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SOME REMARKS ON A PROBLEM OF CHOWLA

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To Paulo Ribenboim on his 80th birthday.

RÉSUMÉ. Lorsque p est un premier et f est une fonction non identiquement nulle à valeurs entières de période p avec $f(p) = 0$ et que $\sum_{n=1}^{p} f(n) = 0$, alors S. Chowla a conjecturé que

$$\sum_{n=1}^{\infty} \frac{f(n)}{n} \neq 0\,.$$

Cette conjecture fut par la suite prouvée et généralisée par Baker, Birch et Wirsing. Dans leur article conjoint, Baker, Birch et Wirsing affirment que Chowla a aussi résolu le cas particulier que ce dernier a énoncé, mais ne donnent aucune indication sur le genre de démonstration effectuée. En nous basant sur certains articles préalablement écrits par Chowla, nous essayons de reconstruire ce qui aurait pu avoir été la preuve de cette conjecture par Chowla. Nous en profitons pour indiquer comment cette preuve peut être modifiée pour déduire le résultat général de Baker, Birch et Wirsing.

ABSTRACT. If p is prime and f is an integer valued function with period p, f not identically zero, $f(p) = 0$ and $\sum_{n=1}^{p} f(n) = 0$, then Chowla conjectured that

$$\sum_{n=1}^{\infty} \frac{f(n)}{n} \neq 0\,.$$

This conjecture was subsequently proved and generalized by Baker, Birch and Wirsing. In their paper, Baker, Birch and Wirsing state that Chowla had also solved the special case that he posed but gave no indication of what his proof may have been. Based on some of Chowla's earlier papers, we try to reconstruct what may have been Chowla's proof of his conjecture. At the same time, we indicate how this proof can be modified to deduce the general result of Baker, Birch and Wirsing.

1. Introduction

In a paper written in 1964, Sarvadaman Chowla [3] considered the following problem. Let p be a prime number and f an integer-valued arithmetical function with period p not identically zero, such that $f(p) = 0$ and

$$\sum_{n=1}^{p} f(n) = 0\,. \tag{1.1}$$

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Chowla conjectured that under these conditions,

$$\sum_{n=1}^{\infty} \frac{f(n)}{n} \neq 0\,.$$

It is useful to define

$$L(s,f) := \sum_{n=1}^{\infty} \frac{f(n)}{n^s}\,.$$

Then, condition (1.1) allows us to give an analytic continuation of $L(s,f)$ for $\Re(s) > 0$ and Chowla's conjecture can be reformulated as $L(1,f) \neq 0$ under these conditions. Following an argument outlined by Siegel, Chowla proved this conjecture in the case that f is odd (that is, $f(-n) = -f(n)$). Since his argument is very short and elegant, we give it below (see section 2). At the same time, we note that the argument applies in a wider context with minor changes.

In a later paper, Chowla [2] asked if there exists a function $f : \mathbb{Z} \to \mathbb{Q}$ with period q and q prime such that

$$\sum_{n=1}^{\infty} \frac{f(n)}{n} = 0\,. \tag{1.2}$$

The difference now is that $f(q)$ is not required to be zero. One can also investigate the general case when q is not necessarily prime and inquire under what conditions the sum (1.2) is zero. This general question was addressed by Baker, Birch and Wirsing [1] using Baker's theory of linear forms in logarithms. They showed that there is no such function for arbitrary q provided that the function satisfies the condition

$$f(a) = 0 \quad \text{for } 1 < (a,q) < q\,. \tag{1.3}$$

More generally, if f takes values in an algebraic number field K which is disjoint from the q-th cyclotomic field, and satisfies (1.3), then they proved that no such function exists.

After describing Chowla's original question, Baker, Birch and Wirsing wrote in a footnote on page 225 of their paper [1] that "While working on the manuscript, we were informed by Professor Chowla that he had also solved the problem to the extent stated above." It is unclear what the phrase "to the extent stated above" means but judging from the context, it seems to mean that Chowla also solved the problem in the case q is prime. However, Chowla does not seem to have published any work on this to give us an indication of what his methods were, if they were different from those of [1].

