Ann. Sci. Math. Québec **35**, No 2, (2011), 141–152

FIBONACCI SNOWFLAKES

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Dedicated to Paulo Ribenboim.

RÉSUMÉ. Nous étudions les mots sur l'alphabet à 2 lettres $\mathcal{T} = \{\text{L}, \text{R}\}$ qui codent des chemins dans le plan discret $\mathbb{Z} \times i\mathbb{Z}$. Une récurrence permet de construire des chemins simples associés à une classe de polyominos que nous appelons *polyominos de Fibonacci* : ils possèdent une structure de type flocon de neige dont l'aire est donnée par la suite

$$1, 5, 29, 169, 985, 5741, 33461, 195025, 1136689, 6625109, 38613965, \ldots$$

tandis que leur périmètre s'exprime en terme des nombres de Fibonacci $F(3n+1)$. De plus chacun de ces polyominos pave le plan par translation.

ABSTRACT. We study words on the 2-letter alphabet $\mathcal{T} = \{\text{L}, \text{R}\}$ coding simple paths in the discrete plane $\mathbb{Z} \times i\mathbb{Z}$. A recurrence formula allows to build paths that are simple and linked with a class of polyominoes which we call *Fibonacci polyominoes*: they have a snowflake like structure whose area is given by the sequence

$$1, 5, 29, 169, 985, 5741, 33461, 195025, 1136689, 6625109, 38613965, \ldots$$

while their perimeter is expressed in terms of the Fibonacci numbers $F(3n + 1)$. In addition, each polyomino in this class tiles the plane by translation.

1. Introduction

A *path* in the square lattice, identified with $\mathbb{Z} \times i\mathbb{Z}$, is a polygonal path made of the elementary unit translations

$$i = (0, 1), \quad i^2 = (-1, 0), \quad i^3 = (0, -1), \quad i^4 = (1, 0).$$

These paths are conveniently encoded by words on the alphabet $\mathcal{E} = \{i, -i, 1, -1\}$, and we say that w is *closed* if it satisfies $|w|_i = |w|_{-i}$ and $|w|_1 = |w|_{-1}$, *i.e.*, if the two extremities of the polygonal line coincide. For instance, the words in $\mathcal{E}$ corresponding to the paths in Figure 1(a) (nonclosed) and (d) (closed) are

(a) $p_1 = 1\;1\;i\;i\;i\;1\;1\;i\;-1\;-1\;-1\;-i\;-i\;-i\;1\;1\;-i\;-i$,
(d) $p_4 = i\;i\;1\;1\;1\;-i\;-i\;1\;i\;i\;i\;i\;-1\;-1\;-i\;-i\;-i\;-i\;-i\;-1\;i\;-1$.

A *simple path* (as in (b) or (c)) is a word w such that none of its proper factors is a closed path. A *loop* (like (c)) is a nonempty simple closed path, also called *boundary*

Reçu le 28 juillet 2008 et, sous forme définitive, le 18 janvier 2010.

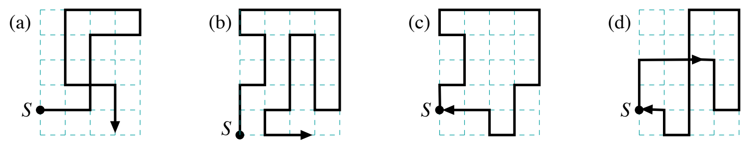

FIGURE 1. Some paths in the discrete plane $\mathbb{Z} \times i\mathbb{Z}$.

word. A loop splits the plane in two regions, where the exterior one is unbounded while the interior one is bounded and called *polyomino*, as depicted in Figure 1(c).

Another way to describe a path on the square grid is to give the sequence of directions at intersections: a path is fully determined by the starting step $\boldsymbol{\alpha} \in \mathcal{E}$ and the sequence of direction indications, that is left (L), right (R), forward (F) or backward (B) moves. In this note we consider a subclass defined on the alphabet $\mathcal{T} = \{\mathrm{L}, \mathrm{R}\}$, and in particular the class of paths obtained from words defined by the recurrence

$$q_n = \begin{cases} q_{n-1}q_{n-2} & \text{if } n \equiv 2 \mod 3, \\ q_{n-1}\overline{q_{n-2}} & \text{if } n \equiv 0, 1 \mod 3. \end{cases}$$

where $q_0 = \varepsilon$ (the empty word), $q_1 = \mathrm{R}$, and the $\overline{\cdot}$ operation consists in exchanging the letters R and L. It turns out that the subsequence q_{3n+1} enables us to construct an infinite sequence $T_F(n)$ of polyominoes, which we call *Fibonacci polyominoes*, with the following properties:

(i) The perimeter, *i.e.*, the length of its contour line, of the polyomino of order n is equal to $4F(3n + 1)$, where $F(n)$ is the Fibonacci sequence, with $F(1) = F(2) = 1$.

