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A STUDY OF FROBENIUS-EULER NUMBERS AND POLYNOMIALS

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In honour of Paulo Ribenboim on the occasion of his 80th birthday.

RÉSUMÉ. Le but principal de cet article est d'étudier les propriétés de base des polynômes et nombres de Frobenius-Euler, pour en tirer diverses formules de récurrence et de convolution, et de discuter d'une certaine congruence de type Kummer permettant l'écriture de ces nombres au moyen de limites p-adiques.

ABSTRACT. The main purpose of this paper is to study the basic properties of ordinary and generalized Frobenius-Euler numbers and polynomials, from which we deduce various recurrence and convolution formulas and discuss a certain Kummer-type congruence allowing the expression of these numbers by means of p-adic limits.

1. Introduction

The classical Euler numbers E_n and polynomials $E_n(x)$, which are very important in number theory, combinatorics and other branches of mathematics, are defined by, respectively, the generating functions

$$(1.1)\qquad \begin{cases} F(t) := \dfrac{2}{e^t+1} = \displaystyle\sum_{n=0}^{\infty} E_n \frac{t^n}{n!}, \\ F(t,x) := F(t)e^{xt} = \displaystyle\sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}. \end{cases}$$

These were extended by Frobenius in 1910 to the numbers $H_n(u)$ and polynomials $H_n(u,x)$ associated to an algebraic number $u \neq 1$. The ordinary Frobenius-Euler numbers $H_n(u)$ and the polynomials $H_n(u,x)$ associated to u are defined by, respectively, the generating functions

$$(1.2)\qquad \begin{cases} F(u,t) := \dfrac{1-u}{e^t-u} = \displaystyle\sum_{n=0}^{\infty} H_n(u) \frac{t^n}{n!}, \\ F(u,t,x) := F(u,t)e^{xt} = \displaystyle\sum_{n=0}^{\infty} H_n(u,x) \frac{t^n}{n!}. \end{cases}$$

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As easily seen,

$$H_n(u^{-1},x) = (-1)^n H_n(u,1-x) = (1-u)x^n + (-1)^n u H_n(u,x)$$

and these numbers and polynomials satisfy the following recurrence relations. With the initial conditions $H_0(u) = 1$ and $H_0(u,x) = 1$, we have for $n \geq 1$,

$$\begin{cases} (H(u)+1)^n = uH_n(u), \\ (H(u)+x)^n = H_n(u,x), \\ (H(u,x)+1)^n = uH_n(u,x) + (1-u)x^n. \end{cases}$$

Here we used the symbolic umbral notation, though we replaced $H^k(u,x)$ by $H_k(u,x)$ and $H^k(u)$ by $H_k(u)$, after expanding in full by means of the binomial theorem.

Recently, the above numbers were further extended to the generalized Frobenius-Euler numbers analogously to the generalized Bernoulli numbers (*cf.* [7]). Let χ be a primitive Dirichlet character with conductor $f = f_\chi$. The generalized Frobenius-Euler numbers $H_{n,\chi}(u)$ attached to an algebraic number $u \neq 1$ are defined by the generating function

$$F_\chi(u,t) := \sum_{a=0}^{f-1} \frac{(1-u^f)\chi(a)e^{at}u^{f-1-a}}{e^{ft}-u^f} = \sum_{n=0}^{\infty} H_{n,\chi}(u)\frac{t^n}{n!}. \tag{1.3}$$

When $\chi = 1$, we know $F_\chi(u,t) = F(u,t)$ and $H_{n,\chi}(u) = H_n(u)$.

Similarly to $H_n(u,x)$, the generalized Frobenius-Euler polynomials, denoted by $H_{n,\chi}(u,x)$, are defined by

$$F_\chi(u,t,x) := F_\chi(u,t)e^{xt} = \sum_{n=0}^{\infty} H_{n,\chi}(u,x)\frac{t^n}{n!}. \tag{1.4}$$

Then we can easily see that

$$\begin{cases} H_{n,\chi}(u,x) = \ (H_\chi(u)+x)^n := \displaystyle\sum_{i=0}^{n} \binom{n}{i} H_{i,\chi}(u)x^{n-i}, \\ H_{n,\chi}(u,x+y) = \ (H_\chi(u,x)+y)^n := \displaystyle\sum_{i=0}^{n} \binom{n}{i} H_{i,\chi}(u,x)y^{n-i}. \end{cases} \tag{1.5}$$

Basic and important properties of these numbers and polynomials were studied by many mathematicians including Carlitz [1,2], Kim [4], Shiratani [6], Tsumura [8], Young [9] and others. Further, several types of p-adic analytic interpolation functions associated with $H_n(u)$ and $H_{n,\chi}(u)$ were constructed and their specific properties were investigated by Kim *et al.* [3], Kozuka [5], Shiratani-Yamamoto [7] and Tsumura [8].