In this short note, our goal is to give a very short and simple proof of this result in the case q is prime as discussed in the original setting of Chowla's question. We will not need Baker's theory if f is restricted to being an odd function. This was known to Chowla. Baker's theory will enter into our proof only in a rudimentary way to deal with even functions and finally to fuse the odd case and even case together. This presentation may approximate the proof Chowla had in mind, since it uses ideas that were familiar to him and draws upon his earlier work. Moreover, it uses Baker's theory in a "minimal" way.

At the outset, let us remark that if $q = 2$, then it is easy to see that

$$L(1, f) = f(1) \log 2$$

is non-zero if and only if $f(1) \neq 0$. Since $f(1)+f(2) = 0$, this means that $L(1, f) = 0$ if and only if f is identically zero. Henceforth, we assume $q \neq 2$.

2. The Chowla-Siegel theorem

We begin by giving the proof (with some minor variations) of Chowla and Siegel [3] when f is an odd function. The essential idea here is to use the familiar cotangent expansion:

$$\pi \cot \pi x = \sum_{n \in \mathbb{Z}} \frac{1}{n+x},$$

which is valid for $x \notin \mathbb{Z}$.

Theorem 1. *Let K be an algebraic number field which is disjoint from the q-th cyclotomic field. Let $f : \mathbb{Z}/q\mathbb{Z} \to K$ be odd, that is, $f(-n) = -f(n)$. Suppose that $f(n) = 0$ whenever $(n, q) > 1$. Then,*

$$\sum_{n=1}^{\infty} \frac{f(n)}{n} \neq 0,$$

unless f is identically zero.

Remark. As noted earlier, only the case q prime and K is the rational number field is considered in [3]. However, a careful study shows that the argument gives a proof of the theorem stated above.

Proof. Let

$$S_k = \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} \frac{f(kn)}{n} \cdot$$

Since f is odd,

$$S_1 = 2 \sum_{n=1}^{\infty} \frac{f(n)}{n} \cdot$$

We will show that if $S_1 = 0$, then f is identically zero. To this end, we observe that

$$S_k = \sum_{a \,(\mathrm{mod}\, q)} f(ka) \sum_{\substack{n \equiv a \,(\mathrm{mod}\, q) \\ n \neq 0}} \frac{1}{n}$$

and the inner sum is

$$\frac{1}{q} \sum_{t \in \mathbb{Z}} \frac{1}{t + a/q} = \frac{\pi}{q} \cot \frac{\pi a}{q} = \frac{2\pi}{qi} \left(\frac{1}{2} + \frac{1}{\zeta^a - 1} \right),$$

where $\zeta = e^{2\pi i/q}$. Thus,

$$iS_k = \sum_{a \,(\mathrm{mod}\, q)} f(ka) \frac{2\pi}{q} \left(\frac{1}{2} + \frac{1}{\zeta^a - 1} \right).$$

Since k is coprime to q we have

$$\sum_{(a,q)=1} f(ka) = \sum_{(a,q)=1} f(a) = 0\,.$$

Thus, the first sum in the above expression for S_k disappears and we deduce that

$$\frac{iqS_k}{2\pi} = \sum_{a\,(\mathrm{mod}\,q)} \frac{f(ka)}{\zeta^a - 1}\,.$$