(ii) The area is given by the sequence (A001653 in [12])

$$1, 5, 29, 169, 985, 5741, 33461, 195025, 1136689, 6625109, 38613965, \ldots$$

defined by the recurrence

$$A(0) = 1,\ A(1) = 5; \quad A(n) = 6A(n-1) - A(n-2), \quad \text{for } n > 1,$$

this sequence being the subsequence of odd index of the Pell numbers defined by

$$P(0) = 0,\ P(1) = 1; \quad P(n) = 2P(n-1) + P(n-2), \quad \text{for } n > 1.$$

Geometry on a square grid therefore reveals yet another connection between Fibonacci and Pell numbers. Pell numbers are of course closely related to the continued fraction expansion of $\sqrt{2}$ and to the so-called *Pell equation* $X^2 - 2Y^2 = \pm 1$, for $X, Y \in \mathbb{N}$. They are described on page 55 in the book "The new book of prime number records" by Paulo Ribenboim [11], in connection with primality testing. These polyominoes are remarkable in that they tile the plane as we shall see later on.

2. Preliminaries

We restrict our study to words on the alphabet $\mathcal{T} = \{\mathrm{L}, \mathrm{R}\}$ and start by defining some useful functions We have already met the function $\overline{\cdot}$ which exchanges the letters R and L. The *reversal* $\widetilde{w}$ of $w = w_1w_2 \cdots w_n$ is the word $\widetilde{w} = w_nw_{n-1} \cdots w_1$, and words satisfying $w = \widetilde{w}$ are called *palindromes*. The words w of $\mathcal{E}^*$ or $\mathcal{T}^*$ satisfying

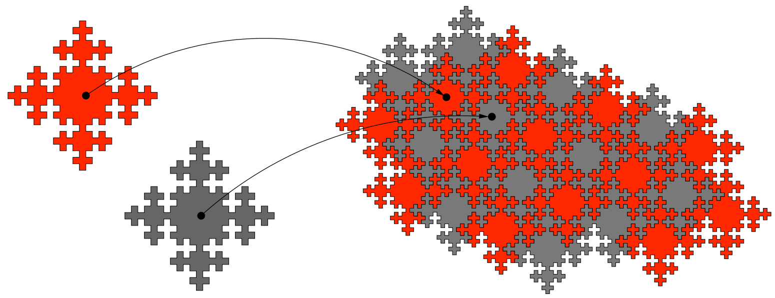

FIGURE 2. A tesselation with a Fibonacci polyomino.

$\widehat{w} = \overline{\widetilde{w}} = w$ are called *antipalindromes*. The *winding number* $\Delta : \mathcal{T}^* \to \mathbb{Z}$ is defined by $\Delta(w) = |w|_{\text{L}} - |w|_{\text{R}}$ (see [4]). The *empty word* ε satisfies $\Delta(\varepsilon) = 0$, and we also have the following trivial properties:

$$(1) \qquad \begin{cases} \Delta(uv) &= \Delta(u) + \Delta(v), \\ \Delta(w) &= \Delta(\widetilde{w}), \\ \Delta(\overline{w}) &= -\Delta(w) = \Delta(\widehat{w}), \\ \Delta(w\text{L}) &= \Delta(w) + 1, \\ \Delta(w\text{R}) &= \Delta(w) - 1. \end{cases}$$

Observe that if w is an antipalindrome, then $\Delta(w) = 0$.

Let us consider the (right) action $\mathcal{E} \times \mathcal{T}^* \to \mathcal{E}$ by setting for each $\boldsymbol{\alpha} \in \mathcal{E}$:

(i) $\boldsymbol{\alpha}\text{L} = i \cdot \boldsymbol{\alpha}, \ \boldsymbol{\alpha}\text{R} = -i \cdot \boldsymbol{\alpha},$

(ii) $\boldsymbol{\alpha}\varepsilon = \boldsymbol{\alpha},$

where $\cdot$ is the usual complex product, often omitted when operands are complex numbers. Clearly, we have $\boldsymbol{\alpha}(uv) = (\boldsymbol{\alpha}u)v$ by associativity.

Theorem 2.1. *We have $\boldsymbol{\alpha}w = i^{\Delta(w)}\boldsymbol{\alpha}$.*

Proof. By induction. Assume the claim true for w. Then

$$\boldsymbol{\alpha}w\text{L} = (\boldsymbol{\alpha}w)\text{L} = i^{\Delta(w)}\boldsymbol{\alpha}\text{L} = i^{\Delta(w)+1}\boldsymbol{\alpha} = i^{\Delta(w\text{L})}\boldsymbol{\alpha},$$

and the result is true for $w\text{L}$. Similarly for $w\text{R}$.