Throughout this paper, we denote by $\mathbb{Q}$, $\mathbb{C}$, $\mathbb{Q}_p$ and $\mathbb{C}_p$, with p a prime number, the field of rational numbers, the complex number field, the field of p-adic rational numbers and the p-adic completion of the algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}$, respectively. We fix an embedding of the algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$ into $\mathbb{C}_p$. Also we denote by $|\cdot|_p$ the p-adic absolute value on $\mathbb{C}_p$ normalized by $|p|_p = p^{-1}$.

The main purpose of this paper is to study basic properties of Frobenius-Euler numbers and polynomials. In Section 2 we deduce various recurrence and convolution formulas for these numbers and polynomials. In Section 3 we discuss arithmetic properties of generalized Frobenius-Euler numbers and derive a certain Kummer-type congruence applying the expression of these numbers by means of p-adic limit.

2. Recurrence and convolution formulas

At first, we would like to present the most basic recurrence relations.

Proposition 2.1. *For* $m, n \geq 1$,

$$(H_\chi(u)+mf)^n - u^{mf} H_{n,\chi}(u) = (1-u^f)\sum_{a=0}^{f-1}\chi(a)\sum_{i=0}^{m-1}(if+a)^n u^{(m-i)f-1-a}. \tag{2.1}$$

In particular,

$$(H(u)+m)^n - u^m H_n(u) = (1-u)\sum_{i=1}^{m-1} i^n u^{m-1-i}. \tag{2.2}$$

Proof. Consider the identity

$$\begin{aligned} F_\chi(u,t)(e^{mft}-u^{mf}) &= \left(\sum_{a=0}^{f-1}(1-u^f)\chi(a)e^{at}u^{f-a-1}\right)\left(\sum_{i=0}^{m-1}e^{ift}u^{(m-1-i)f}\right) \\ &= (1-u^f)\sum_{a=0}^{f-1}\chi(a)\sum_{i=0}^{m-1}e^{(if+a)t}u^{(m-i)f-1-a}. \end{aligned}$$

Then we have

$$\begin{aligned} (H_\chi(u)+mf)^n &- u^{mf}H_{n,\chi}(u) \\ &= \left[\frac{d^n}{dt^n}\left(F_\chi(u,t)(e^{mft}-u^{mf})\right)\right]_{t=0} \\ &= (1-u^f)\left[\frac{d^n}{dt^n}\left(\sum_{a=0}^{f-1}\chi(a)\sum_{i=0}^{m-1}e^{(if+a)t}u^{(m-i)f-1-a}\right)\right]_{t=0} \\ &= (1-u^f)\sum_{a=0}^{f-1}\chi(a)\sum_{i=0}^{m-1}(if+a)^n u^{(m-i)f-1-a}, \end{aligned}$$

which completes the proof of (2.1). Formula (2.2) is just a special case of $\chi = 1$. □

For brevity, put for $n, r \geq 0$,

$$b_r^n(u,f) := \left[\frac{d^n}{dt^n}(e^{ft}-u^f)^r\right]_{t=0} = \sum_{i=0}^{r}(-1)^{r-i}\binom{r}{i}(if)^n u^{(r-i)f},$$

with the convention $0^0 = 1$, and $b_r^n(u) := b_r^n(u,1)$.

Proposition 2.2. *For* $n, r \geq 1$, *we have*

$$
\begin{aligned}
&\sum_{i=0}^{n} \binom{n}{i} b_r^{n-i}(u,f) H_{i,\chi}(u) \\
&\qquad = (1-u^f) \sum_{a=0}^{f-1} \chi(a) u^{f-1-a} \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b_{r-1}^{i}(u,f) \\
&\qquad = (1-u^f) \sum_{a=0}^{f-1} \chi(a) \sum_{j=0}^{r-1} (-1)^{r-1-j} \binom{r-1}{j} (jf+a)^n u^{(r-j)f-1-a},
\end{aligned} \tag{2.3}
$$

and in particular,

$$
\begin{aligned}
\sum_{i=0}^{n} \binom{n}{i} b_r^{n-i}(u) H_i(u) &= (1-u) b_{r-1}^{n}(u,1) \\
&= (1-u) \sum_{j=0}^{r-1} \binom{r-1}{j} j^n (-u)^{r-1-j}.
\end{aligned} \tag{2.4}
$$

