Ann. Sci. Math. Québec **34** (2010), no 1, 63–71

INTERVALS BETWEEN FAREY FRACTIONS IN THE LIMIT OF INFINITE LEVEL

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Dedicated to Professor Paulo Ribenboim on the occasion of his 80th birthday.

RÉSUMÉ. La séquence de Farey modifiée est formée, à chaque niveau k, de fractions rationelles $r_k^{(n)}$, avec $n = 1, 2, \ldots, 2^k + 1$. Nous considérons $I_k^{(e)}$, la longueur totale d'une partie des intervalles alternant entre les fractions de Farey qui sont nouvelles (*i.e.* qui apparaissent pour la première fois) au niveau k :

$$I_k^{(e)} := \sum_{i=1}^{2^{k-2}} \left(r_k^{(4i)} - r_k^{(4i-2)} \right).$$

Nous démontrons que $\liminf_{k\to\infty} I_k^{(e)} = 0$, et conjecturons que $\lim_{k\to\infty} I_k^{(e)} = 0$. Cette propriété géométrique simple des fractions de Farey se montre assez subtile, et ne paraît pas avoir une interprétation évidente. La conjecture est équivalente à dire que $\lim_{k\to\infty} S_k = 0$, où S_k est la somme des carrés inverses des dénominateurs nouveaux au niveau k, *i.e.*, $S_k := \sum_{n=1}^{2^{k-1}} \left(1/\left(d_k^{(2n)}\right)^2 \right)$. Notre preuve emploie des bornes pour la longueur des intervalles entre les fractions de Farey en fonction des longueurs de leurs intervalles « parents » aux plus bas niveaux.

ABSTRACT. The modified Farey sequence consists, at each level k, of rational fractions $r_k^{(n)}$, with $n = 1, 2, \ldots, 2^k + 1$. We consider $I_k^{(e)}$, the total length of (one set of) alternate intervals between Farey fractions that are new (*i.e.*, appear for the first time) at level k:

$$I_k^{(e)} := \sum_{i=1}^{2^{k-2}} \left(r_k^{(4i)} - r_k^{(4i-2)} \right).$$

We prove that $\liminf_{k\to\infty} I_k^{(e)} = 0$, and conjecture that $\lim_{k\to\infty} I_k^{(e)} = 0$. This simple geometrical property of the Farey fractions turns out to be surprisingly subtle, with no apparent simple interpretation. The conjecture is equivalent to the fact that $\lim_{k\to\infty} S_k = 0$, where S_k is the sum over the inverse squares of the new denominators at level k, *i.e.*, $S_k := \sum_{n=1}^{2^{k-1}} \left(1/\left(d_k^{(2n)}\right)^2 \right)$. Our result makes use of bounds for Farey fraction intervals in terms of their "parent" intervals at lower levels.

Reçu le 19 mai 2009 et, sous forme définitive, le 30 novembre 2009.

1. Introduction

The Farey fractions (modified Farey sequence) may be defined as $r_k^{(n)} := \frac{n_k^{(n)}}{d_k^{(n)}}$, with gcd $\left(n_k^{(n)}, d_k^{(n)}\right) = 1$, where n denotes the *order* of the Farey fraction at level k. Level $k = 0$ consists of the two fractions $\left\{\frac{0}{1}, \frac{1}{1}\right\}$. Succeeding levels are generated by keeping all the fractions from level k in level $k+1$, and including new fractions. The new fractions at level $k+1$ are defined via

$$d_{k+1}^{(2n)} := d_k^{(n)} + d_k^{(n+1)} \quad \text{and} \quad n_{k+1}^{(2n)} := n_k^{(n)} + n_k^{(n+1)},$$

so that

$$\begin{cases} k=0: & \left\{\frac{0}{1}, \frac{1}{1}\right\}, \\ k=1: & \left\{\frac{0}{1}, \frac{1}{2}, \frac{1}{1}\right\}, \\ k=2: & \left\{\frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1}\right\}, \textit{etc.} \end{cases}$$

Note that $n = 1, \ldots, 2^k+1$. It follows that the fractions at a given level are in increasing order. The Farey fractions may also be defined using products of the 2×2 matrices $A = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$ and $B = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ (see [4] for details).

