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L-VALUES FOR BIQUADRATIC EXTENSIONS AND THE FITTING IDEAL OF THE TAME KERNEL

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Dedicated to Paulo Ribenboim on the occasion of his 80th birthday.

RÉSUMÉ. Soit $\mathcal{E}/F$ une extension de Galois totalement réelle de corps de nombres telle que le groupe de Galois G est isomorphe au groupe de Klein d'ordre 4 et supposons que la conjecture de Birch-Tate est vérifiée pour tous les corps compris entre F et $\mathcal{E}$, inclusivement. Soit S un ensemble fini de premiers de F contenant les premiers infinis et tous ceux qui se ramifient dans $\mathcal{E}$, soit $S_{\mathcal{E}}$ l'ensemble des premiers de $\mathcal{E}$ au-dessus de ceux de S, et soit $\mathcal{O}_{\mathcal{E}}^S$ l'anneau des $S_{\mathcal{E}}$-entiers de $\mathcal{E}$. Lorsque $\mathcal{E}$ peut être judicieusement plongé dans des extensions diédrales de F, nous montrons que l'idéal de Fitting de $K_2(\mathcal{O}_{\mathcal{E}}^S)$ et un certain idéal de Stickelberger supérieur dans $\mathbb{Z}[G]$ sont tous les deux d'indice un ou deux dans leur somme.

ABSTRACT. Fix a Galois extension $\mathcal{E}/F$ of totally real number fields such that the Galois group G is isomorphic to the Klein four group and assume that the Birch-Tate conjecture holds in all the intermediate fields between F and $\mathcal{E}$, inclusive. Let S be a finite set of primes of F containing the infinite primes and all those which ramify in $\mathcal{E}$, let $S_{\mathcal{E}}$ denote the primes of $\mathcal{E}$ lying above those in S, and let $\mathcal{O}_{\mathcal{E}}^S$ denote the ring of $S_{\mathcal{E}}$-integers of $\mathcal{E}$. When $\mathcal{E}$ can be embedded in dihedral extensions of F in certain ways, we show that the Fitting ideal of $K_2(\mathcal{O}_{\mathcal{E}}^S)$ and a higher Stickelberger ideal in $\mathbb{Z}[G]$ both have index one or two in their sum.

1. Introduction

Fix an abelian Galois extension of number fields $\mathcal{E}/F$ and let G denote the Galois group. Also fix a finite set S of primes of F which contains all of the infinite primes of F and all of the primes which ramify in $\mathcal{E}$. Associated with this data is a Stickelberger function, $\theta_{\mathcal{E}/F}^S(s)$, a meromorphic function of s with values in the group ring $\mathbb{C}[G]$. It can be defined when the real part of s is greater than 1, as a product over the (finite) primes $\mathfrak{p}$ of F that are not in S. Let $\mathrm{N}\mathfrak{p}$ denote the absolute norm of the ideal $\mathfrak{p}$ and $\sigma_{\mathfrak{p}} \in G$ denote the Frobenius automorphism of $\mathfrak{p}$. Then

$$\theta_{\mathcal{E}/F}^S(s) = \prod_{\text{prime } \mathfrak{p} \notin S} \left(1 - \frac{1}{\mathrm{N}\mathfrak{p}^s} \sigma_{\mathfrak{p}}^{-1}\right)^{-1}.$$

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This extends meromorphically to all of $\mathbb{C}$. When $\mathcal{E} = F$, the function $\theta^S_{F/F}(s)$ is simply the identity automorphism of F times $\zeta^S_F(s)$, the Dedekind zeta-function of F with Euler factors for the primes in S removed.

The function $\theta^S_{\mathcal{E}/F}(s)$ is connected with the arithmetic of the number fields $\mathcal{E}$ and F in ways one would like to make as precise as possible. The ring of S-integers $\mathcal{O}^S_F$ of F is defined to be the set of elements of F whose valuation is non-negative at every prime not in S. Similarly, define the ring $\mathcal{O}^S_{\mathcal{E}}$ of S-integers of $\mathcal{E}$ to be the set of elements of $\mathcal{E}$ whose valuation is non-negative at every prime not in $S_{\mathcal{E}}$, the set of all primes of $\mathcal{E}$ which lie above some prime in S. The function $\zeta^S_F(s)$ may be viewed as the zeta-function of the Dedekind domain $\mathcal{O}^S_F$.

We will study the "higher Stickelberger element" $\theta^S_{\mathcal{E}/F}(-1)$, which lies in $\mathbb{Q}[G]$ by the theorem of Klingen-Siegel [11], and is related to the algebraic K-group $K_2(\mathcal{O}^S_{\mathcal{E}})$. Denoting the valuation at a finite prime $\mathfrak{p}$ of $\mathcal{E}$ by $v_\mathfrak{p}$, the group $K_2(\mathcal{O}^S_{\mathcal{E}})$ may be described as the subgroup of $K_2(\mathcal{E})$ consisting of all elements whose image under the map which sends $\{\gamma, \alpha\}_{\mathcal{E}}$ to the tame symbol

$$(\gamma, \alpha)_\mathfrak{p} = -1^{v_\mathfrak{p}(\gamma)v_\mathfrak{p}(\alpha)}\gamma^{v_\mathfrak{p}(\alpha)}/\alpha^{v_\mathfrak{p}(\gamma)} \pmod{\mathfrak{p}}$$

is trivial in the residue field modulo $\mathfrak{p}$ for every prime $\mathfrak{p}$ not in $S_{\mathcal{E}}$. This group $K_2(\mathcal{O}^S_{\mathcal{E}})$ is known to be finite by [2] and [7], and will be called the *S-tame kernel of $\mathcal{E}$*. It contains the tame kernel $K_2(\mathcal{O}_{\mathcal{E}})$ as a subgroup.