Proof. Consider the equality

$$
\begin{aligned}
&F_\chi(u,t)(e^{ft}-u^f)^r \\
&\quad = \left((1-u^f) \sum_{a=0}^{f-1} \chi(a) e^{at} u^{f-1-a} \right) (e^{ft}-u^f)^{r-1} \\
&\quad = (1-u^f) \sum_{a=0}^{f-1} \chi(a) \sum_{j=0}^{r-1} (-1)^{r-1-j} \binom{r-1}{j} e^{(jf+a)t} u^{(r-j)f-1-a}.
\end{aligned}
$$

Using this identity, we can obtain (2.3) by the same method as stated in the proof of Proposition 2.1. The recurrence (2.4) is nothing but a special case when $\chi = 1$. □

Let $A(n,k)$, with $n, k \geq 0$, be the Eulerian number defined by

$$A(n,k) := \sum_{j=0}^{k} (-1)^j \binom{n+1}{j} (k-j)^n.$$

Using generalized binomial coefficients, these numbers appear in the expansion

$$x^n = \sum_{k=0}^{n} A(n,k) \binom{x+n-k}{n}, \qquad n = 0, 1, 2, \ldots,$$

and they satisfy, with the initial conditions $A(0,0) = 1$, $A(n,0) = 0$ for $n > 0$, and $A(n,k) = 0$ for $k > n$,

$$
\begin{cases}
A(n,k) = A(n, n-k+1) \qquad \text{with } n, k \geq 0, \\
A(n+1,k) = kA(n,k) + (n-k+2)A(n,k-1).
\end{cases} \tag{2.5}
$$

Lemma 2.3. *For* $n, g \geq 1$ *and* $\alpha \in \mathbb{C}$, *we have*

$$\frac{d^n}{dt^n}\frac{1}{e^{gt}-\alpha} = \frac{(-g)^n \sum_{j=1}^{n} A(n,j)\alpha^{j-1}e^{(n+1-j)gt}}{(e^{gt}-\alpha)^{n+1}}. \tag{2.6}$$

Proof. We shall give the proof by induction on n. By direct calculations, it is easy to confirm that (2.6) is true for $n = 1, 2$. Assume that (2.6) holds for $n = k$. Denoting by $P_k(\alpha, g, t)$ the numerator on the right-hand side of (2.6), in which we replaced n by k, we have

$$\frac{d^{k+1}}{dt^{k+1}}\frac{1}{e^{gt}-\alpha} = \frac{d}{dt}\frac{P_k(\alpha,g,t)}{(e^{gt}-\alpha)^{k+1}} = \frac{P_k'(\alpha,g,t)(e^{gt}-\alpha) - P_k(\alpha,g,t)(k+1)ge^{gt}}{(e^{gt}-\alpha)^{k+2}},$$

where $P_k'(\alpha, g, t) := \frac{d}{dt}P_k(\alpha, g, t)$. Here the numerator becomes, by using (2.5),

$$\begin{aligned}
&P_k'(\alpha,g,t)(e^{gt}-\alpha) - P_k(\alpha,g,t)(k+1)ge^{gt}\\
&\quad= \left((-g)^k \sum_{j=1}^{k}(k+1-j)gA(k,j)\alpha^{j-1}e^{(k+1-j)gt}\right)(e^{gt}-\alpha)\\
&\qquad- \left((-g)^k \sum_{j=1}^{k}A(k,j)\alpha^{j-1}e^{(k+1-j)gt}\right)\left((k+1)ge^{gt}\right)\\
&\quad= (-g)^{k+1}\sum_{i=1}^{k+1}\left(iA(k,i) + (k+2-i)A(k,i-1)\right)\alpha^{i-1}e^{(k+2-i)gt}\\
&\quad= (-g)^{k+1}\sum_{i=1}^{k+1}A(k+1,i)\alpha^{i-1}e^{(k+2-i)gt} = P_{k+1}(\alpha,g,t),
\end{aligned}$$

which shows that (2.6) holds for $n = k+1$. □

For simplification, set

$$\begin{cases} P_0(\alpha, g) := 1, \\ P_r(\alpha, g) := P_r(\alpha, g, 0) = (-g)^r \sum_{j=1}^{r} A(r,j)\alpha^{j-1} \quad \text{for } r \geq 1. \end{cases}$$

As explicit expressions of $H_{n,\chi}(u)$ and $H_n(u)$, we can state the following.

