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A REMARK ON THE GENERALIZED RAMANUJAN–NAGELL EQUATION $x^2 - D = k^n$

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Dedicated to Professor Paulo Ribenboim on his 80th birthday.

RÉSUMÉ. Dans cet article, nous montrons, en utilisant des arguments élémentaires ainsi qu'un résultat sur l'approximation diophantienne, que l'équation donnée dans le titre a au plus $6 \log |3.2D| / \log k + 8$ solutions (x, n).

ABSTRACT. In this note, using elementary arguments and a result of Diophantine approximation, we prove that the Diophantine equation in the title has at most $6 \log |3.2D| / \log k + 8$ solutions (x, n).

1. Introduction

The Diophantine equation

$$x^2 + 7 = 2^n \tag{1}$$

is called *the Ramanujan-Nagell equation*. In 1960, Nagell [11] proved that the only positive integer solutions to equation (1) are

$$(x, n) = (1, 3),\ (3, 4),\ (5, 5),\ (11, 7),\ (181, 15).$$

The *Generalized Ramanujan-Nagell equation* is the Diophantine equation

$$x^2 + D = k^n, \quad \text{with } x \geq 1,\ \ n \geq 1 \ \text{ and } \ \gcd(D, k) = 1. \tag{2}$$

The literature on the generalized Ramanujan-Nagell equation is very rich. One aspect of the study of equation (2) is to determine the integer solutions (x, k, n). In 1850, Lebesgue [8] proved that the above equation has no solutions when $D = 1$. In 1965, Chao Ko [4] proved that the only solution of equation (2) with $D = -1$ is $x = 3$ and $k = 2$. J.H.E. Cohn [5] solved the above equation for several values of the parameter D in the range $1 \leq D \leq 100$. Some of the remaining values of D in that range were covered by Mignotte and De Weger in [10], while the remaining ones were considered in the recent paper [3]. Recently, several authors have become interested in the case when only the prime factors of D are specified. For example, the cases when D is a fixed product of a few powers of primes were studied. See [1], [2], and [9] for the recent surveys on this type of equations.

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In this paper, D is a nonzero integer and k is a positive integer. We consider the Diophantine equation

$$x^2 - D = k^n, \quad \text{with } x \geq 1, \ \ n \geq 1 \ \text{ and } \ \gcd(D,k) = 1. \tag{3}$$

We denote by $N(D,k)$ the number of solutions of equation (3). One can see, for example, [7] for a history on $N(D,k)$. In [7], Le proved the following result.

Theorem 1.1. *Let $\omega(D)$ be the number of distinct prime factors of $|D|$. Then*

$$N(D,k) \leq \begin{cases} 2^{\omega(D)+1} & \text{if } D < 0, \\ 2^{\omega(D)+1} + 1 & \text{if } D > 0. \end{cases} \tag{4}$$

The aim of this paper is to sharpen Le's result by using Diophantine approximations and properties on continued fractions to prove the following result.

Theorem 1.2. *There are at most 8 solutions (x,n) satisfying $k^n > 4^5 D^6$.*

From Theorem 1.2, we deduce the following result.

Corollary 1.3. *We have $N(D,k) \leq \frac{6 \log|3.2D|}{\log k} + 8$.*

2. Lemmas

Let us prove the following lemma.

Lemma 2.1. *For $j = 1,2,3$, let (x_j, n_j) be any three solutions of (3) such that $n_1 < n_2 < n_3$ and $2 \nmid n_j$. Suppose that $k^{n_1} > 4^5 D^6$. Then*

$$n_3 > n_2 + \frac{2}{3} n_1.$$

Proof. From equation (3), we get

$$\begin{aligned} \left| \frac{x_2}{k^{(n_2-n_1)/2}} - \sqrt{k^{n_1}} \right| &= \left| \frac{x_2}{k^{(n_2-n_1)/2}} + \sqrt{k^{n_1}} \right|^{-1} \cdot \frac{|D|}{k^{n_2-n_1}} \\ &< \frac{|D|}{\sqrt{k^{n_1}} k^{n_2-n_1}} \\ &< \frac{1}{3.2 k^{n_1/3} \left(k^{(n_2-n_1)/2}\right)^2}. \end{aligned} \tag{5}$$