Another piece of the arithmetic interpretation of $\theta^S_{\mathcal{E}/F}(-1)$ involves a group of roots of unity. Let μ_∞ denote the group of all roots of unity in an algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$ containing $\mathcal{E}$, and let $\mathcal{G}$ denote the Galois group of $\overline{\mathbb{Q}}/\mathbb{Q}$. Define $W_2 = W_2(\overline{\mathbb{Q}})$ to be the $\mathbb{Z}[\mathcal{G}]$-module whose underlying group is μ_∞, with the action of $\gamma \in \mathcal{G}$ on $\omega \in W_2$ given by $\omega^\gamma = \gamma^2(\omega)$. For any subfield L of $\overline{\mathbb{Q}}$, let $W_2(L)$ be the submodule fixed under this action by the Galois group of $\overline{\mathbb{Q}}$ over L. Then $W_2(\mathcal{E})$ naturally becomes a $\mathbb{Z}[G]$-module, where the action of G arises by lifting elements of G to $\mathcal{G}$ and then using the action of $\mathcal{G}$ just defined. One easily sees that the G-fixed submodule $W_2(\mathcal{E})^G$ equals $W_2(F)$. We use the notation $w_2(L) = |W_2(L)|$, which we note is finite for any algebraic number field L.

Our approach makes use of the conjecture of Birch and Tate (see Section 4 in [12]), which gives a precise arithmetic interpretation of $\zeta^S_F(-1)$. We state a form of it for an arbitrary finite set S which is easily seen to be equivalent to the original conjecture for the set S containing just the infinite primes (see Corollary 3.3 of [10]).

Conjecture 1.1 (Birch-Tate). *Suppose that F is totally real and that the finite set S contains the infinite primes of F. Then*

$$\zeta^S_F(-1) = (-1)^{|S|}\frac{|K_2(\mathcal{O}^S_F)|}{w_2(F)}.$$

Results on Iwasawa's main conjecture in [6] and [14] lead to the following (see [4]).

Proposition 1.2. *The Birch-Tate conjecture holds if F is abelian over $\mathbb{Q}$, and the odd part holds for all totally real F.*

Kolster [3] has shown that the 2-part of the Birch-Tate conjecture for F would follow from the 2-part of Iwasawa's Main conjecture for F.

For any module M over a commutative ring A, we let $\mathrm{Ann}_A(M)$ denote the annihilator of M in A. If M is a finitely generated A-module, we denote the *Fitting ideal* of M over A by $\mathrm{Fit}_A(M)$; it is the ideal of A generated by the determinants of all square matrices representing relations among a set of generators of A. The following result is proved in Theorem 1.3 of [8]. The notation $\overline{G}$ for the Galois group reflects that in later applications this group will in fact be a homomorphic image of a group G.

Proposition 1.3. *Let E/F be a relative quadratic extension of totally real number fields, with Galois group $\overline{G}$. Let S contain the infinite primes and those which ramify in E/F. Assume that the 2-part of the Birch-Tate conjecture holds for E and for F. Then the (first) Fitting ideal of $K_2(\mathcal{O}_E^S)$ as a $\mathbb{Z}[\overline{G}]$-module is*

$$\mathrm{Fit}_{\mathbb{Z}[\overline{G}]}(K_2(\mathcal{O}_E^S)) = \mathrm{Ann}_{\mathbb{Z}[\overline{G}]}(W_2(E))\theta_{E/F}^S(-1).$$

More specifically, this ideal equals its extension to the maximal order of $\mathbb{Q}[\overline{G}]$ if and only if it is not principal, and this happens exactly when E is not the first layer of the cyclotomic $\mathbb{Z}_2$-extension of F. Without the assumption of the Birch-Tate conjecture, the ideals $\mathrm{Fit}_{\mathbb{Z}[\overline{G}]}(K_2(\mathcal{O}_E^S))$ and $\mathrm{Ann}_{\mathbb{Z}[\overline{G}]}(W_2(E))\theta_{E/F}^S(-1)$ have the same extension to $\mathbb{Z}[1/2][\overline{G}]$.

In this paper, we investigate the relationship between the Fitting ideal

$$\mathrm{Fit}_{\mathcal{E}/F}^S(1) = \mathrm{Fit}_{\mathbb{Z}[G]}(K_2(\mathcal{O}_{\mathcal{E}}^S))$$

and the corresponding higher Stickelberger ideal

$$\mathrm{Stick}_{\mathcal{E}/F}^S(-1) = \mathrm{Ann}_{\mathbb{Z}[G]}(W_2(\mathcal{E}))\theta_{\mathcal{E}/F}^S(-1)$$

when G is the Klein four group. The theorem of Deligne and Ribet [1] guarantees that $\mathrm{Stick}_{\mathcal{E}/F}^S(-1)$ is an ideal in the integral group ring $\mathbb{Z}[G]$.