In particular,

$$\frac{iqS_1}{2\pi} = \sum_{a\,(\mathrm{mod}\,q)} \frac{f(a)}{\zeta^a - 1}\,.$$

This calculation also evaluates $L(1,\chi)$ when χ is an odd Dirichlet character modulo q. Now the right hand side is an algebraic number. Applying the Galois automorphism $\zeta \mapsto \zeta^{k'}$ where $kk' \equiv 1\,(\mathrm{mod}\,q)$, we see that $S_1 = 0$ implies $S_k = 0$ for all $(k,q) = 1$. Note that this step is valid if f is K-valued and K is disjoint from the q-th cylcotomic field. Hence,

$$\begin{aligned} 0 &= \sum_{k\,(\mathrm{mod}\,q)} \overline{\chi}(k) S_k \\ &= \sum_{k\,(\mathrm{mod}\,q)} \overline{\chi}(k) \sum_{a\,(\mathrm{mod}\,q)} \frac{f(ka)}{\zeta^a - 1} \\ &= \sum_{a,k\,(\mathrm{mod}\,q)} f(ka)\, \frac{\overline{\chi}(ka)\chi(a)}{\zeta^a - 1}\,. \end{aligned}$$

Now put $ka \equiv b\,(\mathrm{mod}\,q)$ to obtain

$$0 = \sum_{b\,(\mathrm{mod}\,q)} f(b)\overline{\chi(b)} \sum_{k\,(\mathrm{mod}\,q)} \frac{\chi(k'b)}{\zeta^{k'b} - 1}\,.$$

For fixed b, the number $k'b$ runs over all coprime residue classes (mod q) as k runs over all coprime residue classes (mod q). Thus, the inner sum is

$$\sum_{t\,(\mathrm{mod}\,q)} \frac{\chi(t)}{\zeta^t - 1} = \frac{iqL(1,\chi)}{\pi}$$

for χ odd. In particular, by virtue of the fact that $L(1,\chi) \neq 0$, we deduce that for χ odd,

$$\sum_{b\,(\mathrm{mod}\,q)} f(b)\overline{\chi}(b) = 0\,. \tag{2.1}$$

Now (2.1) is also true if χ is even since

$$\begin{aligned} \sum_{b\,(\mathrm{mod}\,q)} f(b)\overline{\chi}(b) &= \sum_{b\,(\mathrm{mod}\,q)} f(-b)\overline{\chi}(-b) \\ &= -\sum_{b\,(\mathrm{mod}\,q)} f(b)\overline{\chi}(b)\,. \end{aligned}$$

In any case, we have

$$\sum_{b\,(\mathrm{mod}\,q)} f(b)\chi(b) = 0$$

for all χ (mod q). Thus,

$$\begin{aligned} 0 &= \sum_{\chi} \chi(a)\left(\sum_{b\,(\mathrm{mod}\,q)} f(b)\overline{\chi}(b)\right) \\ &= \sum_{b\,(\mathrm{mod}\,q)} f(b)\left(\sum_{\chi}\chi(a)\overline{\chi}(b)\right) \\ &= \phi(q)f(a), \end{aligned}$$

by the orthogonality relations. Hence f is identically zero. □

As alluded to above, there are several noteworthy features in the above proof. First is that the argument works if f is K-valued and $K \cap \mathbb{Q}(\zeta_q) = \mathbb{Q}$. The second is that the function is only supported on the coprime arguments. That is, f is "Dirichlet type" in the sense of [6].

The remainder of the paper will be devoted to showing that with some minor modifications, the above strategy can be used to deal with the Chowla question in the case q is prime. This may have been the proof that Chowla had in mind when he wrote to the authors of [1] that he had also solved his question "to the extent stated above."

3. A review of Baker's theorem

Baker's fundamental theorem is the following one.

Theorem 2. *If $\alpha_1, \ldots, \alpha_n \in \overline{\mathbb{Q}}\backslash\{0\}$, and $\beta_1, \ldots, \beta_n \in \overline{\mathbb{Q}}$, then*

$$\beta_1 \log \alpha_1 + \cdots + \beta_n \log \alpha_n$$

is either zero or transcendental. The latter case arises if $\log \alpha_1, \ldots, \log \alpha_n$ are linearly independent over $\mathbb{Q}$ and $\beta_1, \ldots, \beta_n$ are not all zero.

In particular, if $\log \alpha_1, \ldots, \log \alpha_n$ are linearly independent over $\mathbb{Q}$, then they are linearly independent over $\overline{\mathbb{Q}}$.