Proposition 2.4. *For* $n \geq 0$, *we get*

$$H_{n,\chi}(u) = \sum_{a=0}^{f-1}\chi(a)u^{f-1-a}\sum_{r=0}^{n}\binom{n}{r}\frac{P_r(u^f, f)a^{n-r}}{(1-u^f)^r}, \tag{2.7}$$

and in particular $H_0(u) = 1$ *and for* $n \geq 1$,

$$H_n(u) = \frac{1}{(u-1)^n}\sum_{j=1}^{n}A(n,j)u^{j-1}. \tag{2.8}$$

Proof. By using Leibniz's rule and Lemma 2.3, we obtain from the definition of $F_\chi(u,t)$ in (1.3) that

$$\begin{aligned} H_{n,\chi}(u) &= \left[\frac{d^n}{dt^n}F_\chi(u,t)\right]_{t=0} \\ &= (1-u^f)\sum_{a=0}^{f-1}\chi(a)u^{f-1-a}\left[\frac{d^n}{dt^n}\frac{e^{at}}{e^{ft}-u^f}\right]_{t=0} \\ &= (1-u^f)\sum_{a=0}^{f-1}\chi(a)u^{f-1-a}\sum_{r=0}^{n}\binom{n}{r}\left[\frac{d^r}{dt^r}\frac{1}{e^{ft}-u^f}\cdot\frac{d^{n-r}}{dt^{n-r}}e^{at}\right]_{t=0} \\ &= (1-u^f)\sum_{a=0}^{f-1}\chi(a)u^{f-1-a}\sum_{r=0}^{n}\binom{n}{r}\frac{P_r(u^f,f)a^{n-r}}{(1-u^f)^{r+1}} \\ &= \sum_{a=0}^{f-1}\chi(a)u^{f-1-a}\sum_{r=0}^{n}\binom{n}{r}\frac{P_r(u^f,f)a^{n-r}}{(1-u^f)^{r}}, \end{aligned}$$

and this implies (2.7). For (2.8), consider the special case with $\chi = 1$. □

Incidentally, considering the special case where $u=-1$, we see that

$$tF(-1,t) = \frac{2t}{e^t+1} = \sum_{n=0}^{\infty} G_n\frac{t^n}{n!},$$

where the G_n's are the Genocchi numbers. Let B_n be the Bernoulli number in the even suffix notation defined by

$$\frac{t}{e^t-1} = \sum_{n=0}^{\infty}\frac{B_n}{n!}t^n.$$

Then, noticing that $B_{n+1} = G_{n+1} = E_n = 0$ if $n \geq 2$ is even, we obtain from (1.1) and (2.8) the well-known formula

$$\sum_{i=1}^{n}(-1)^i A(n,i) = \frac{2^n G_{n+1}}{n+1} = \frac{2^{n+1}(1-2^{n+1})B_{n+1}}{n+1} = 2^n E_n.$$

The next proposition is a special (shortened) recurrence relation for ordinary Frobenius-Euler numbers that allows, for any given $k \geq 1$, to compute $H_{n+k}(u)$ from $H_k(u), H_{k+1}(u), \ldots, H_{k+n-1}(u)$.

Proposition 2.5. *For any* $n, k \geq 1$, *we have*

$$\sum_{i=0}^{n}\binom{n}{i}b_{k+1}^{n-i}(u)H_{k+i}(u) = (1-u)(-1)^k\sum_{j=1}^{k}A(k,j)u^{j-1}(k+1-j)^n. \tag{2.9}$$

Proof. Using Lemma 2.3, we have

$$(e^t-u)^{k+1}\frac{d^k}{dt^k}F(u,t) = (1-u)(-1)^k\sum_{j=1}^{k}A(k,j)u^{j-1}e^{(k+1-j)t}. \tag{2.10}$$

Therefore it can be shown, by Leibniz's rule, that

$$\left[\frac{d^n}{dt^n}\left((e^t-u)^{k+1}\frac{d^k}{dt^k}F(u,t)\right)\right]_{t=0}$$
$$=\sum_{i=0}^{n}\binom{n}{i}\left[\frac{d^{n-i}}{dt^{n-i}}(e^t-u)^{k+1}\cdot\frac{d^{k+i}}{dt^{k+i}}F(u,t)\right]_{t=0}$$
$$=\sum_{i=0}^{n}\binom{n}{i}b_{k+1}^{n-i}(u)H_{k+i}(u),$$

and also

$$\left[\frac{d^n}{dt^n}\sum_{j=1}^{k}A(k,j)u^{j-1}e^{(k+1-j)t}\right]_{t=0}=\sum_{j=1}^{k}A(k,j)u^{j-1}(k+1-j)^n,$$