2. Biquadratic extensions

From now on, we let $\mathcal{E}/F$ be a biquadratic extension of totally real number fields with intermediate fields E_1, E_2 and E_3 and assume that $\mathcal{E}$ is contained in $\mathbb{R}$.

One particular quadratic extension of F plays a special role, namely the first layer $F^{(1)}$ of the cyclotomic $\mathbb{Z}_2$-extension of F. This extension may be described by setting $\zeta_k = e^{2\pi i/k}$ and choosing the largest positive integer n such that $\pi_F = 2 + \zeta_{2^n} + \zeta_{2^n}^{-1}$ lies in F. Then $F^{(1)} = F\left(\sqrt{\pi_F}\right)$, and $\sqrt{\pi_F} = \zeta_{2^{n+1}} + \zeta_{2^{n+1}}^{-1}$, whose absolute norm is a power of 2.

The following two results appear in Theorem 4.5, Theorem 4.8 and Proposition 4.9 in [9]. Our goal will be to obtain new general applications of these. Here we let e_0 be the idempotent in $\mathbb{Q}[G]$ corresponding to the trivial character of G, and let e_i be the idempotent corresponding to the non-trivial character whose kernel fixes E_i. Then $\mathcal{S} = \mathbb{Z}e_0 + \mathbb{Z}e_1 + \mathbb{Z}e_2 + \mathbb{Z}e_3$ is the maximal order of $\mathbb{Q}[G]$. It is easy to compute that $4\mathcal{S}$ has index 16 in $\mathbb{Z}[G]$, and hence $\mathbb{Z}[G]$ has index 16 in $\mathcal{S}$. The symbol $k_2^S(F)$ will denote the order of $K_2(\mathcal{O}_F^S)$, and $k_2^S(E_i)^-$ will denote the order of the submodule of elements in $K_2(\mathcal{O}_{E_i}^S)$ that are inverted by the non-trivial automorphism of E_i over F.

Theorem 2.1 (Comparison Theorem for a biquadratic extension not containing $\boldsymbol{F^{(1)}}$). *Suppose that $\mathcal{E}/F$ is biquadratic, and that $\mathcal{E}$ does not contain $F^{(1)}$. If the intersection of the images in $K_2(\mathcal{O}_\mathcal{E}^S)$ of $K_2(\mathcal{O}_{E_1}^S)$ and $K_2(\mathcal{O}_{E_3}^S)$ does not equal the image of $K_2(\mathcal{O}_F^S)$, and likewise for the intersection of the images in $K_2(\mathcal{O}_\mathcal{E}^S)$ of $K_2(\mathcal{O}_{E_1}^S)$ and $K_2(\mathcal{O}_{E_2}^S)$, then either*

(a) $\mathrm{Fit}^S_{\mathcal{E}/F}(1) = \mathrm{Fit}^S_{\mathcal{E}/F}(1)\mathcal{S}$, *or*

(b) $\mathrm{Fit}^S_{\mathcal{E}/F}(1)$ *has index* 2 *in* $\mathrm{Fit}^S_{\mathcal{E}/F}(1)\mathcal{S}$.

Suppose further that the Birch-Tate conjecture holds for F and the $E_i's$. Then

$$\mathrm{Fit}^S_{\mathcal{E}/F}(1)\mathcal{S} = \mathrm{Stick}^S_{\mathcal{E}/F}(-1)\mathcal{S},$$

which contains $\mathrm{Stick}^S_{\mathcal{E}/F}(-1)$ of index 2. In case (a), *$\mathrm{Stick}^S_{\mathcal{E}/F}(-1)$ is contained in $\mathrm{Fit}^S_{\mathcal{E}/F}(1)$ with index 2. In case* (b), *$\mathrm{Stick}^S_{\mathcal{E}/F}(-1)$ and $\mathrm{Fit}^S_{\mathcal{E}/F}(1)$ both have index 2 in $\mathrm{Fit}^S_{\mathcal{E}/F}(1)\mathcal{S} = \mathrm{Stick}^S_{\mathcal{E}/F}(-1)\mathcal{S}$, and thus have the same index in $\mathbb{Z}[G]$. Assuming that the Birch-Tate conjecture also holds for $\mathcal{E}$, the index of $\mathrm{Stick}^S_{\mathcal{E}/F}(-1)$ in $\mathbb{Z}[G]$ equals the order of $K_2(\mathcal{O}_E^S)$ in both cases.*

Theorem 2.2 (Comparison Theorem for a biquadratic extension containing $\boldsymbol{F^{(1)}}$). *Suppose that $\mathcal{E}/F$ is biquadratic, and that $E_1 = F^{(1)}$.*