Our main result concerns the sum of lengths of half of the intervals between "new" Farey fractions. Theorem 1 proves that the "lim inf" of this sum vanishes in the limit of infinite level k. Based on numerical evidence, we also conjecture that the limit of this sum vanishes. This very simple geometric property of the Farey fractions is not very apparent. The intervals chosen are alternating, and there seems no obvious reason why the sum of their lengths should vanish in this limit.

In this paper, we focus on the "Farey tree", which means retaining only the 2^{k-1} *even* Farey fractions at each level $k > 1$: these are exactly the new fractions at each level. In our notation they are of even order, *i.e.*, $r_k^{(2n)}$ so for each level $k > 1$ we obtain the set

$$\left\{r_k^{(2n)} \,|\, n = 1, \ldots, 2^{k-1}\right\}.$$

The lengths of the intervals between even (new) Farey fractions at every level $k > 1$ are denoted

$$I_k^{(n)} := \left(r_k^{(2n)} - r_k^{(2n-2)}\right) > 0, \tag{1}$$

where $n = 2, 3, 4, \ldots, 2^{k-1}$. In what follows, for brevity, we abuse the terminology slightly and refer to $I_k^{(n)}$ as an *interval*. When n itself is even (*i.e.*, $n = 2, 4, \ldots, 2^{k-1}$), we refer to these as *even intervals*. (Note that there are 2^{k-2} even intervals at level k.) The complementary intervals in the unit interval $[0, 1]$, that is, those with n odd ($n = 3, 5, \ldots, 2^{k-1} - 1$), including the two extra intervals at the ends of the unit interval, namely $\left(r_k^{(2)} - r_k^{(1)}\right)$ and $\left(r_k^{(2^k+1)} - r_k^{(2^k)}\right)$, are the *odd intervals*. (Note that odd and even intervals alternate.) From the definition of the Farey fractions it is easy to verify that each of the extra intervals has length $1/(k+1)$. Thus we combine them and define

$$I_k^{(1)} := 2/(k+1), \tag{2}$$

see Figure 1. (As a result there are 2^{k-2} odd intervals at level k, that is the same as the number of even intervals.)

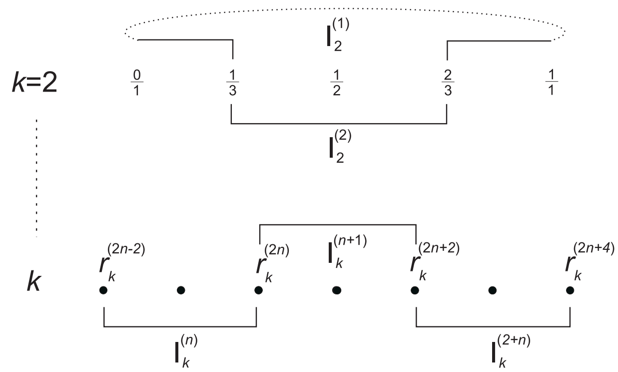

FIGURE 1. Definition of Farey fraction intervals. Even (odd) intervals are shown via a line below (above) the fractions. (Note that in the lower diagram, n is even.) The dashed line in the upper diagram indicates the combining of the two "extra" intervals into $I_k^{(1)}$. In this figure and those below the intervals are not to scale.

Next we define the total length of even intervals at level k

$$I_k^{(e)} := \sum_{i=1}^{2^{k-2}} I_k^{(2i)}, \tag{3}$$

and similarly for the odd intervals

$$I_k^{(o)} := \sum_{i=1}^{2^{k-2}} I_k^{(2i-1)}. \tag{4}$$

It follows that

$$I_k^{(e)} + I_k^{(o)} = 1, \tag{5}$$

for all $k \geq 2$.

The quantity

$$S_k := \sum_{n=1}^{2^{k-1}} \frac{1}{\left(d_k^{(2n)}\right)^2},$$

is the sum over the inverse squares of the new denominators at level k. As we will see, S_k is closely related to $I_k^{(e)}$.

In the next section, by identifying intervals at different levels and examining their evolution from level to level, we prove our main result.

Theorem 1. *We have*

$$\liminf_{k\to\infty} I_k^{(e)} = 0.$$

Numerical evidence (see the last paragraph in Section 2) then leads us to the following conjecture.