which yield (2.9) in view of (2.10). □

Putting $u=-1$ in (2.9), we get a linear recurrence relation of arbitrary length for Genocchi, Bernoulli and Euler numbers. Indeed, since $A(k,j)=A(k,k+1-j)$ for $j=1,2,\dots,k$,

$$\sum_{i=0}^{n}\binom{n}{i}b_{k+1}^{n-i}(-1)H_{k+i}(-1)=2\sum_{j=1}^{k}(-1)^{k+j-1}A(k,j)(k+1-j)^n$$
$$=2\sum_{j=1}^{k}(-1)^{j}A(k,j)j^n,$$

where

$$\begin{cases} b_r^m(-1)=\displaystyle\sum_{j=0}^{r}\binom{r}{j}j^m, \\[2ex] H_r(-1)=\dfrac{G_{r+1}}{r+1}=\dfrac{2(1-2^{r+1})B_{r+1}}{r+1}=E_r. \end{cases}$$

The following convolution identities for Frobenius-Euler numbers are the analogue of some formulas of Euler for Bernoulli numbers and Euler numbers, namely

$$\begin{cases} (B+B)^n:=\displaystyle\sum_{i=0}^{n}\binom{n}{i}B_iB_{n-i}=-nB_{n-1}-(n-1)B_n, \\[2ex] (E+E)^n:=\displaystyle\sum_{i=0}^{n}\binom{n}{i}E_iE_{n-i}=2(E_n+E_{n+1}). \end{cases}$$

Proposition 2.6. *Let* $n \geq 1$ *and* $u \neq 0, 1$. *Then*

$$(2.11)\qquad \begin{aligned}(H(u)+H(u))^n &:= \sum_{i=0}^{n} \binom{n}{i} H_i(u) H_{n-i}(u) \\ &= \frac{u-1}{u}\left(H_n(u)+H_{n+1}(u)\right).\end{aligned}$$

Proof. Considering the identity

$$(2.12)\qquad F(u,t)^2 = \frac{u-1}{u}\left(F(u,t)+\frac{d}{dt}F(u,t)\right),$$

we have, by Leibniz's rule,

$$\begin{aligned}\frac{d^n}{dt^n}F(u,t)^2 &= \sum_{i=0}^{n}\binom{n}{i}\frac{d^i}{dt^i}F(u,t)\cdot\frac{d^{n-i}}{dt^{n-i}}F(u,t) \\ &= \frac{u-1}{u}\left(\frac{d^n}{dt^n}F(u,t)+\frac{d^{n+1}}{dt^{n+1}}F(u,t)\right).\end{aligned}$$

Here, setting $t = 0$, we get immediately (2.11). □

Similarly, as a convolution identity for Frobenius-Euler polynomials, we can state the following.

Proposition 2.7. *For* $n \geq 1$ *and* $u \neq 0, 1$, *we have*

$$\begin{aligned}(H(u,x)+H(u,x))^n &:= \sum_{i=0}^{n}\binom{n}{i}H_i(u,x)H_{n-i}(u,x) \\ &= \frac{u-1}{u}\left((1-2x)H_n(u,2x)+H_{n+1}(u,2x)\right).\end{aligned}$$

Proof. The proof can be given by a similar method to that of Proposition 2.6. That is, instead of (2.12), considering the identity

$$F(u,t,x)^2 = \frac{u-1}{u}\left((1-2x)F(u,t,2x)+\frac{d}{dt}F(u,t,2x)\right),$$

we have only to carry out the same calculation as done above. □

3. Basic properties and Kummer-type congruence

In this section, we study basic properties of generalized Frobenius-Euler numbers and polynomials and discuss a certain Kummer-type congruence on these numbers.

Proposition 3.1. *We have*

$$(3.1)\qquad F_\chi(u,t,x) = \sum_{a=0}^{f-1}\chi(a)u^{f-1-a}F\left(u^f, ft, \frac{a+x}{f}\right),$$

and therefore

$$H_{n,\chi}(u,x) = f^n \sum_{a=0}^{f-1} \chi(a) u^{f-1-a} H_n\left(u^f, \frac{a+x}{f}\right). \tag{3.2}$$

Proof. From the definitions of $F(u,t,x)$ in (1.2) and $F_\chi(u,t,x)$ in (1.4) we obtain, since $e^{(a+x)t} = e^{(\frac{a+x}{f})ft}$,

$$\begin{aligned} F_\chi(u,t,x) &= \sum_{a=0}^{f-1} \chi(a) u^{f-1-a} \frac{(1-u^f)e^{(\frac{a+x}{f})ft}}{e^{ft}-u^f} \\ &= \sum_{a=0}^{f-1} \chi(a) u^{f-1-a} F\left(u^f, ft, \frac{a+x}{f}\right), \end{aligned}$$

which proves (3.1). To prove (3.2), compare the coefficient of t^n on both sides of this equality. □

Setting $x = 0$ in (3.2), we obtain immediately

$$H_{n,\chi}(u) = f^n \sum_{a=0}^{f-1} \chi(a) u^{f-1-a} H_n\left(u^f, \frac{a}{f}\right).$$

As a symmetric property of $F_\chi(u,t,x)$, we can state the following.