(i) *Then*

$$\mathrm{Fit}^S_{\mathcal{E}/F}(1) \supset 2\mathrm{Fit}^S_{\mathcal{E}/F}(1)\mathcal{S}$$

and $\dfrac{\mathrm{Fit}^S_{\mathcal{E}/F}(1)}{2\mathrm{Fit}^S_{\mathcal{E}/F}(1)\mathcal{S}}$ must be one of three $\mathbb{F}_2$-subspaces of $\dfrac{\mathrm{Fit}^S_{\mathcal{E}/F}(1)\mathcal{S}}{2\mathrm{Fit}^S_{\mathcal{E}/F}(1)\mathcal{S}}\cdot$ (We cannot say that all three occur.) The bases for these subspaces are:

(a) $\{k_2^S(F)e_0 + k_2^S(E_1)^-e_1,\ k_2^S(E_2)^-e_2,\ k_2^S(E_3)^-e_3\}$,

(b) $\{k_2^S(F)e_0 + k_2^S(E_1)^-e_1 + k_2^S(E_2)^-e_2,\ k_2^S(E_2)^-e_2 + k_2^S(E_3)^-e_3\}$,

(c) $\{k_2^S(F)e_0 + k_2^S(E_1)^-e_1,\ k_2^S(E_2)^-e_2 + k_2^S(E_3)^-e_3\}$.

(ii) *Now assume that the Birch-Tate conjecture holds for F and each $E_i's$. Then*

$$\mathrm{Fit}^S_{\mathcal{E}/F}(1)\mathcal{S} = \mathrm{Stick}^S_{\mathcal{E}/F}(-1)\mathcal{S},$$

which contains $\mathrm{Stick}^S_{\mathcal{E}/F}(-1)$ of index 4. If case (a) *occurs, $\mathrm{Stick}^S_{\mathcal{E}/F}(-1)$ lies in $\mathrm{Fit}^S_{\mathcal{E}/F}(1)$ with index 2. If case* (b) *occurs, $\mathrm{Fit}^S_{\mathcal{E}/F}(1) = \mathrm{Stick}^S_{\mathcal{E}/F}(-1)$. If case* (c) *occurs, $\mathrm{Stick}^S_{\mathcal{E}/F}(-1)$ and $\mathrm{Fit}^S_{\mathcal{E}/F}(1)$ have the same index in $\mathbb{Z}[G]$. If the intersection of the images in $K_2(\mathcal{O}_\mathcal{E}^S)$ of $K_2(\mathcal{O}_{E_1}^S)$ and $K_2(\mathcal{O}_{E_3}^S)$ does not equal the image of $K_2(\mathcal{O}_F^S)$, then case* (c) *does not occur.*

(iii) *If the Birch-Tate conjecture holds for $\mathcal{E}$, then the index of $\mathrm{Stick}^S_{\mathcal{E}/F}(-1)$ in $\mathbb{Z}[G]$ equals the order of $K_2(\mathcal{O}_E^S)$ in all cases.*

Proposition 2.3. *Suppose that F is totally real and $E = F(\sqrt{d})$ for some non-zero, totally positive $d \in F$. Then the kernel of the natural map $\iota_{E/F} : K_2(F) \to K_2(E)$ is generated by the symbol $\{-1, d\}_F$.*

Proof. Suppose that ω is in the kernel. So $\iota_{E/F}(\omega) = 1$. Applying the transfer $\mathrm{Tr}_{E/F}$ shows that $\omega^2 = 1$. Then by Theorem 6.1 in [13], $\omega = \{-1, a\}_F$ for some non-zero $a \in F$. Hence our assumption is that $\{-1, a\}_E = 1$. Thus $a \in F$ lies in the Tate kernel of E. Since E is totally real, the Tate kernel of E is generated by the squares in $E^\times$ and π_E for which $E(\sqrt{\pi_E})$ is the first layer $E^{(1)}$ of the cyclotomic $\mathbb{Z}_2$ extension of E (see Proposition 2.4 in [5]). Thus $F(\sqrt{a}) \subset E^{(1)} = F(\sqrt{d}, \sqrt{\pi_E})$.

Firstly, if $E = F^{(1)}$, then $E^{(1)}/F$ is cyclic, being contained in the cyclotomic $\mathbb{Z}_2$-extension of F. So $F(\sqrt{a}) \subset F(\sqrt{d}) = E = F^{(1)} = F(\sqrt{\pi_F})$, the only quadratic extension of F in $E^{(1)}$. By Kummer theory, a and d must both equal a power of π_F times a square in $F^\times$. Thus they both lie in the Tate kernel of F, and

$$\omega = \{-1, a\}_F = 1 = \{-1, d\}_F.$$

So the kernel of $\iota_{E/F}$ is trivial and we are done in this case.

Secondly, if $E \neq F^{(1)}$, then we have $\pi_E = \pi_F \in F$. This time, Kummer theory implies that a lies in the subgroup of $F^\times$ generated by the squares, along with d and π_F. Since $\{-1, \pi_F\}_F$=1, we see that $\{-1, a\}_F$ is a power of $\{-1, d\}_F$. □

Proposition 2.4. *Let F be a totally real algebraic number field. Let $E_1 = F(\sqrt{d_1})$ and $E_3 = F(\sqrt{d_3})$ be distinct totally real quadratic extensions of F with composite $\mathcal{E}$, and assume that $E_2 = F(\sqrt{d_1 d_3}) \neq F^{(1)}$. If $\{-1, \alpha_1\}_{\mathcal{E}} = \{-1, \alpha_3\}_{\mathcal{E}} = \iota_{\mathcal{E}/F}(\omega)$, for some $\alpha_1 \in E_1$, $\alpha_3 \in E_3$ and $\omega \in K_2(F)$, then $\omega^2 = 1$.*