The next property is obvious and describes the action of $\overline{w}$ on $\boldsymbol{\alpha}$. It follows from the fact that $\boldsymbol{\alpha}w$ and $\boldsymbol{\alpha}\overline{w}$ are symmetric with respect to the axis defined by $\boldsymbol{\alpha}$.

Corollary 2.2. *Let $w \in \mathcal{T}^*$ and $\boldsymbol{\alpha} \in \mathcal{E}$. Then,*

$$\boldsymbol{\alpha}\overline{w} = \begin{cases} \ \ \boldsymbol{\alpha}w & \text{if } \Delta(w) \equiv 0 \bmod 2, \\ -\boldsymbol{\alpha}w & \text{otherwise.} \end{cases}$$

To each pair $(\boldsymbol{\alpha}, w) \in \mathcal{E} \times \mathcal{T}^*$, corresponds a polygonal path Γ whose first side is $\boldsymbol{\alpha}$ and whose subsequent sides are obtained by reading the instructions L or R of w.

The sequence $z_0 = 0, z_1 = \boldsymbol{\alpha}, \ldots, z_{|w|+1}$ of vertices of Γ is computed by

$$z_{\ell+1} = \boldsymbol{\alpha} \sum_{k=1}^{\ell} i^{\Delta(w[1,k])}, \quad \text{for } 0 \leq \ell \leq |w|,$$

where $w[1,k] = w_1 w_2 \cdots w_k$, denotes the prefix of length k of w (by convention $w[1,0] = \varepsilon$). For sake of simplicity, we denote $\overrightarrow{\boldsymbol{\alpha} w}$ the vector $\overrightarrow{z_0, z_{|w|+1}}$. Notice that Δ trivially extends to paths. Each closed path is identified with the word w, dropping the starting step $\boldsymbol{\alpha}$. By abuse of terminology we often identify w with a path, and if we want to distinguish, we say that w *represents* some path y. The composition $\widehat{\cdot} = \overline{\cdot} \circ \widetilde{\cdot}$ is interpreted as follows: if $w \in \mathcal{T}^*$ is a path, then $\widehat{w}$ is the reverse path which runs backwards. We recall from [4] a property of closed paths.

Lemma 2.3. *If a path $w \in \mathcal{T}^*$ is closed then $\Delta(w) \equiv 0 \bmod 3$.*

Given a nonempty word $w = w_1 w_2 \cdots w_n \in \mathcal{T}^*$, let w- denote the word obtained by removing the last letter, that is, $w\text{-} = w_1 w_2 \cdots w_{n-1}$.

Theorem 2.4. *Let $w \in \mathcal{T}^*$ and $\boldsymbol{\alpha} \in \mathcal{E}$. The following properties hold:*

(i) *If $\boldsymbol{\alpha} w = \pm i \boldsymbol{\alpha}$ then $w^3 w$- is a closed path.*

(ii) *If $\boldsymbol{\alpha} w = -\boldsymbol{\alpha}$ then ww- is a closed path.*

(iii) *If $\boldsymbol{\alpha} w = \boldsymbol{\alpha}$ then either w- is a closed path, so that w^n is bounded as $n \to \infty$, or w- is open and $|\overrightarrow{\boldsymbol{\alpha} w^n}| = cn$.*

Proof. (i) In this case, the initial step $\boldsymbol{\alpha}$ is rotated by an angle of $\pm\pi/2$, illustrated in Figure 3 with an angle of $+\pi/2$ counterclockwise. Taking four copies of w corre-

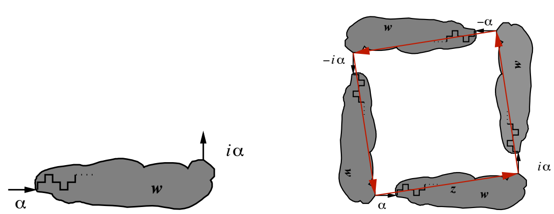

FIGURE 3. Case (i) with a $+\pi/2$ angle.

sponds to a closed polygonal path where the first side $\boldsymbol{\alpha}$ corresponds to the last side $\boldsymbol{\alpha}$. This is illustrated with an angle $+\pi/2$ in the figure on the right. So that the computation of $\boldsymbol{\alpha} w^3 w$- amounts to compute $\mathbf{z} + i\mathbf{z} + i^2\mathbf{z} + i^3\mathbf{z} = 0$, where $\mathbf{z} = \overrightarrow{\boldsymbol{\alpha} w\text{-}}$.