Proposition 3.2. *If* $\chi \neq 1$, *then*

$$F_\chi(u,-t,-x) = \chi(-1)u^{f-2} F_\chi(u^{-1},t,x), \tag{3.3}$$

and hence

$$(-1)^n H_{n,\chi}(u,-x) = \chi(-1)u^{f-2} H_{n,\chi}(u^{-1},x). \tag{3.4}$$

Proof. From the definition of $F_\chi(u,t,x)$ in (1.4), we have, since $\chi(k) = \chi(m)$ if $k \equiv m \pmod f$,

$$\begin{aligned} F_\chi(u,-t,-x) &= \sum_{a=0}^{f-1} \frac{(1-u^f)\chi(a)e^{(x-a)t}u^{f-1-a}}{e^{-ft}-u^f} \\ &= \sum_{a=0}^{f-1} \frac{(1-u^{-f})\chi(a)e^{(f-a+x)t}u^{f-1-a}}{e^{ft}-u^{-f}} \\ &= \sum_{a=0}^{f-1} \frac{(1-u^{-f})\chi(-1)\chi(f-a)e^{(f-a+x)t}u^{f-1-a}}{e^{ft}-u^{-f}} \\ &= \sum_{b=1}^{f} \frac{(1-u^{-f})\chi(-1)\chi(b)e^{(b+x)t}u^{b-1}}{e^{ft}-u^{-f}} \\ &= \chi(-1)u^{f-2} \sum_{b=1}^{f} \frac{(1-u^{-f})\chi(b)e^{(b+x)t}u^{-(f-1-b)}}{e^{ft}-u^{-f}} \\ &= \chi(-1)u^{f-2} F_\chi(u^{-1},t,x), \end{aligned}$$

which proves (3.3). Also, comparing the coefficient of t^n on both sides of this equality, we can deduce (3.4). □

Proposition 3.3. *For any* $l \geq 1$, *we have*

$$F_\chi(u,t,x-(l-1)f)-u^f F_\chi(u,t,x-lf) = (1-u^f)\sum_{a=0}^{f-1}\chi(a)u^{f-1-a}e^{(a+x-lf)t}, \tag{3.5}$$

hence

$$\begin{aligned} &H_{n,\chi}(u,x-(l-1)f)-u^f H_{n,\chi}(u,x-lf) \\ &\qquad\qquad = (1-u^f)\sum_{a=0}^{f-1}\chi(a)u^{f-1-a}\,(a+x-lf)^n\,. \end{aligned} \tag{3.6}$$

Proof. From the definition of $F_\chi(u,t,x)$ we have

$$\begin{aligned} F_\chi(u,t,x-(l-1)f)&-u^f F_\chi(u,t,x-lf) \\ &= \sum_{a=0}^{f-1}\frac{(1-u^f)\chi(a)e^{(a+x-(l-1)f)t}u^{f-1-a}}{e^{ft}-u^f} \\ &\quad - u^f\sum_{a=0}^{f-1}\frac{(1-u^f)\chi(a)e^{(a+x-lf)t}u^{f-1-a}}{e^{ft}-u^f} \\ &= \sum_{a=0}^{f-1}\frac{(1-u^f)\chi(a)e^{(a+x-lf)t}u^{f-1-a}}{e^{ft}-u^f}(e^{ft}-u^f) \\ &= (1-u^f)\sum_{a=0}^{f-1}\chi(a)u^{f-1-a}e^{(a+x-lf)t}, \end{aligned}$$

which implies (3.5). Also (3.6) follows from (3.5) by equating the coefficient of t^n. □

Proposition 3.4. *Let* $\chi \neq 1$. *For any* $k \geq 1$, *we have*

$$H_{n,\chi}(u,kf)-u^{kf}H_{n,\chi}(u) = (1-u^f)\sum_{b=0}^{kf-1}\chi(b)b^n u^{kf-1-b}. \tag{3.7}$$