Conjecture 1. *We have*

$$\lim_{k\to\infty} I_k^{(e)} = 0.$$

Section 3.2 contains some further remarks concerning this conjecture.

2. Proof of Theorem 1

In this section we prove Theorem 1, *i.e.*, $\liminf_{k\to\infty} I_k^{(e)} = 0$. The key step is Lemma 7, which bounds an arbitrary odd interval in terms of its "parent" even interval at a lower level. In addition, at the end of the section we present numerical evidence for Conjecture 1.

It is convenient to use the full set of Farey fractions, even though only the even ones enter $I_k^{(e)}$ (see (3)). As mentioned, including $\frac{0}{1}$ and $\frac{1}{1}$, there are $2^k + 1$ fractions at level $k \geq 1$. In our notation, at a given level $k \geq 1$, the even-numbered fractions are new, having been "born" at that level, while the odd-numbered ones are kept from the preceding level. Recall, also, that the intervals in (3) are exactly the $I_k^{(n)}$ with n even.

Lemma 1. *For any $k > 2$ we have the bounds*

$$\frac{2}{k+1} + \frac{2}{3}\sum_{j=1}^{k-2} I_{k-j}^{(e)} \frac{2j+3}{(j+1)(j+2)} \leq I_k^{(o)} < 1, \tag{6}$$

and

$$\frac{2}{k+1} + \sum_{j=1}^{k-2} I_{k-j}^{(e)} \frac{3}{2j+3} \geq I_k^{(o)}. \tag{7}$$

The rightmost inequality in (6) follows immediately from (5). Now clearly $I_k^{(e)} > 0$ for any finite value of k. Also, if there were an $\epsilon > 0$ such that $I_k^{(e)} \geq \epsilon$ for all k, the left-hand side of (6) would diverge as $k \to \infty$. Thus Lemma 1 implies Theorem 1.

We prove Lemma 1 by understanding how even and odd intervals evolve from level k to level $k+1$ (see Figure 2). To proceed, first note that the n-th Farey fraction at level k will be at order $2n-1$ in level $k+1$. Therefore we have the following result.

Lemma 2. *The transformation of the order for any Farey fraction in going from level k to $k+l$ is*

$$n \to 2^l(n-1) + 1.$$

The first step in our argument involves identifying intervals at successive levels. We do this via their "middle" Farey fractions. In a slight abuse of notation, let r_M be the "middle" Farey fraction of the interval $I_k^{(n)}$ at level k, *i.e.*, $r_M = r_k^{(2n-1)}$ (see (1)). We use this to identify any set of intervals at different levels with the same "middle" fraction r_M. Thus, since r_M is necessarily of odd order, an interval at level k is the "parent" of the odd interval at level $k+1$ with the same r_M.

It follows that any even interval $I_k^{(2m)}$ at level k will produce a (necessarily smaller) odd interval at level $k+1$, with $r_M = r_k^{(4m-1)} = r_{k+1}^{(8m-3)}$. Similarly, any odd interval produces an (odd) interval with the same r_M at the next level (see Figure 2). In addition, there are new even intervals that are born at each level. Their "middle" fractions are the ends of the even intervals from the previous level.

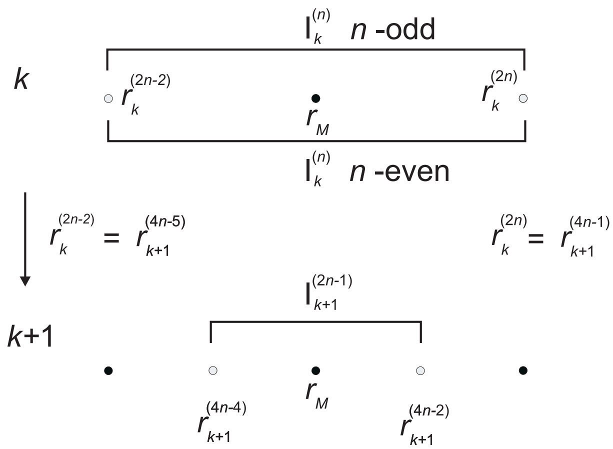

FIGURE 2. Interval changes between levels of Farey fractions. The new Farey fractions (*i.e.*, those "born" at level $k+1$) are depicted by empty circles. Note that any interval, even or odd, at level k gives rise to an odd interval at level $k+1$.