Proof. Applying the transfer map $\mathrm{Tr}_{\mathcal{E}/E_1}$ and using its standard properties gives

$$\begin{aligned} 1 &= \{1, \alpha_1\}_{E_1} = \{-1, \alpha_1\}_{E_1}^2 = \{-1, \alpha_1^2\}_{E_1} = \{-1, \mathrm{N}_{\mathcal{E}/E_1}(\alpha_1)\}_{E_1} \\ &= \mathrm{Tr}_{\mathcal{E}/E_1}(\{-1, \alpha_1\}_{\mathcal{E}}) = \mathrm{Tr}_{\mathcal{E}/E_1}(\iota_{\mathcal{E}/E_1}(\iota_{E_1/F}(\omega))) = \iota_{E_1/F}(\omega)^2 = \iota_{E_1/F}(\omega^2). \end{aligned}$$

Thus ω^2 lies in the kernel of $\iota_{E_1/F}$. By Proposition 2.3, ω^2 is a power of $\{-1, d_1\}_F$. The proof of Proposition 2.3 then also shows that $\omega^2 = 1$ if $E_1 = F^{(1)}$, and we are done in this case. By the same argument, ω^2 is a power of $\{-1, d_3\}_F$, and we are done if $E_3 = F^{(1)}$. In the remaining case, assume by way of contradiction that $\omega^2 \neq 1$. Then we must have $\{-1, d_1\}_F = \omega^2 = \{-1, d_3\}_F$, so that $d_1 d_3$ is in the Tate kernel of F. As in the proof of Proposition 2.3, this implies that $E_2 = F(\sqrt{d_1 d_3})$ lies in $F^{(1)}$, contradicting our assumptions. □

Lemma 2.5. *Let $\mathcal{E}$ be a biquadratic extension of F, with intermediate subfields E_1, E_2 and E_3. If M is an extension of $\mathcal{E}$ which is Galois over E_1 and E_3, then M is Galois over F.*

Proof. Fix an automorphism σ_3 of the normal closure of M over F which restricts to the non-trivial automorphism of $\mathcal{E}/E_3$. Now if σ is any automorphism of the normal closure of M over F, the restriction of σ to $\mathcal{E}$ must fix E_i for some i. If σ fixes E_1 or E_3, then $\sigma(M) = M$, since M is Galois over E_1 and E_3. Otherwise, σ must restrict to the non-trivial automorphism of $\mathcal{E}/E_2$. In this case, $\sigma_3\sigma$ fixes E_1, so that $\sigma_3\sigma(M) = M$. Thus $\sigma(M) = \sigma_3^{-1}(M) = M$, and hence the only conjugate of M over F is itself, and M/F is Galois. □

For the next proposition, we denote the dihedral group of order 8 by D_8.

Proposition 2.6. *Let F be a totally real algebraic number field. Let $E_1 = F(\sqrt{d_1})$ and $E_3 = F(\sqrt{d_3})$ be distinct totally real quadratic extensions of F with composite $\mathcal{E}$, and assume that $\mathcal{E} \neq E_1^{(1)}$. Then there exists an element in $K_2(\mathcal{E})$ which is simultaneously the image of elements of order 2 in $K_2(E_1)$ and $K_2(E_3)$, but not the image of an element in $K_2(F)$, if and only if $\mathcal{E}$ lies in a D_8 Galois extension M of F which is biquadratic over E_1, biquadratic over E_3, and cyclic over $E_2 = F(\sqrt{d_1d_3})$. In particular, if such an element exists and is the image of $\{-1,\alpha_3\}_{E_3}$, then we can choose $M = \mathcal{E}(\sqrt{\alpha_3})$. Conversely, if such a field $M = \mathcal{E}(\sqrt{\alpha})$ exists, then $\{-1,\alpha\}_{\mathcal{E}}$ is an element satisfying the specified description.*

Proof. Let $\pi_{E_1} \in E_1$ be the canonical element such that $E_1(\sqrt{\pi_{E_1}}) = E_1^{(1)}$. Under our assumption that $E_1^{(1)} \neq \mathcal{E}$, we have $\mathcal{E}^{(1)} = \mathcal{E}(\sqrt{\pi_{E_1}})$. Equivalently,

$$\pi_{\mathcal{E}} = \pi_{E_1}.$$

Suppose that there exist elements $\{-1,\alpha_1\}_{E_1}$ and $\{-1,\alpha_3\}_{E_3}$ with the same non-trivial image in $K_2(\mathcal{E})$. Then α_1 and α_3 are not in the Tate kernel of $\mathcal{E}$, but α_1/α_3 is. We have seen that this Tate kernel is $\langle\pi_{\mathcal{E}}\rangle \cdot (\mathcal{E}^\times)^2$. So $M = \mathcal{E}(\sqrt{\alpha_3})$ is a quadratic extension of $\mathcal{E}$, and is clearly biquadratic over E_3 since $\alpha_3 \in E_3$. Also since $\pi_{\mathcal{E}} = \pi_{E_1}$, we see that $\alpha_3 = \pi_{E_1}^t\gamma^2\alpha_1$ for some integer t and $\gamma \in \mathcal{E}^\times$. Thus

$$M = \mathcal{E}\left(\sqrt{\alpha_3}\right) = \mathcal{E}\left(\sqrt{\pi_{E_1}^t\alpha_1}\right),$$

with $\pi_{E_1}^t\alpha_1 \in E_1$, so M is also biquadratic over E_1. Then by Lemma 2.5, M is Galois over F.