Proof. In particular, set $x=(l+i-1)f$ for $i \geq 1$ in (3.6). Then we obtain

$$\begin{aligned} &H_{n,\chi}(u,if)-u^f H_{n,\chi}(u,(i-1)f) \\ &\qquad\qquad = (1-u^f)\sum_{a=0}^{f-1}\chi(a)u^{f-1-a}(a+(i-1)f)^n. \end{aligned} \tag{3.8}$$

Making use of (3.8) with $i = 1, 2, \ldots, k$, it follows that, since $H_{n,\chi}(u) = H_{n,\chi}(u, 0)$,

$$\begin{aligned} &H_{n,\chi}(u, kf) - u^{kf} H_{n,\chi}(u) \\ &\qquad = \sum_{i=1}^{k} u^{(k-i)f} \Big(H_{n,\chi}(u, if) - u^f H_{n,\chi}(u, (i-1)f) \Big) \\ &\qquad = (1 - u^f) \sum_{a=0}^{f-1} \sum_{i=1}^{k} u^{(k-i+1)f-1-a} \chi(a + (i-1)f)(a + (i-1)f)^n \\ &\qquad = (1 - u^f) \sum_{b=0}^{kf-1} \chi(b) b^n u^{kf-1-b}, \end{aligned}$$

which completes the proof of (3.7). □

Here and in what follows, we assume that u is algebraic over $\mathbb{Q}_p$ with $|1-u^f|_p \geq 1$. The next proposition was already shown by Tsumura [8], however we will give here another proof based on (3.7).

Proposition 3.5. *In the field* $\mathbb{Q}_p(u)$,

$$\frac{u}{1 - u^f} H_{n,\chi}(u) = \lim_{N \to \infty} \sum_{b=0}^{fp^N - 1} \chi(b) b^n \frac{u^{fp^N - b}}{1 - u^{fp^N}}, \tag{3.9}$$

where the limit on the right-hand side means p*-adic limit.*

Proof. Put $k = p^N$ in (3.7). Hence

$$H_\chi^n(u, p^N f) - u^{p^N f} H_{n,\chi}(u) = (1 - u^f) \sum_{b=0}^{p^N f - 1} \chi(b) b^n u^{p^N f - 1 - b}. \tag{3.10}$$

From (1.5), the left-hand side of (3.10) becomes

$$H_{n,\chi}(u, p^N f) - u^{p^N f} H_{n,\chi}(u) = (1 - u^{p^N f}) H_{n,\chi}(u) + \sum_{i=0}^{n-1} \binom{n}{i} H_{i,\chi}(u)(p^N f)^{n-i},$$

and then (3.10) gives

$$\begin{aligned} \frac{u}{1 - u^f} H_{n,\chi}(u) = &\sum_{b=0}^{fp^N - 1} \chi(b) b^n \frac{u^{fp^N - b}}{1 - u^{fp^N}} \\ &- \frac{u}{(1 - u^f)(1 - u^{fp^N})} \sum_{i=0}^{n-1} \binom{n}{i} H_{i,\chi}(u)(p^N f)^{n-i}. \end{aligned}$$

Since $|1 - u^f|_p \geq 1$ implies $|1 - u^{fp^N}|_p \geq 1$ for any $N \geq 1$ and $\lim_{N \to \infty} p^N = 0$, we get

$$\frac{u}{1 - u^f} H_{n,\chi}(u) = \lim_{N \to \infty} \sum_{b=0}^{fp^N - 1} \chi(b) b^n \frac{u^{fp^N - b}}{1 - u^{fp^N}},$$

which completes the proof. □

In the following proposition, we want to deduce a general form of Kummer-type congruence for generalized Frobenius-Euler numbers referring to the same method as mentioned in [6]. It should be noted that, using properties of p-adic integrals and measures, more general and stronger Kummer-type congruences for these and other related numbers have been already obtained by Young [9].

Given a sequence $\{a_n\}$, let Δ_r be a linear difference operator defined by

$$\Delta_r a_n := a_{n+r} - a_n.$$

The powers of Δ_r are defined by $\Delta_r^0 := id$ and $\Delta_r^k := \Delta_r \circ \Delta_r^{k-1}$ for $k \geq 1$.

Proposition 3.6. *Let n and k be positive integers and let c be also a positive integer divisible by $p^e(p-1)$ $(e \geq 0)$. Then we obtain, for the sequence $\{H_{n,\chi}(u)\}$ in the field $\mathbb{Q}_p(u)$,*

$$\Delta_c^k H_{n,\chi}(u) \equiv 0 \pmod{p^M},$$

where $M := \min\{n, k(e+1)\}$.