Property (ii) is similar with a $\pm\pi$ rotation. As for (iii), if w- is not closed, then

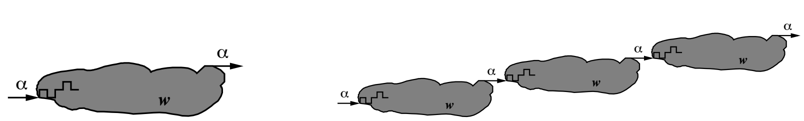

FIGURE 4. Case (iii).

obviously, $\boldsymbol{\alpha} w = \boldsymbol{\alpha}$ represents a nontrivial translation of $\boldsymbol{\alpha}$, and therefore the two endpoints of the polygonal path corresponding to $\boldsymbol{\alpha} w^n$ are cn apart for some constant $c > 0$, as shown in Figure 4. Finally, if w- is closed, then clearly the polygonal path corresponding to w^n is bounded. ✥

The above proof in case (i) uses the rotation among the isometric transformations. As we shall see later these transformations are useful for describing our Fibonacci polyominoes. In particular we have the following properties.

Lemma 2.5. *Let $w \in \mathcal{T}^*, \boldsymbol{\alpha} \in \mathcal{E}$ and $y \in \boldsymbol{\alpha}\mathcal{E}^*$ such that $y = (\boldsymbol{\alpha}, w)$. Then the following statements are equivalent:*

(i) $\widehat{y} = (i^2, w)$.

(ii) y *is a palindrome.*

(iii) w *is an antipalindrome.*

Lemma 2.6. *Let $w \in \mathcal{T}^*$ and Γ its corresponding polygonal line. Let $\boldsymbol{\alpha} \in \mathcal{E}$ and M be the midpoint of the vector $\mathbf{z} = \overrightarrow{\boldsymbol{\alpha} w}$.*

(i) w *is a palindrome if and only if the perpendicular to $\overrightarrow{\boldsymbol{\alpha} w}$ at M is a symmetry axis for Γ.*

(ii) w *is a antipalindrome if and only if Γ is symmetric with respect to M.*

Observe that if w is an antipalindrome then the algebraic area between $\mathbf{z}$ and Γ is null.

3. Fibonacci snowflakes

Recall that the sequence $(q_n)_{n\in\mathbb{N}}$ in $\mathcal{T}^*$ is defined by $q_0 = \varepsilon$, $q_1 = \mathtt{R}$ and

$$q_n = \begin{cases} q_{n-1}q_{n-2} & \text{if } n \equiv 2 \mod 3, \\ q_{n-1}\overline{q_{n-2}} & \text{if } n \equiv 0, 1 \mod 3, \end{cases}$$

whenever $n \geq 2$. The first terms of $(q_n)_{n\in\mathbb{N}}$ are

$q_0 = \varepsilon$	$q_3 =$ RL	$q_6 =$ RLLRLLRR
$q_1 =$ R	$q_4 =$ RLL	$q_7 =$ RLLRLLRRLRRLR
$q_2 =$ R	$q_5 =$ RLLRL	$q_8 =$ RLLRLLRRLRRLRRLLRLLRR .

Note that $|q_n| = F_n$ is the n-th Fibonacci number. Moreover, the paths q_n present strong symmetry properties, as shown by the next lemma.

Proposition 3.1. *Let $n \in \mathbb{N}$. There exist an antipalindrome t, two palindromes p, r and a letter $a \in \{\mathtt{L}, \mathtt{R}\}$ such that $q_{3n+1} = ta$, $q_{3n+2} = pa$ and $q_{3n+3} = r\overline{a}$.*

Proof. By induction on n. For $n = 0$, we have indeed $q_1 = \varepsilon \cdot \mathtt{R}$, $q_2 = \varepsilon \cdot \mathtt{R}$ and $q_3 = \mathtt{R} \cdot \mathtt{L}$. Now, assume that $q_{3n+1} = ta$, $q_{3n+2} = pa$ and $q_{3n+3} = r\overline{a}$ for some antipalindrome t, some palindromes p, r and some letter $a \in \{\mathrm{L}, \mathrm{R}\}$. Then

$$\begin{aligned} q_{3n+4} &= q_{3n+3}\overline{q_{3n+2}} = q_{3n+2}\overline{q_{3n+1}}\,\overline{q_{3n+2}} = pa\overline{t}\overline{a}\overline{p} \cdot \overline{a} \\ q_{3n+5} &= q_{3n+4}q_{3n+3} = q_{3n+3}\overline{q_{3n+2}}q_{3n+3} = r\overline{a}\overline{p}\overline{a}r \cdot \overline{a} \\ q_{3n+6} &= q_{3n+5}\overline{q_{3n+4}} = q_{3n+4}q_{3n+3}\overline{q_{3n+4}} = pa\overline{t}\overline{a}\overline{p}\overline{a}r\overline{a}\overline{p}\overline{a}tap \cdot a. \end{aligned} \tag{2}$$

Since $pa\overline{t}\overline{a}\overline{p}$ is an antipalindrome and $r\overline{a}\overline{p}\overline{a}r$, $pa\overline{t}\overline{a}\overline{q}\overline{a}r\overline{a}\overline{p}\overline{a}tap$ are palindromes, the result follows. ∎

The proof above also shows that the winding number can take only three values.

