we prove the following.

Proposition 3.1. *For $x \geq Q^{25}$ and $Q \geq 2$, we have*

$$\sum_{\frac{1}{2}Q<q\leq Q} \sum_{\chi\neq\chi_0} \left| \frac{L'}{L}(1,\chi) + \Phi_\chi(x) \right| \ll \pi^*(Q)(\log Q)\,.$$

We remark that in [6], Lemma 2, a similar result is proved without the averaging over q. Also, we remark that the proof below shows that if there are no "exceptional" zeros (defined below), then the contribution from (3.1) need not be included and thus, the estimate of the proposition above may be replaced with $(\log Q)^{14}$.

Proof. Denote by $N(\sigma, T, q)$ the number of zeros $\rho = \beta + i\gamma$ of the product

$$\prod_{\chi_0\neq\chi \mod q} L(s,\chi)$$

with $\beta \geq \sigma$ and $|\gamma| \leq T$. (Recall that q is prime and so the nontrivial characters are primitive.) Let us also set

$$N(\sigma,T,Q) = \sum_{\frac{1}{2}Q<q\leq Q} N(\sigma,T,q)\,.$$

The proof proceeds exactly as in the proof of [6], Lemma 2, except that we use the zero-density estimate of Montgomery [7] that for $\sigma \geq 4/5$, we have

$$N(\sigma,T,Q) \ll (Q^2T)^{\frac{5}{2}(1-\sigma)}(\log Q^2T)^{13}$$

uniformly for $Q \geq 1$ and $T \geq 2$. We begin with the equality (5.4.1) of [6], namely

$$\frac{L'}{L}(1,\chi) + \Phi_\chi(x) = \frac{1}{x-1}\sum_\rho \frac{x^\rho - 1}{\rho(1-\rho)} + \mathbf{O}\Big(\frac{\log x}{x}\Big),$$

valid for any $\chi \neq \chi_0$.

Let $\chi_1 = \chi_{1,q} \neq \chi_0$ denote the unique quadratic character modulo q. It is well-known (see, for example, [3], §14) that there is an effective and absolute constant $c_6 > 0$ such that for any $\chi \pmod q$, $L(s,\chi)$ has at most one zero $\rho = \beta + i\gamma$ with

$$\beta \geq 1 - \frac{c_6}{\log q},\ |\gamma| \leq 2\,.$$

Moreover, if such a zero exists, then $\chi = \chi_1$ and this zero is real and simple. Let us call such a zero exceptional and denote it by $\beta_1 = \beta_{1,q}$. By a theorem of Page (see for example [3], p. 95), there is a positive constant c_7 so that there is at most one quadratic character χ to a prime modulus $q \leq Q$ for which $L(s,\chi)$ has a real zero β satisfying

$$\beta > 1 - \frac{c_7}{\log Q}\,.$$

From Sublemma 5.4.3, (5.4.4) and (5.4.6) of [6], we note that if the sum over zeros is truncated to zeros ρ with $|\gamma| \le T$, we get an inequality

$$\frac{L'}{L}(1,\chi) + \Phi_\chi(x) \ll \frac{1}{x-1}\sum_{|\gamma|\le T}\left|\frac{x^\rho - 1}{\rho(1-\rho)}\right| + \frac{\log qT}{T} + \frac{(\log q)^2}{x} + \frac{\log x}{x}.$$

Summing this over $\chi \ne \chi_0$ modulo q and then over prime moduli $q \in (\frac{1}{2}Q, Q]$, the error terms contribute an amount which is

$$\ll Q\pi^*(Q)\left\{\frac{\log QT}{T} + \frac{(\log Q)^2}{x} + \frac{\log x}{x}\right\}.$$

If we choose $T > Q^2$ and $x > Q^2(\log Q)$, this amount is bounded.

To estimate the sum over zeros, we first consider the contribution of exceptional zeros. By Page's theorem, with at most one exception, we have

$$\frac{1}{\beta_{1,q}(1-\beta_{1,q})} \ll \log Q$$

for all $q \le Q$. By Siegel's theorem, for any $\epsilon > 0$, we have

$$\frac{1}{\beta_{1,q}(1-\beta_{1,q})} \ll q^\epsilon$$

for *every* q. Using the second estimate for the possible exception to Page's theorem, we deduce that the contribution of exceptional zeros to the sum is

$$\ll \pi^*(Q)(\log Q) + Q^\epsilon. \tag{3.1}$$

We remark that in place of the bound q^ϵ given by Siegel's theorem, we could also use the effective bound $q^{\frac{1}{2}}(\log q)$.