We now note that M is not abelian over F. If it were, then $E_3(\sqrt{\alpha_3})/F$ would be Galois with group isomorphic to $\mathrm{Gal}(M/E_1)$, which is the Klein four group. Thus we would have $E_3(\sqrt{\alpha_3}) = E_3(\sqrt{a})$ for some $a \in F$. Consequently, $\alpha_3 = a\eta^2$ for some $\eta \in E_3^\times$. This would imply that $\{-1,\alpha_3\}_{E_3} = \{-1,a\}_{E_3}$, the image of $\{-1,a\}_F \in K_2(F)$, so that $\{-1,\alpha_3\}_{\mathcal{E}} = \{-1,a\}_{\mathcal{E}}$. As we are assuming this is not the case, we must conclude that M/F is a Galois extension of degree 8 containing the non-Galois extension $E_3(\sqrt{\alpha_3})/F$ of degree 4, and the only possible Galois group is D_8. This proves one implication of the proposition.

Conversely, suppose that an extension $M = \mathcal{E}(\sqrt{\alpha})$ of the specified type exists. Since M is biquadratic over E_3, we must have

$$M = \mathcal{E}(\sqrt{\alpha_3}),$$

for some $\alpha_3 \in \mathcal{E}_3$. By Kummer theory, α_3 must be α times a square in $\mathcal{E}$, so that $\{-1,\alpha\}_{\mathcal{E}} = \{-1,\alpha_3\}_{\mathcal{E}}$. We show that the element $\{-1,\alpha_3\}_{\mathcal{E}}$ satisfies the desired conditions. First of all, it is the image of $\{-1,\alpha_3\}_{E_3}$. At the same time, M is also biquadratic over E_1, so

$$M = \mathcal{E}\big(\sqrt{\alpha_1}\big),$$

for some $\alpha_1 \in E_1$ and, as above, we have $\{-1,\alpha\}_{\mathcal{E}} = \{-1,\alpha_3\}_{\mathcal{E}}$, which is clearly the image of $\{-1,\alpha_1\}_{E_1} \in K_2(E_1)$. It remains to show that $\{-1,\alpha_3\}_{\mathcal{E}}$ is not the image of an element $\omega \in K_2(F)$. We proceed by contradiction. If such an ω existed, then by Proposition 2.4 it would be of the form $\omega = \{-1,a\}_F$, for some $a \in F$. The condition that $E_2 \neq F^{(1)}$ in that proposition is satisfied under our assumption that $\mathcal{E} \neq E_1^{(1)}$. Then we would have $\{-1,a\}_{\mathcal{E}} = \{-1,\alpha_1\}_{\mathcal{E}}$, so that α_1/a is in the Tate kernel of $\mathcal{E}$. This yields that $\alpha_1 = a\pi_{E_1}^t\eta^2$, for some integer t and some $\eta \in \mathcal{E}$. Thus

$$M = \mathcal{E}\big(\sqrt{\alpha_1}\big) \subset \mathcal{E}\big(\sqrt{a}, \sqrt{\pi_{E_1}}\big).$$

But this field is abelian over F, while M is not, and we obtain the desired contradiction to complete the proof. □

Corollary 2.7. *Let F be a totally real algebraic number field. Let $E_1 = F\big(\sqrt{d_1}\big)$ and $E_3 = F\big(\sqrt{d_3}\big)$ be distinct totally real quadratic extensions of F with composite $\mathcal{E}$, and assume that $\mathcal{E}$ does not contain $F^{(1)}$. Then there is no element of order 2 in $K_2(\mathcal{E})$ which is simultaneously the image of elements of order 2 in each of $K_2(E_1)$, $K_2(E_2)$ and $K_2(E_3)$.*

Proof. The hypothesis on $F^{(1)}$ implies that $E_1^{(1)} \neq \mathcal{E} \neq E_2^{(1)}$. If such an element existed, it would be of the form $\{-1,\alpha_3\}_{\mathcal{E}}$, with $\alpha_3 \in E_3$. By Proposition 2.6, $\mathcal{E}\big(\sqrt{\alpha_3}\big)$ would be both biquadratic and cyclic over E_2, a contradiction. □

Now let S_2 denote the set of dyadic primes of F, *i.e.*, the primes lying above 2. Note that the tame symbol $(-1,\alpha)_{\mathfrak{p}}$ is trivial for all $\alpha \in \mathcal{E}$ when $\mathfrak{p}$ is dyadic, since -1 is congruent to 1 modulo any dyadic prime. Thus $K_2(\mathcal{O}_E^S) = K_2(\mathcal{O}_{\mathcal{E}}^{S\cup S_2})$.