Corollary 3.2. *Let $n \in \mathbb{N}$. Then we have*

(i) $\Delta(q_{3n}) = 0$,

(ii) $\Delta(q_{3n+1}) = \Delta(q_{3n+2}) = (-1)^{n+1}$.

Proof. (i) By definition we have $q_{3n+3} = q_{3n+1}q_{3n} \cdot \overline{q_{3n+1}}$. Equations (1) imply that $\Delta(q_{3n+3}) = \Delta(q_{3n})$, and the claim follows by induction.

(ii) By definition we have $q_{3n+2} = q_{3n+1}.q_{3n}$ so that $\Delta(q_{3n+2}) = \Delta(q_{3n+1})$ by what precedes. Since $q_{3n+1} = ta$ for some antipalindrome t, it follows from Equations (1) that

$$\Delta(q_{3n+1}) = \Delta(ta) = \Delta(a) = \pm 1.$$

Finally, Equation (2) shows that $\Delta(q_{3n+1}) = -\Delta(q_{3n+4})$, and $\Delta(q_4) = -1$ permits to conclude the proof. ∎

We first show that the coordinates of the vector $\overrightarrow{\boldsymbol{\alpha} q_n}$ are expressed in terms of Pell numbers.

Lemma 3.3. *Let $\boldsymbol{\alpha} \in \mathcal{E}$. Then for all $n \in \mathbb{N}$, we have*

$$\begin{cases} \overrightarrow{\boldsymbol{\alpha} q_{3n}} &= \boldsymbol{\alpha} \cdot (P(n), (-1)^n P(n)) \\ \overrightarrow{\boldsymbol{\alpha} q_{3n+1}} &= \boldsymbol{\alpha} \cdot (P(n+1), (-1)^n P(n)) \\ \overrightarrow{\boldsymbol{\alpha} q_{3n+2}} &= \boldsymbol{\alpha} \cdot (P(n) + P(n+1), 0) \end{cases}$$

where $\cdot$ is the usual complex product.

Proof. By induction on n. For $n = 0$, we have $\overrightarrow{\boldsymbol{\alpha} q_0} = (0,0)$, $\overrightarrow{\boldsymbol{\alpha} q_1} = \boldsymbol{\alpha} \cdot (1,0)$ and $\overrightarrow{\boldsymbol{\alpha} q_2} = \boldsymbol{\alpha} \cdot (1,0)$. Assume that the claim is true for $k = 0, 1, 2, \ldots, 3n+3$. Then, since $q_{3n+4} = q_{3n+3}\overline{q_{3n+2}}$ we have, by Corollary 3.2(i) that $\Delta(q_{3n+3}) = 0$, so that the action of q_{3n+3} leaves the direction $\boldsymbol{\alpha}$ unchanged. Passing to vectors we have

$$\begin{aligned} \overrightarrow{\boldsymbol{\alpha} q_{3n+4}} &= \overrightarrow{\boldsymbol{\alpha} q_{3n+3}} + \overrightarrow{\boldsymbol{\alpha} \overline{q_{3n+2}}} \\ &= \boldsymbol{\alpha} \cdot \left(\overrightarrow{1 q_{3n+3}} + \overline{\overrightarrow{1 q_{3n+2}}} \right) \\ &= \boldsymbol{\alpha} \cdot \left((P(n+1), (-1)^{n+1} P(n+1)) + \overline{(P(n) + P(n+1), 0)} \right) \\ &= \boldsymbol{\alpha} \cdot \left(2P(n+1) + P(n), (-1)^{n+1} P(n+1) \right) \\ &= \boldsymbol{\alpha} \cdot \left(P(n+2), (-1)^{n+1} P(n+1) \right). \end{aligned}$$

The two other cases are similar and are left to the reader. ✣

The first values of endpoint coordinates of the paths $q_{\overline{n}}$ starting from the origin with initial step $\boldsymbol{\alpha} = (1, 0)$ are :

n	0	1	2	3	4	5
$q_{\overline{3n}}$	$(0,0)$	$(1,-1)$	$(2,2)$	$(5,-5)$	$(12,12)$	$(29,-29)$
$q_{\overline{3n+1}}$	$(1,0)$	$(2,-1)$	$(5,2)$	$(12,-5)$	$(29,12)$	$(70,-29)$
$q_{\overline{3n+2}}$	$(1,0)$	$(3,0)$	$(7,0)$	$(17,0)$	$(41,0)$	$(99,0)$

Lemma 3.3 is illustrated in Figure 5 below, where Q_n denotes the smallest rectangle box containing $q_{\overline{n}}$. This leads to the geometric construction of the paths q_n, using the