As an application of this result, we prove the following variant of a result from [5].

Lemma 3. *Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be positive algebraic numbers. If $c_0, c_1, \ldots, c_n$ are algebraic numbers with $c_0 \neq 0$, then*

$$c_0\pi + \sum_{j=1}^{n} c_j \log \alpha_j$$

is a transcendental number and hence non-zero.

Proof. Let S be such that $\{ \log \alpha_j : j \in S \}$ be a maximal $\mathbb{Q}$-linearly independent subset of

$$\{ \log \alpha_1, \dots, \log \alpha_n \}.$$

We write $\pi = -i \log(-1)$. We can rewrite our linear form as

$$-ic_0 \log(-1) + \sum_{j \in S} d_j \log \alpha_j,$$

for algebraic numbers d_j. By Baker's theorem, this is either zero or transcendental. The former case cannot arise if we show that the elements of

$$\{ \log(-1),\ \log \alpha_j : j \in S \}$$

are linearly independent over $\mathbb{Q}$. But this is indeed the case since the equality

$$b_0 \log(-1) + \sum_{j \in S} b_j \log \alpha_j = 0$$

for integers b_0, b_j, with j running through S, implies that

$$\prod_{j \in S} \alpha_j^{2b_j} = 1,$$

which in turn implies $b_j = 0$ for all $j \in S$ since the elements of $\{ \alpha_j : j \in S \}$ are multiplicatively independent. Consequently, $b_0 = 0$. This completes the proof. □

We will apply this result in the following way. For an arbitrary function

$$f : (\mathbb{Z}/q\mathbb{Z})^* \to \mathbb{Q},$$

we write it as the sum of an even function f_e and an odd function f_o so that

$$\sum_{n=1}^{\infty} \frac{f(n)}{n} = \sum_{n=1}^{\infty} \frac{f_o(n)}{n} + \sum_{n=1}^{\infty} \frac{f_e(n)}{n}.$$

The first sum is a multiple of π by the Chowla-Siegel theorem. Moreover, it is zero if and only if f_o is identically zero. The second sum is a linear form in logarithms of positive algebraic numbers, by the calculation of the next section. Thus, by the lemma, we are done if f_o is not identically zero. Therefore, we can assume that f is even.

4. The even case

We will now consider the case that f is even and assume that $f(q) = 0$ with q prime. We begin with a few general observations. Let us write f as a finite Fourier transform:

$$f(n) = \sum_{b \,(\mathrm{mod}\, q)} \hat{f}(b) \zeta^{bn},$$

so that by the inversion formulas,

$$\hat{f}(n) = \frac{1}{q} \sum_{b \,(\mathrm{mod}\, q)} f(b) \zeta^{-bn}.$$

Thus,

$$\sum_{n=1}^{\infty}\frac{f(n)}{n}=\sum_{n=1}^{\infty}\frac{1}{n}\left(\sum_{a\,(\mathrm{mod}\,q)}\hat{f}(a)\zeta^{an}\right)$$

$$=\sum_{a\,(\mathrm{mod}\,q)}\hat{f}(a)\left(\sum_{n=1}^{\infty}\frac{\zeta^{an}}{n}\right).$$

The inner sum is $-\log(1-\zeta^a)$. Thus,

$$\sum_{n=1}^{\infty}\frac{f(n)}{n}=-\sum_{a\,(\mathrm{mod}\,q)}\hat{f}(a)\log(1-\zeta^a) \tag{4.1}$$

and this formula is valid for general f.