To estimate the sum over non-exceptional zeros, we consider the related sum

$$\tilde{S}(x,Q,T) = \sum_{\frac{1}{2}Q<q\le Q}\ \sum_{\substack{\rho\in Z_q\\ |\gamma|\le T}} x^\beta$$

where Z_q denotes the set of non-exceptional zeros of the product

$$\prod_{\chi_0\ne\chi \mod q} L(s,\chi)$$

with real part β in $[0,1]$. The contribution of terms with $\beta \le 4/5$ to $\tilde{S}(x,Q,T)$ is

$$\ll x^{4/5}Q^2T\log(Q^2T) \ll x \tag{3.2}$$

provided $x \ge (Q^2T)^6$. The sum over remaining zeros is

$$-\int_{4/5}^1 x^\sigma d_\sigma N(\sigma,T,Q) = x^{4/5}N(4/5,T,Q) + \int_{4/5}^1 (x^\sigma \log x)N(\sigma,T,Q)d\sigma.$$

As in (3.2), the first term is $\mathbf{O}(x)$. The second term is

$$\ll (\log x)(Q^2T)^{5/2}(\log Q^2T)^{13}\frac{x}{(Q^2T)^{5/2}}\frac{1}{\log x/Q^2T}$$

and for $x \ge (Q^2T)^6$, this gives

$$\tilde{S}(x,Q,T) \ll x(\log Q^2T)^{13}. \tag{3.3}$$

Now, we return to the estimation of the original sum over non-exceptional zeros

$$\frac{1}{x-1}\sum_{\frac{1}{2}Q<q\leq Q}\sum_{\substack{\rho\in Z_q\\|\gamma|\leq T}}\left|\frac{x^\rho-1}{\rho(1-\rho)}\right|.$$

By [6], (5.4.4), we have

$$\frac{1}{x-1}\sum_{\frac{1}{2}Q<q\leq Q}\sum_{\substack{\rho\in Z_q\\|\gamma|\leq T}}\left|\frac{1}{\rho(1-\rho)}\right| \ll \pi^*(Q)Q(\log Q)^2/x \ll 1$$

for $x\geq Q^2\log Q$. Thus, it remains to consider

$$\sum_{\frac{1}{2}Q<q\leq Q}\sum_{\substack{\rho\in Z_q\\|\gamma|\leq T}}\frac{x^{\beta-1}}{|\rho(1-\rho)|}.$$

We see that the contribution of zeros with real part $\beta\leq 4/5$ is

$$\mathbf{O}(\pi^*(Q)Q(\log Q)^2/x^{1/5}).$$

Assume $x\geq (Q^2T)^6$. From (3.3), we see that the contribution of zeros with imaginary part $|\gamma|\leq 2$ is

$$\ll \frac{1}{x}(\log Q)\tilde{S}(x,Q,2) \ll (\log Q)^{14}.$$

The contribution of zeros with imaginary part satisfying $2^j<|\gamma|\leq 2^{j+1}$ is

$$\leq \frac{1}{x}\frac{1}{2^{2j}}\tilde{S}(x,Q,2^{j+1}).$$

Summing this over j with $2\leq 2^{j+1}\leq T$, and using (3.3), we see that the entire sum is

$$\ll (\log Q)^{13}.$$

Now choosing $T=Q^2(\log Q)$ (say), and $x\geq Q^{25}$, the result follows. □

4. Proof of the Theorem

We have

$$\gamma_q = \gamma - \sum_{\chi\neq\chi_0}\Phi_\chi(x) + \sum_{\chi\neq\chi_0}\left(\frac{L'}{L}(1,\chi)+\Phi_\chi(x)\right).$$

Taking the absolute value of both sides and using the triangle inequality, we have

$$|\gamma_q| \leq \gamma + \left|\sum_{\chi\neq\chi_0}\Phi_\chi(x)\right| + \left|\sum_{\chi\neq\chi_0}\left(\frac{L'}{L}(1,\chi)+\Phi_\chi(x)\right)\right|.$$

Now summing both sides over q and using Proposition 2.1 and Proposition 3.1, the result follows.

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V. K. Murty, Dept. of Math., U. of Toronto, 40 St. George Street, Toronto, ON, M5S 2E4, Canada
murty@math.toronto.edu