Since

$$\frac{1}{3.2 k^{n_1/3} \left(k^{(n_2-n_1)/2}\right)^2} < \frac{1}{2 \left(k^{(n_2-n_1)/2}\right)^2},$$

we have, by the classical Legendre's Theorem on Diophantine approximation, that $\frac{x_2}{k^{(n_2-n_1)/2}}$ is a convergent in the simple continued fraction expansion of $\sqrt{k^{n_1}}$. One can apply the same method to the solution (x_3, n_3). Moreover, if p_m/q_m is the m-th convergent of $\sqrt{k^{n_1}}$, then

$$\left| \sqrt{k^{n_1}} - \frac{p_m}{q_m} \right| > \frac{1}{(a_{m+1}+2) q_m^2}, \tag{6}$$

where a_{m+1} is the $(m+1)$-st partial quotient of $\sqrt{k^{n_1}}$ (see *e.g.* [6]). Since

$$\gcd\left(x_2, k^{(n_2-n_1)/2}\right) = \gcd\left(x_3, k^{(n_3-n_1)/2}\right) = 1,$$

it follows that if

$$\frac{x_2}{k^{(n_2-n_1)/2}} = \frac{p_b}{q_b} \quad \text{and} \quad \frac{x_3}{k^{(n_3-n_1)/2}} = \frac{p_c}{q_c},$$

then

$$x_2 = p_b, \quad k^{(n_2-n_1)/2} = q_b, \quad x_3 = p_c \ \text{ and } \ k^{(n_3-n_1)/2} = q_c.$$

Thus, combining equations (5) and (6) yields to

$$a_{b+1} > 3.2k^{n_1/3} - 2 > k^{n_1/3}.$$

Also, since $p_c \geq p_{b+1} > a_{b+1}p_b$, we obtain

$$k^{(n_3-n_1)/2} > k^{(n_2-n_1)/2} \cdot k^{n_1/3}.$$

Therefore, we have

$$n_3 > n_2 + \frac{2}{3}n_1. \tag{7}$$

□

Now, we recall a result due to Tzanakis-Wolfskill [12].

Lemma 2.2. *Suppose k is not a square and (x, n) is a solution of* (3) *which satisfies $k^n \geq 4^{1+s/r}|D|^{2+s/r}$ for some $r, s \in \mathbb{N}$. Then, we have*

$$\left|\frac{x'}{k^{n'/2}} - 1\right| > \frac{8}{2187k^{n(3+\nu/2)}}\left(\frac{81k^n}{4}\right)^{1/s} k^{-n'(1+\nu)/2},$$

for any $x', n' \in \mathbb{N}$ with $2 \nmid n'$, where ν satisfies $k^{n\nu} = 9(81k^n/4)^{r/s}$.

Proof. This is [12, Theorem I.2] with $a = 1$. □

Now, we prove the following result.

Lemma 2.3. *Suppose that k is not a square and equation* (3) *has a solution (x, n) such that $k^n > 4^5D^6$. Then every solution (x', n') of* (3) *with $2 \nmid n'$ satisfies $n' < 12n$.*

Proof. We take $r = 1,\ s = 4$ in Lemma 2.2. It follows that $k^n > 4^5D^6$ and

$$\left|\frac{x'}{k^{n'/2}} - 1\right| > \frac{8}{2187k^{n(3+\nu/2)}}\left(\frac{81k^n}{4}\right)^{1/4} k^{-n'(1+\nu)/2}, \tag{8}$$

where

$$\nu = \frac{\log 9}{\log k^n} + \frac{1}{4} + \frac{\log(81/4)}{4\log k^n} < \frac{1}{4} + \frac{2.8}{\log k^n} < \frac{1}{2}.$$