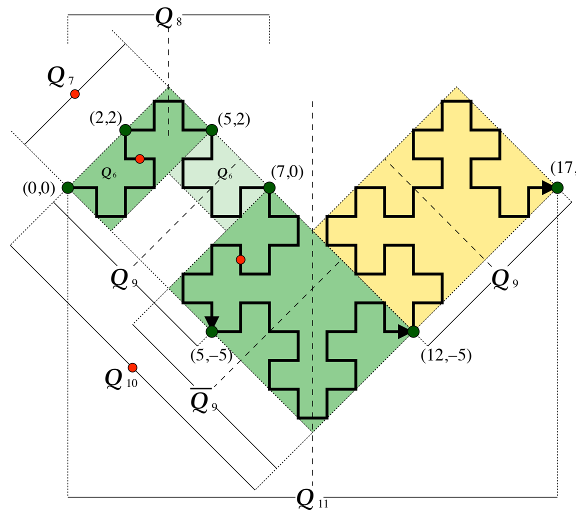

FIGURE 5. The geometric construction of the Fibonacci paths.

symmetries deduced from Proposition 3.1 and Lemma 2.6,

$$Q_n = \begin{cases} Q_{n-1}Q_{n-2} & \text{if } n \equiv 2 \mod 3, \\ Q_{n-1}\overline{Q_{n-2}} & \text{if } n \equiv 0, 1 \mod 3, \end{cases}$$

where the sides of the box are determined by the origin and the endpoint of $q_{\overline{n}}$. Clearly each box is contained in the next one (by prefix property), and the sides have a $\pm\frac{\pi}{4}$ angle since Q_{3n} is so. Observe also that this implies that Q_{3n+2} has a vertical symmetry axis for every n given by the line $z = \frac{P(n)+P(n+1)}{2}$.

Theorem 3.4. *For all $n \in \mathbb{N}$, the paths q_n satisfy the following properties:*

(i) $(q_{3n+1})^3 q_{\overline{3n+1}}$ *is closed;*

(ii) $(q_{3n+2})^3 q_{\overline{3n+2}}$ *is closed;*

(iii) q_n *is non-intersecting;*

(iv) $(q_{3n+1})^3 q_{\overline{3n+1}}$ *is non-intersecting.*

Proof. (i) Since $\Delta(ta) = \Delta(a) = \pm 1$, we are in case (i) of Theorem 2.4 and the result follows.

(ii) As in (i).

(iii) Since by definition q_n is a prefix of q_{n+1}, it suffices to prove that q_{3n} is non-intersecting for all $n \in \mathbb{N}$. First, we show by induction that

$$q_{\bar{3}n} = q_{3n-1}\overline{q_{\bar{3}n-2}} = q_{3n-2}q_{\bar{3}n-1}. \tag{3}$$

Indeed, it is clearly true for $n = 1$. Then, we have

$$\begin{aligned} q_{\bar{3}n+3} = q_{3n+2}\overline{q_{\bar{3}n+1}} &= q_{3n+1}q_{3n} \cdot \overline{q_{\bar{3}n+1}} = q_{3n+1}q_{3n} \cdot \overline{q_{\bar{3}n}}q_{\bar{3}n-1} \\ &= q_{3n+1}q_{3n} \cdot \overline{q_{\bar{3}n-1}}q_{3n-2} \cdot q_{\bar{3}n-1} \\ &= q_{3n+1} \cdot q_{3n}\overline{q_{\bar{3}n-1}} \cdot q_{3n-2}q_{\bar{3}n-1} \\ &= q_{3n+1} \cdot q_{3n+1} \cdot q_{\bar{3}n} \qquad \text{(by induction hypothesis)} \\ &= q_{3n+1} \cdot q_{\bar{3}n+2} \qquad \text{(by definition).} \end{aligned}$$

We finish the proof by induction. Assume that $q_{\bar{3}n-1}$ is non-intersecting, which implies that $q_{\bar{3}n-k}$ is also non-intersecting for all $k \in \mathbb{N}$ such that $1 \leq k \leq 3n-1$. Passing to boxes, we have from Equation (3)

$$Q_{3n} = Q_{3n-1}\overline{Q_{3n-2}} = Q_{3n-2}Q_{3n-1},$$

where there is no intersection in neither left or right box Q_{3n-1}.