At level $k = 2$, we have one even interval, which lies between the two "extra" intervals comprising $I_2^{(1)}$ at the ends of the unit interval. It follows that all odd intervals at level $k > 2$ (except $I_k^{(1)}$) are born from even intervals at some previous level. Further, every odd interval shrinks from a level to the next level while preserving its "middle" Farey fraction. This establishes the following result.

Lemma 3. *For any level $k > 2$, the unit interval is covered by a set of $2^{k-1} - 1$ alternating even and odd intervals plus the two "end" intervals comprising $I_k^{(1)}$. The even intervals are "newborn", while each of the odd intervals (except $I_k^{(1)}$) is the offspring of an even interval born at a previous level.*

The next step is to determine what fraction of a given interval at level k remains at level $k+1$. In doing this, it is useful to recall that the difference between any two successive fractions at a given level is $r_k^{(n+1)} - r_k^{(n)} = \frac{1}{d_k^{(n+1)} d_k^{(n)}}$ (see for example [4, 2]).

Now consider an arbitrary even interval $I_{k+2}^{(2n)}$ at level $k+2$, for $k \geq 0$. (Note that $n = 1, 2, \ldots, 2^k$.) Its "middle" fraction $r_M = r_{k+2}^{(4n-1)}$ is of odd order, and was therefore carried over from level $k+1$, where it is indexed as $r_M = r_{k+1}^{(2n)}$. This fraction is even, and therefore newborn at level $k+1$. Hence the neighboring fractions to its left and right, $r_{k+1}^{(2n-1)}$ and $r_{k+1}^{(2n+1)}$, respectively, are odd. These two fractions, therefore, appear at level k as $r_k^{(n)}$ and $r_k^{(n+1)}$, respectively.

Now the denominators of odd order fractions carry over from the previous level, *i.e.*, $d_{k+1}^{(2n-1)} = d_k^{(n)}$, while those at even order (since they belong to "new" Farey fractions) are the sum of their neighbors, *i.e.*,

$$d_{k+1}^{(2n)} = d_{k+1}^{(2n-1)} + d_{k+1}^{(2n+1)} = d_k^{(n)} + d_k^{(n+1)}.$$

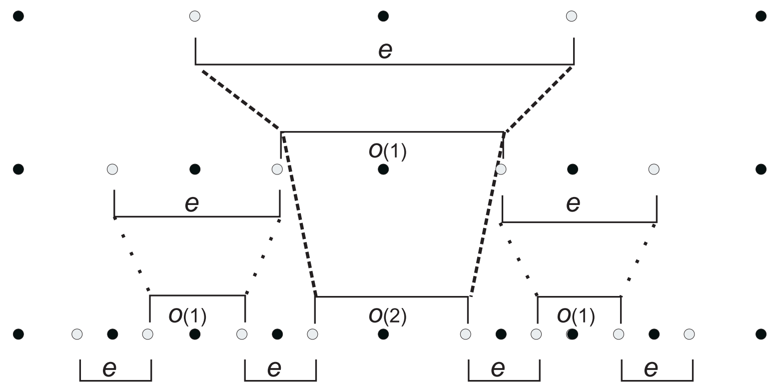

FIGURE 3. Even and odd interval lineage. The numbers indicate age of a given odd interval.

Putting these things together with Lemma 3 gives the following.

Lemma 4. *For any $k \geq 0$ and $n = 1, 2, \ldots, 2^k$, i.e., any even interval at level $k+2$, we have*

$$I_{k+2}^{(2n)} = \frac{3}{(d_k^{(n)} + 2d_k^{(n+1)})(2d_k^{(n)} + d_k^{(n+1)})}. \tag{8}$$

Furthermore, for any $l \geq 1$,

$$I_{k+l+1}^{(2^{l-1}(2n-1)+1)} = \frac{2l+1}{(ld_k^{(n)} + (l+1)d_k^{(n+1)})((l+1)d_k^{(n)} + ld_k^{(n+1)})}, \tag{9}$$

where, for $l > 1$, (9) includes, at level $k+l+1$, all descendants of the 2^k even intervals at level k.