Now it is clear that if f is even, then so is $\hat{f}$ and conversely. Since q is an odd prime and $\hat{f}$ is even, we can pair up a and $-a$ in the summation by noting that

$$(1-\zeta^a)(1-\zeta^{-a})=4\left(\sin\frac{\pi a}{q}\right)^2$$

to conclude

$$\sum_{n=1}^{\infty}\frac{f(n)}{n}=-\sum_{1\le a<q/2}\hat{f}(a)\log\left(4\sin^2\frac{\pi a}{q}\right). \tag{4.2}$$

Moreover, since $f(q)=0$, we have

$$0=f(q)=\sum_{b=1}^{q-1}\hat{f}(b)=2\sum_{1\le b<q/2}\hat{f}(b)$$

and we see that

$$\sum_{1\le a<q/2}\hat{f}(a)=0\,.$$

Thus,

$$\sum_{n=1}^{\infty}\frac{f(n)}{n}=-2\sum_{1\le a<q/2}\hat{f}(a)\log\left(\sin\frac{\pi a}{q}\right).$$

Consequently, we can rewrite (4.2) as

$$\sum_{n=1}^{\infty}\frac{f(n)}{n}=-2\sum_{1<a<q/2}\hat{f}(a)\log\left(\frac{\sin\pi a/q}{\sin\pi/q}\right).$$

Now (4.2) is a linear form in logarithms and one can apply Baker's theory to it. The essential point is that if q is prime, the set of numbers

$$\frac{\sin\pi a/q}{\sin\pi/q},\quad 1<a<q/2,$$

is a multiplicatively independent set of units of $\mathbb{Q}(\zeta)$ (see for example Lemma 8.1 of [7]). By Baker's theorem, the elements

$$\log\left(\frac{\sin \pi a/q}{\sin \pi/q}\right), \quad 1 < a < q/2,$$

are linearly independent over $\overline{\mathbb{Q}}$. Thus,

$$\sum_{n=1}^{\infty} \frac{f(n)}{n} = 0$$

if and only if $\hat{f}$ is identically zero. That is, if and only if f is identically zero. This completes the proof in the case f is even and $f(q) = 0$.

5. Concluding remarks

As noted at the end of section 3, this completes the solution of Chowla's problem in the case q is prime and $f(q) = 0$. In all likelihood, this was the solution Chowla had to resolve his problem. By a small variation, one can also treat the case that $f(q) \neq 0$. Indeed, consider the function g defined as $g(1) = g(2) = \cdots = g(q-1) = 1$ and $g(q) = -(q-1)$. A quick calculation shows that for $1 \leq a \leq q-1$,

$$\hat{g}(a) = \frac{1}{q}\left(\sum_{b=1}^{q-1} \zeta^{-ba} - (q-1)\right) = -1,$$

so that by formula (4.1), we infer that

$$L(1, g) = \sum_{a=1}^{q-1} \log(1 - \zeta^a)\,.$$

Since

$$\prod_{a=1}^{q-1} (1 - \zeta^a) = q,$$

we conclude that $L(1, g) = \log q$. Now let f be any function defined mod q with $f(0) \neq 0$ and suppose that $L(1, f) = 0$. Then, defining

$$F = (q-1)f + f(0)g,$$

we see that $F(0) = 0$. Thus, on the one hand,

$$L(1, F) = f(0)L(1, g) = f(0) \log q\,.$$

On the other hand, $L(1, F) = L(1, F_e) + L(1, F_o)$ with F_e and F_o the even and odd parts of F respectively. By our earlier discussion, $L(1, F_o)$ is an algebraic multiple of π and $L(1, F_e)$ is a linear form in logarithms of multiplicatively independent units in the q-th cyclotomic field. By Baker's theorem, we conclude that the logarithms of multiplicatively independent units, $\log(-1)$ and $\log q$ are linearly dependent over $\mathbb{Q}$. But this means that q can be written as a product of units, which is a contradiction. Hence, if $L(1, f) = 0$, we must have $f(0) = 0$ and this reduces to the case already considered in the previous section.

It is interesting to note that this argument can be extended to treat the general case of q composite. In fact, using Ramachandra's units along with this minor variation, it is possible to deduce a modest generalization of the Baker-Birch-Wirsing theorem with only a nominal appeal to Baker's theory to the extent used in this paper. This is the approach we used in [4]. The details are given there.

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