(iv) Theorem 2.4 implies that $(q_{3n+1})^3 q_{\bar{3}n+1}$ is non-intersecting if $(q_{3n+1})^2$ is non-intersecting. But we have

$$\begin{aligned} (q_{3n+1})^2 &= q_{3n}\overline{q_{3n-1}} \cdot q_{3n}\overline{q_{3n-1}} \\ &= q_{3n} \cdot \overline{q_{3n-1}}q_{3n-1} \cdot \overline{q_{3n-2}}\,\overline{q_{3n-1}}, \end{aligned}$$

and since $\overline{q_{3n-1}}q_{3n-1}$ is a factor of $\overline{q_{3n+2}}$, a non-intersecting path by (iii), the claim follows. ✣

A Fibonacci snowflake or *tile of order* n is a polyomino $T_F(n)$ represented by the word $\phi_n = (q_{3n+1})^3 q_{\bar{3}n+1}$, where $n \in \mathbb{N}$. The first Fibonacci snowflakes are shown in Figure 6.

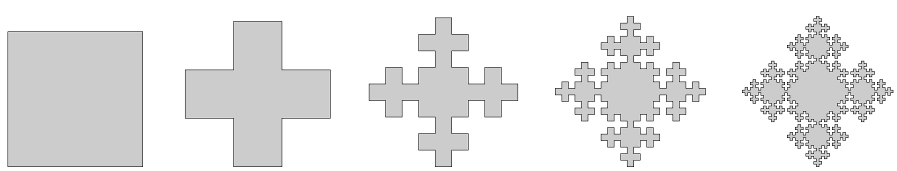

FIGURE 6. Fibonacci snowflakes of order $n = 0, 1, 2, 3, 4$.

We have already mentioned that the perimeter $L(n)$ of the snowflake of rank n is

$$L(n) = 4F(3n+1) = \frac{4}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{3n+1} - \frac{4}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{3n+1}.$$

Let $(A(n))_{n\in\mathbb{N}}$ be the sequence of areas of the Fibonacci snowflakes. Its first values are

$$1, 5, 29, 169, 985, 5741, 33461, 195025, 1136689, 6625109, 38613965, \ldots,$$

which we shall prove to be the odd index numbers in the Pell sequence whose first values are

$$0, 1, 2, 5, 12, 29, 70, 169, 408, 985, 2378, 5741, 13860, 33461, 80782, 195025, \ldots$$

Recall that the sequence of Pell numbers satisfies the following identity [2]

$$P(2n+1) = P(n+1)^2 + P(n)^2. \tag{4}$$

We are now ready to prove our result on the area of the Fibonacci snowflakes.

Theorem 3.5. *We have* $A(n) = P(2n+1)$.

Proof. The parallelogram determined by the word $(q_{3n+1})^3 q_{\overline{3n+1}}$ is a square (Theorem 2.4(i)), and by Lemma 2.6(ii) the area of $T_F(n)$ is equal to the area of the square determined by

$$\overrightarrow{\boldsymbol{\alpha} q_{\overline{3n+1}}} = \big(P(n+1), \pm P(n)\big).$$

Hence $A(n) = P(n+1)^2 + P(n)^2$ which is equal to $P(2n+1)$ by Equation (4). ✠

It follows that $A(n)$ satisfies the recurrence formulas

$$A(0) = 1,\ A(1) = 5; \quad A(n) = 6A(n-1) - A(n-2), \quad \text{for } n > 1.$$

and that its general term is

$$A(n) = \frac{\sqrt{2}+1}{2\sqrt{2}}\left(3+2\sqrt{2}\right)^n + \frac{\sqrt{2}-1}{2\sqrt{2}}\left(3-2\sqrt{2}\right)^n.$$

This sequence is already known as defining the numbers n such that $2n^2-1$ is a square (see A001653 in [12]) but also for counting diverse combinatorial objects.

Fibonacci snowflakes reveal a new connection between the Fibonacci numbers of index $3n+1$ and odd-index Pell numbers. It is worth noticing that the Markoff numbers also link the two sequences but in a different way: indeed, all odd-index Pell numbers and all odd-index Fibonacci numbers are particular Markoff numbers, a fact already observed by Frobenius [7]. The reader is refered to [5, 10] for recent developments in that direction.

Other connections may be found in combinatorics. For instance, the number of two-stack sortable permutations which avoid the pattern 132 is the Pell number P_n, while the number of two-stack sortable permutations avoiding both 132 and 4123 is $F_{n+4} - 2n - 2$ [6].

A measure of the complexity of the snowflakes. We compute now the limit ratio $\lim_{n\to\infty} \frac{A(n)}{H(n)}$ between the area of the Fibonacci snowflake and the area of its convex hull, which is half of the area $S(n)$ of the smallest square having sides parallel to the axes and containing it. The squared tiles are shown in Figure 7 for $n = 2, 3$, where $(A, B) = (P(n+1), (-1)^n P(n))$ are the coordinates of $\overrightarrow{\boldsymbol{\alpha} q_{\overline{3n+1}}}$ given by Lemma 3.3. It is easy to compute $S(n)$:

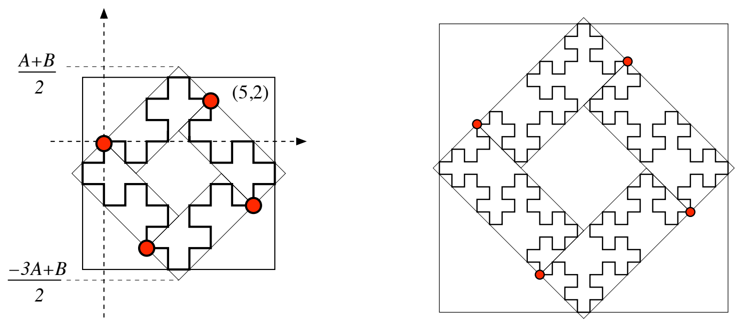

FIGURE 7. Squared Fibonacci snowflakes of order $n = 2, 3, 4$.

$$S(n) = \left(\frac{A+B}{2} - \frac{-3A+B}{2} - 1\right)^2 = (2A-1)^2 = \big(2P(n+1)-1\big)^2.$$

Then the limit ratio is computed as follows.

$$\begin{aligned}
\lim_{n\to\infty} \frac{A(n)}{H(n)} &= 2 \lim_{n\to\infty} \frac{P(2n+1)}{(2P(n+1)-1)^2} \\
&= 2 \lim_{n\to\infty} \frac{P(n+1)^2 + P(n)^2}{(2P(n+1)-1)^2} && \text{(by Equation (4))} \\
&= 2 \lim_{n\to\infty} \frac{P(n+1)^2 + P(n)^2}{4P(n+1)^2 - 4P(n+1) + 1} \\
&= \frac{2}{4}\left(1 + \left(\lim_{n\to\infty} \frac{P(n)}{P(n+1)}\right)^2\right) \\
&= 2 - \sqrt{2} \approx 0.585786438.
\end{aligned}$$

On the other hand, the value of the ratio

$$\lim_{n\to\infty} \frac{\log(A(n))}{\log(L(n))} \approx 0.964825683\ldots$$

seems to be linked, loosely speaking, to the complexity of the shape of the boundary of a given nonempty interior set. For a square it equals 2, while for a long and thin rectangle it is practically 1. In the present case the value $0.96482\ldots$ is a strong hint towards the complicated behavior of our snowflakes.

4. Last remarks

Fibonacci snowflakes are somehow related to the Fibonacci fractals found in [8]. They also form a subclass of tiles used for tiling the plane. Indeed, each Fibonacci tile may be used for tiling the whole plane with translated copies of it, as shown in Figure 2 and 9. More precisely, there exist a class of tiles, called *pseudo-squares*, characterized by the equation (due to Beauquier and Nivat [1])

$$\mathbf{b}(P) \equiv A \cdot B \cdot \widehat{A} \cdot \widehat{B},$$

where $\mathbf{b}(P) \in \mathcal{T}^*$ is a simple closed path. Each such factorization defines the homologous sides of the tile, and hence the way to assemble them.

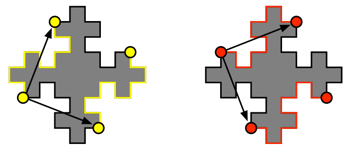

FIGURE 8. Two distinct factorizations.

For instance, the tile $T_F(2)$ in Figure 8 may be assembled as shown on the right. It was conjectured in [9] that a pseudo-square polyomino has at most two distinct nontrivial factorizations, and the Fibonacci tiles are an infinite family having this property [3], providing two distinct tesselations as shown in Figure 9.

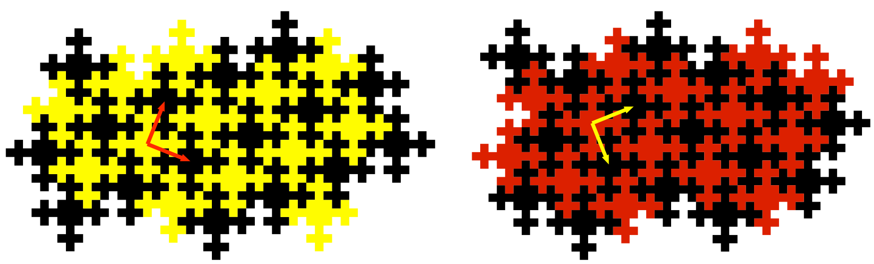

FIGURE 9. Distinct tesselations with $T_F(2)$.

Acknowledgements. The research of S. Brlek was funded by an NSERC (Canada) grant. He also benefited from the support of CNRS (France) during his sabbatical stay at the LaBRI laboratory in Bordeaux. Both A. Blondin Massé and S. Labbé were supported by an Alexander Graham Bell Canada Graduate NSERC scholarship and benefited form the "Programme Frontenac de bourses à la mobilité" (FQRNT, Québec) and French Governement.

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