Lemma 4 is illustrated in Figure 3. Now (except for $I_k^{(1)}$) every odd interval (for $k > 2$) is the descendant of a unique even interval at some lower level. Therefore (9) is valid for all $2^{k+l-1}-1$ odd intervals at any level $k+l+1$ with $k \geq 0$ and $l > 1$, omitting $I_k^{(1)}$. Consider the identity $2^{k+l-1} - 1 = \sum_{i=2}^{k+l} 2^{i-2}$. In this context it expresses the number of odd intervals at level $k+l+1$ in terms of a sum over the numbers of (even) "parent" intervals at each lower level i, with $2 \leq i \leq k+l$.

At this point, we consider the ratio of an arbitrary odd interval, as given by (9), to its parent interval $I_{k+2}^{(2n)}$. For simplicity, we let $m = k+2$ and $j = l-1$, so that $m \geq 2$ and $j > 0$, and relabel the left-hand side of (9) as $I_{m+j}^{([2n,j])}$, to indicate that it is the j-th descendant of the $2n$-th interval at level m (note that $I_m^{([2n,0])} = I_m^{(2n)}$). Then, with $z := \frac{d_k^{(n)}}{d_k^{(n+1)}}$, we find the next lemma.

Lemma 5. *For* $m \geq 2$ *and* $j > 0$,

$$\frac{I_{m+j}^{([2n,j])}}{I_m^{(2n)}} = \frac{(2j+3)}{3} \frac{(1+2z)(2+z)}{((j+1)+(j+2)z)((j+2)+(j+1)z)}. \tag{10}$$

Lemma 5 expresses each successive descendent odd interval in terms of its parent even interval. It leads immediately, via elementary computations, to Lemma 6.

Lemma 6. *For* $j > 0$ *and* $m \geq 2$,

$$\frac{2(2j+3)}{3(j+1)(j+2)} \leq \frac{I_{m+j}^{([2n,j])}}{I_m^{(2n)}} \leq \frac{3}{2j+3}. \tag{11}$$

The lower bound in (11) arises from (10) at $z = 0$ or $z = \infty$, and the upper bound from (10) at $z = 1$. In the case of Farey fraction denominators none of these z values actually occurs. However, the important point is that these bounds are independent of both the parent level m and the initial (parent) even interval.

Let $I_{m+j}^{([o,j])}$ denote the sum of all odd intervals at level $m + j$ that are descendants of the even intervals at an arbitrary level $m \geq 2$.

Lemma 7. *For* $j > 0$ *and* $m \geq 2$,

$$I_m^{(e)} \frac{2}{3} \frac{(2j+3)}{(j+1)(j+2)} \leq I_{m+j}^{([o,j])} \leq I_m^{(e)} \frac{3}{2j+3}. \tag{12}$$

Proof. The result is obtained by summing (11) over the even intervals at level m, and by making use of the definition (3). □

Proof of Lemma 1: The remainder of the proof is as follows. First, relabel the "parent" level in Lemma 7 as $m = k - j$. If we fix $k = m + j > 2$, since m varies over the range $2 \leq m \leq k - 1$, j satisfies $1 \leq j \leq k - 2$. Thus all the odd intervals at an arbitrary level k are included, except the "end" interval $I_k^{(1)}$. The leftmost inequality in (6) and the inequality in (7) then follow directly on summing (12) and using (2). □

Finally, A. Zhigljavsky [7] has verified numerically, up to level $k = 34$, that $I_k^{(e)}$ continues to decrease as k increases. His results are consistent with the result that $I_k^{(e)} \sim 1/\log_2(k)$ as $k \to \infty$ recently proven in [3].

3. Discussion

3.1. Proving Conjecture 1

Conjecture 1 is, as already pointed out, very simple. Nevertheless a proof is apparently quite elusive, at least using the methods employed in this paper. Even establishing that $I_k^{(e)}$ is monotonically decreasing with k, which, given Theorem 1, would be sufficient, appears very non-trivial. However, this conjecture has recently been proved [3] using a measure theoretic argument, and approaches to it have been proposed [7, 5] based, respectively, on the Chacon-Ornstein ergodic theorem and continued fractions.