Proof. By making use of (3.9), it follows that

$$\begin{aligned}
\Delta_c^k H_{n,\chi}(u) &= \Delta_c^k \lim_{N\to\infty} \frac{1-u^f}{1-u^{fp^N}} \sum_{b=0}^{fp^N-1} \chi(b) b^n u^{fp^N-1-b} \\
&= \lim_{N\to\infty} \frac{1-u^f}{1-u^{fp^N}} \sum_{b=0}^{fp^N-1} u^{fp^N-1-b} \Delta_c^k \chi(b) b^n \\
&= \lim_{N\to\infty} \frac{1-u^f}{1-u^{fp^N}} \sum_{b=0}^{fp^N-1} u^{fp^N-1-b} \chi(b) b^n \sum_{i=0}^{k} \binom{k}{i} b^{ic} (-1)^{k-i} \\
&= \lim_{N\to\infty} \frac{1-u^f}{1-u^{fp^N}} \sum_{b=0}^{fp^N-1} u^{fp^N-1-b} \chi(b) b^n (b^c - 1)^k.
\end{aligned}$$

Noting that

$$\begin{cases} p^n \mid b^n & \text{if } p \mid b, \\ p^{k(e+1)} \mid (b^c-1)^k & \text{if } p \nmid b, \end{cases}$$

we get $\Delta_c^k H_{n,\chi}(u) \equiv 0 \pmod{p^M}$, as desired. □

Proposition 3.7. *If $p-1 \mid n$ for $n \geq 1$, then*

$$H_{n,\chi}(u) \equiv \lim_{N\to\infty} \left(1 - \chi(p) \frac{1-u^{fp^{N-1}}}{1-u^{fp^N}}\right) H_{0,\chi}(u) \pmod{p}.$$

Proof. From (3.9), we have

$$H_{n,\chi}(u) = \lim_{N\to\infty} \frac{1-u^f}{1-u^{fp^N}} \sum_{b=0}^{fp^N-1} \chi(b) b^n u^{fp^N-1-b}. \tag{3.11}$$

For any $n \geq 1$ satisfying $p-1 \mid n$, it follows that

$$\begin{aligned}
&\sum_{b=0}^{fp^N-1} \chi(b) b^n u^{fp^N-1-b} \\
&\quad = \sum_{\substack{b=0\\(b,p)\neq 1}}^{fp^N-1} \chi(b) b^n u^{fp^N-1-b} + \sum_{\substack{b=0\\(b,p)=1}}^{fp^N-1} \chi(b) b^n u^{fp^N-1-b} \\
&\quad \equiv \sum_{\substack{b=0\\(b,p)=1}}^{fp^N-1} \chi(b) u^{fp^N-1-b} \\
&\quad \equiv \sum_{b=0}^{fp^N-1} \chi(b) u^{fp^N-1-b} - \sum_{j=0}^{fp^{N-1}-1} \chi(jp) u^{fp^N-1-jp} \\
&\quad \equiv \sum_{k=0}^{p^N-1} \sum_{i=0}^{f-1} \chi(kf+i) u^{fp^N-1-kf-i} - \chi(p) \sum_{l=0}^{p^{N-1}-1} \sum_{i=0}^{f-1} \chi(lf+i) u^{fp^N-1-(lf+i)p} \\
&\quad \equiv \frac{1-u^{fp^N}}{1-u^f} \sum_{i=0}^{f-1} \chi(i) u^{f-1-i} - \frac{1-u^{fp^{N-1}}}{1-u^f} \chi(p) \sum_{i=0}^{f-1} \chi(i) u^{f-1-i} \\
&\quad \equiv \frac{1-u^{fp^N}}{1-u^f} \left(1 - \frac{1-u^{fp^{N-1}}}{1-u^{fp^N}} \chi(p)\right) \sum_{i=0}^{f-1} \chi(i) u^{f-1-i} \pmod{p}.
\end{aligned}$$

So that, using the leading coefficient $H_{0,\chi}(u) = \sum_{i=0}^{f-1} \chi(i) u^{f-1-i}$ of $H_{n,\chi}(u,x)$, we can deduce from (3.11) that

$$H_{n,\chi}(u) \equiv \lim_{N\to\infty} \left(1 - \frac{1-u^{fp^{N-1}}}{1-u^{fp^N}} \chi(p)\right) H_{0,\chi}(u) \pmod{p},$$

which proves the proposition. □

It seems that an explicit condition of u for which $H_{0,\chi}(u)$ does not vanish is unknown. About this topic, we do not enter into details, but it can be shown that if $|u| \geq 2$ or $|u| \leq \frac{1}{2}$, then $H_{0,\chi}(u) \neq 0$.

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