Since (x', n') is a solution of (3) with $2 \nmid n'$, one has

$$\left|\frac{x'}{k^{n'/2}} - 1\right| = \frac{|D|}{k^{n'/2}(k^{n'/2} + x')} < \frac{|D|}{k^{n'}}. \tag{9}$$

Combining equations (8) and (9) then gives

$$\frac{8}{2187k^{n(3+\nu/2)}}\left(\frac{81k^n}{4}\right)^{1/4} k^{-n'(1+\nu)/2} < \frac{|D|}{k^{n'}}.$$

As $|D| < (k^n/4^5)^{1/6}$, we get

$$8\left(\frac{81k^n}{4}\right)^{1/4} k^{n'(1-\nu)/2} < 2187k^{n(3+\nu/2)}|D| < 2187k^{n(3+\nu/2)}(k^n/4^5)^{1/6},$$

and so

$$k^{n'(1-\nu)/2} < 40.6k^{n(3+\nu/2+1/6-1/4)} < k^{n(3+\nu/2+1/6-1/4-1/4)}.$$

Therefore, we obtain

$$n' < \frac{3+\nu/2+1/6-1/2}{(1-\nu)/2} \cdot n \le 35n/3 < 12n. \qquad \square$$

3. The proofs

In this section, we prove our main results.

Proof of Theorem 1.2. We assume that there exist 9 solutions (x_j, n_j), $1 \le j \le 9$, with $k^{n_1} > 4^5D^6$. It is easy to see that if $n \ge n_1$, then n is odd. Indeed, equation (3) can be factored into

$$(x + k^{n/2})(x - k^{n/2}) = D.$$

Therefore, there exist two integers D_1, D_2 with $D_1 > 0$ such that

$$x + k^{n/2} = D_1, \quad x - k^{n/2} = D_2 \quad \text{and} \quad D = D_1D_2.$$

As

$$k^{n/2} = (D_1 - D_2)/2 \le (D_1 + |D_2|)/2 \le |D|,$$

we deduce that $4^5D^6 < k^{n_1} \le k^n < D^2$, which is impossible.

Therefore, by Lemma 2.1, we have

$$n_{j+2} > n_{j+1} + \frac{2}{3}n_j, \qquad 0 \le j \le 7. \tag{10}$$

This implies

$$\begin{aligned} n_9 &> n_8 + \tfrac{2}{3}n_7 > \tfrac{5}{3}n_7 + \tfrac{2}{3}n_6 > \tfrac{7}{3}n_6 + \tfrac{10}{9}n_5 > \tfrac{31}{9}n_5 + \tfrac{14}{9}n_4 \\ &> 5n_4 + \tfrac{62}{27}n_3 > \tfrac{197}{27}n_3 + \tfrac{10}{3}n_2 > \tfrac{287}{27}n_2 + \tfrac{394}{81}n_1 > \tfrac{1255}{81}n_1 \\ &> 15n_1. \end{aligned}$$

But Lemma 2.3 provides $n_9 < 12n_1$. This leads to a contradiction. Therefore, there are at most 8 solutions (x, n) satisfying $k^n > 4^5D^6$. $\square$

We deduce the proof of Corollary 1.3.

Proof of Corollary 1.3. Since there exist at most $\log(4^5D^6)/\log k$ positive integers n satisfying $k^n \le 4^5D^6$, Theorem 1.2 yields the upper bound for the number of solutions for equation (3). $\square$

4. Final remark

In [7], Le showed that the number of positive integer solutions of equation (3) is at most $2^{\omega(D)+1} + \delta$ (where $\delta = 0$ for $D > 0$ and $\delta = 1$ for $D < 0$), see Theorem 1.1. The method of the present paper can sharpen the upper bound to $3 \cdot 2^{\omega(D)-1} + c$, where c is an absolute constant a little bigger than 8.

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