3.2. Relation to physics

The problem treated here arose from previous investigations of the Farey fraction spin chains, a set of statistical mechanical models based on the Farey fractions (see [4], [2] and references therein for details). It follows directly from their definitions that the "Farey tree partition function" $Z_k^F(\beta)$ (see equation (7) in [2]) satisfies $Z_k^F(1) = I_k^{(e)}$, while the "even Knauf partition function" $Z_{k,e}^K(\beta)$ (see the equation after (3) in [2]) satisfies $Z_{k,e}^K(2) = S_k$.

Therefore the inequality (13), proven in [2], can be rewritten as

$$S_k < I_k^{(e)} < 4S_{k-1},$$

which immediately proves the following lemma.

Lemma 8. *Conjecture 1 is equivalent to*

$$\lim_{k\to\infty} S_k = 0.$$

Note that the term "partition function" is used in its statistical mechanical sense here, which in general has no connection with the number theoretic usage.

Since this paper was written, as mentioned in Section 3.1, several proofs of Conjecture 1 using different methods have been proposed. The method employed here does not seem capable of establishing Conjecture 1, however it gives detailed information about the evolution of the intervals not otherwise available.

There are several spin chains known to have the same free energy, and thus the same thermodynamic behavior. They all exhibit a second-order phase transition at non-zero temperature. (The free energy is defined via $f(\beta) := \lim_{k\to\infty}\big(-\log Z_k^F(\beta)/k\beta\big)$, and a phase transition is a singularity in $f(\beta)$; see [2] for more details on these matters.) The Farey tree model is employed in [1] for a study of multifractal behavior associated with chaotic maps exhibiting intermittency. The critical point (phase transition) in this model occurs at $\beta = 1$. Therefore, physically, $Z_k^F(1) = I_k^{(e)}$ is the value of the partition function at the critical point. (The value of the partition function at one point is, however, generally of no physical interest.)

In proving that the free energy of the Farey tree model is the same as the free energy for other Farey statistical models, it was already demonstrated in [2] that as $k \to \infty$, $Z_k^F(\beta) \to 0$ for $\beta > 1$, while $Z_k^F(\beta) \to \infty$ for $\beta < 1$. (This, incidentally, also establishes that the Hausdorff dimension is $\beta_H = 1$.) However, exactly at the critical point, it was only shown that $0 < Z_k^F(1) < 1$. To our knowledge, the result of Theorem 1, extended in [7, 5, 3], is new. Finally, we note that [6] contains some related work, giving results on the large k behavior of the quantity

$$\sigma_k(\beta) := \sum_{i=1}^{2^k} \left(r_k^{(i+1)} - r_k^{(i)}\right)^{\beta},$$

for $\beta > 1$.

Acknowledgments. We are grateful to Don Zagier and A. Zhigljavsky for conversations, and to the latter for numerical results. We also thank the referee.

The second author was supported in part by the National Science Foundation under Grants Nos. DMR-0203589, DMR-0536927 and PHY99-07949. He is also grateful for the hospitality of the Kavli Institute for Theoretical Physics, University of California, Santa Barbara, where part of this work was done.

REFERENCES

[1] M. J. Feigenbaum, I. Procaccia and T. Tél, *Scaling properties of multifractals as an eigenvalue problem*, Phys. Rev. A (3) **39** (1989), no. 10, 5359–5372.

[2] J. Fiala, P. Kleban and A. Özlük, *The phase transition in statistical models defined on Farey fractions*, J. Statist. Phys. **110** (2003), no. 1-2, 73–86.

[3] M. Kesseböhmer and B. O. Stratmann, *A multifractal analysis for Stern-Brocot intervals, continued fractions and Diophantine growth rates*, J. reine angew. Math. **605** (2007), 133–163.

[4] P. Kleban and A. Özlük, *A Farey fraction spin chain*, Comm. Math. Phys. **203** (1999), no. 3, 635–647.

[5] N. Moshchevitin, private communication.

[6] N. Moshchevitin and A. Zhigljavsky, *Entropies of the partitions of the unit interval generated by the Farey tree*, Acta Arith. **115** (2004), no. 1, 47–58.

[7] A. Zhigljavsky, private communication.

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