Theorem 2.8 (Main Theorem for $\mathcal{E}$ containing $F^{(1)}$). *Suppose that $\mathcal{E}/F$ is a biquadratic extension of totally real number fields with the Birch-Tate conjecture holding for F and the three relative quadratic extensions of F in $\mathcal{E}$. Assume that one of these intermediate fields is $E_1 = F^{(1)}$. Let S contain the infinite primes of F and the primes that ramify in $\mathcal{E}$. Suppose that $\mathcal{E}$ can be embedded in a D_8 extension M of F which is biquadratic over E_1 and unramified over F outside of $S \cup S_2$. Then $\mathrm{Stick}_{\mathcal{E}/F}^S(-1)$ is contained in $\mathrm{Fit}_{\mathcal{E}/F}^S(1)$ with index 1 or 2.*

Proof. We let E_2 and E_3 be the other relative quadratic extensions of F in $\mathcal{E}$ with M cyclic over E_2 and biquadratic over E_3. Note that $\mathcal{E} \neq E_1^{(1)}$, for then $\mathcal{E}$ would be cyclic over F. The conditions of Proposition 2.6 are met, so there exists an element $\{-1,\alpha\}_{\mathcal{E}} \in K_2(\mathcal{E})$ which is the image of some $\{-1,\alpha_1\}_{E_1} \in K_2(E_1)$ and of some $\{-1,\alpha_3\}_{E_1} \in K_2(E_3)$, but not in the image of $K_2(F)$. Since we have $M = \mathcal{E}\big(\sqrt{\alpha}\big)$ unramified over F outside of $S \cup S_2$, α has even valuation at all primes above those not in $S \cup S_2$. This implies that $\{-1,\alpha\}_{\mathcal{E}}$ lies in the S-tame kernel $K_2(\mathcal{O}_{\mathcal{E}}^S)$ of $\mathcal{E}$. Since $\mathcal{E}/E_i$ is unramified outside the primes above S for each i, we also find that $\{-1,\alpha_1\}_{E_1} \in K_2(\mathcal{O}_{E_1}^S)$ and $\{-1,\alpha_3\}_{E_3} \in K_2(\mathcal{O}_{E_3}^S)$. Now Theorem 2.2 applies to give the result. □

Theorem 2.9 (Main Theorem for $\mathcal{E}$ not containing $F^{(1)}$). *Suppose that $\mathcal{E}/F$ is a biquadratic extension of totally real number fields with the Birch-Tate conjecture holding for F and the relative quadratic extensions E_1, E_2 and E_3 of F in $\mathcal{E}$. Assume that $F^{(1)}$ is not contained in $\mathcal{E}$. Let S contain the infinite primes of F and the primes that ramify in $\mathcal{E}$. Suppose that $\mathcal{E}$ can be embedded in a D_8 extension M of F which is cyclic over E_2 and unramified over F outside of $S \cup S_2$, and also in a D_8 extension M' of F which is cyclic over E_3 and unramifed over F outside of $S \cup S_2$. Then either $\mathrm{Stick}^S_{\mathcal{E}/F}(-1)$ is contained in $\mathrm{Fit}^S_{\mathcal{E}/F}(1)$ with index 1 or 2, or $\mathrm{Stick}^S_{\mathcal{E}/F}(-1)$ and $\mathrm{Fit}^S_{\mathcal{E}/F}(1)$ are both of index 2 in $\mathrm{Stick}^S_{\mathcal{E}/F}(-1) + \mathrm{Fit}^S_{\mathcal{E}/F}(1)$.*

Proof. Proposition 2.6 implies that there is an element $\{-1, \alpha\}_{\mathcal{E}}$ in the intersection of the images in $K_2(\mathcal{E})$ of $K_2(E_1)$ and $K_2(E_3)$ but not in the image of $K_2(\mathcal{O}^S_F)$. As in the proof of Theorem 2.8, the fact that M is unramified outside S implies that $\{-1, \alpha\}_{\mathcal{E}}$ lies in the intersection of the images in $K_2(\mathcal{O}^S_{\mathcal{E}})$ of $K_2(\mathcal{O}^S_{E_1})$ and $K_2(\mathcal{O}^S_{E_3})$ but not in the image of $K_2(\mathcal{O}^S_F)$. The same argument with M replaced by M' then shows that we may apply Theorem 2.1 and obtain the desired conclusion. □

3. Applications

For easy reference, we first record some standard facts in a lemma.

Lemma 3.1. *Suppose that E/F is a relative quadratic extension and that α and β lie in $E^\times$. Then*

(a) *$E(\sqrt{\alpha}) = E(\sqrt{\beta})$ if and only if $\alpha\beta$ is a square in E;*

(b) *$E(\sqrt{\alpha})/F$ is a Galois extension if and only if the relative norm of α is a square in E;*

(c) *$E(\sqrt{\alpha})/F$ is a biquadratic extension if and only if α is not a square in E and the relative norm of α is a square in F.*

Proof. First, (a) follows from Kummer theory or an easy exercise, while (b) follows from (a) upon taking β to be the conjugate of α over F.

For (c), suppose that the extension is biquadratic. Then $E(\sqrt{\alpha}) = E(\sqrt{a})$ for some $a \in F$. The implication follows upon applying (a) and taking the norm. For the converse, let c^2 be the norm of α. The automorphisms sending $\sqrt{\alpha}$ to its conjugates $\pm c/\sqrt{\alpha}$ both have order two, so cannot lie in a cyclic group. □

Proposition 3.2. *Let E_1 be a totally real number field which is a relative quadratic extension of F. Let r be a totally positive non-square element of F, which is the norm of an integral element $\alpha_1 \in E_1$ such that $E_3 := F(\sqrt{r})$ is not contained in $E_1^{(1)}$. Set $\mathcal{E} = E_1(\sqrt{r})$, and let S contain all of the infinite primes of F, all of the primes that ramify in E_1, and all of the primes dividing r. Then $M = \mathcal{E}(\sqrt{\alpha_1})$ is a D_8 extension of F which is unramified outside of $S \cup S_2$ and cyclic over E_2.*

Proof. The hypotheses clearly guarantee that E_3 is a relative quadratic extension of F, distinct from E_1. Thus $\mathcal{E}$ is a biquadratic extension of F, and we denote the third relative quadratic extension of F in $\mathcal{E}$ by E_2. Since $F(\sqrt{r})$ is not contained in $E_1^{(1)}$, it is clear that $\mathcal{E} \neq E_1^{(1)}$, and since $\alpha_1 \in E_1$, M is biquadratic over E_1. Because the relative norm of α_1 from $\mathcal{E}$ to E_3 is r, which is a square in E_3, M is biquadratic over E_3, by Lemma 3.1. The relative norm of α_1 from $\mathcal{E}$ to E_2 is again r, which is a square in $\mathcal{E}$, but not a square in E_2. For this would imply that $\sqrt{r} \in E_2$, and consequently $E_2 = F(\sqrt{r}) = E_3$, a contradiction. Thus M is a cyclic extension of E_2. By Lemma 2.3, we conclude that M is a Galois extension of F. By Lemma 3.1 again, the extension $E_1(\sqrt{\alpha_1})$ is not Galois over F, and one finds that M must be a D_8 extension of F. Since α_1 is integral of norm r, the ramified primes of $M = \mathcal{E}(\sqrt{\alpha_1})$ over $\mathcal{E}$ are divisors of $2r$, and thus lie above primes in $S \cup S_2$. □

The following corollary strengthens and generalizes Corollary 5.3 of [9]. By Proposition 1.2, all of the assumptions of the Birch-Tate conjecture in both of these corollaries are satisfied when $\mathcal{E}$ is absolutely abelian, for example if $F = \mathbb{Q}$.

Corollary 3.3. *Let F be a totally real field and let $E_1 = F^{(1)}$. Also let α_1 be an integral element of E_1 whose norm to F is a totally positive non-square element r such that $E_3 = F(\sqrt{r}) \neq E_1$. Put $\mathcal{E} = E_1 \cdot E_3$, and let S contain all of the infinite primes of F, and all of the primes that ramify in $\mathcal{E}/F$. Assume that the Birch-Tate conjecture holds for F and that the quadratic extensions of F in E. Then we have*

$$\mathrm{Stick}^S_{\mathcal{E}/F}(-1) \subset \mathrm{Fit}^S_{\mathcal{E}/F}(1),$$

and the index is 1 *or* 2.

Proof. This follows from Proposition 3.2 and Theorem 2.8. □

Corollary 3.4. *Let F be a totally real field and $d \in F$ be a totally positive integral element such that there is a unit $\epsilon_1 \in E_1 = F(\sqrt{d})$ whose relative norm to F is -1. Fix an integral element $\alpha_1 \in E_1$ whose relative norm r in F is totally positive, and not a square in E_1. Set $E_3 = F(\sqrt{r})$ and $\mathcal{E} = E_1 \cdot E_3$, so $E_2 = F(\sqrt{rd})$, and suppose that $F^{(1)}$ is not contained in $\mathcal{E}$. Let S contain all of the infinite primes of F, all of the primes that divide rd, and all of the dyadic primes that ramify in $\mathcal{E}/F$. Assume that the Birch-Tate conjecture holds for F and for the quadratic extensions of F in E. Then either $\mathrm{Stick}^S_{\mathcal{E}/F}(-1)$ is contained in $\mathrm{Fit}^S_{\mathcal{E}/F}(1)$ with index 1 or 2, or $\mathrm{Stick}^S_{\mathcal{E}/F}(-1)$ and $\mathrm{Fit}^S_{\mathcal{E}/F}(1)$ are both of index 2 in $\mathrm{Stick}^S_{\mathcal{E}/F}(-1) + \mathrm{Fit}^S_{\mathcal{E}/F}(1)$.*

Proof. By Proposition 3.2, $M = \mathcal{E}(\sqrt{\alpha_1})$ is a D_8 extension of F which is unramified outside of $S \cup S_2$ and cyclic over E_2. Now consider $\alpha_1' = \alpha_1 \epsilon_1 \sqrt{d}$. The norm of α_1' is $r(-1)(-d) = rd$, and $E_2 = F(\sqrt{rd})$. This time, Proposition 3.3 shows that M' is a D_8 extension of F which is cyclic over E_3 and unramified outside of $S \cup S_2$. The result follows from Theorem 2.9. □

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