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DISTRIBUTION MODULO 1 AND THE LEXICOGRAPHIC WORLD

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In honour of Paulo Ribenboim on the occasion of his 80th birthday.

RÉSUMÉ. Nous donnons une description complète des intervalles de longueur minimale contenant toutes les parties fractionnaires $\{\xi 2^n\}$, pour un certain nombre réel positif ξ, et pour tout $n \geq 0$.

ABSTRACT. We give a complete description of the minimal intervals containing all fractional parts $\{\xi 2^n\}$ for some positive real number ξ, and for all $n \geq 0$.

1. Introduction

In the paper [20], Mahler defined the set of Z-numbers by

$$\left\{\xi \in \mathbb{R} \;\middle|\; \xi > 0,\ \forall n \geq 0,\ 0 \leq \left\{\xi \left(\frac{3}{2}\right)^n\right\} < \frac{1}{2}\right\},$$

where $\{z\}$ is the fractional part of the real number z. Mahler proved that this set is at most countable. It is still an open problem to prove that this set is actually empty. More generally, given a real number $\alpha > 1$ and an interval $(x, y) \subset (0, 1)$ one can ask whether there exists $\xi > 0$ such that, for all $n \geq 0$, one has $x \leq \{\xi\alpha^n\} < y$ (or the variant $x \leq \{\xi\alpha^n\} \leq y$). Flatto, Lagarias and Pollington [14, Theorem 1.4] proved that if $\alpha = p/q$, with p, q coprime integers and $p > q \geq 2$, then any interval (x, y) such that for some $\xi > 0$ one has $\{\xi(p/q)^n\} \in (x, y)$ for all $n \geq 0$ must satisfy $y - x \geq 1/p$. Recently Bugeaud and Dubickas [8] characterized irrational numbers ξ such that for a fixed integer $b \geq 2$ all the fractional parts $\{\xi b^n\}$ belong to a closed interval of length $1/b$. Before stating their theorem we need a definition.

Definition 1. Given two real numbers α and ρ, with $\alpha \geq 0$, we denote by

$$\boldsymbol{s}_{\alpha,\rho} := (s_{\alpha,\rho}(n))_{n\geq 0} \quad \text{and} \quad \boldsymbol{s}'_{\alpha,\rho} := (s'_{\alpha,\rho}(n))_{n\geq 0}$$

the sequences defined by

$$s_{\alpha,\rho}(n) = \lfloor (n+1)\alpha + \rho \rfloor - \lfloor n\alpha + \rho \rfloor \quad \text{and} \quad s'_{\alpha,\rho}(n) = \lceil (n+1)\alpha + \rho \rceil - \lceil n\alpha + \rho \rceil,$$

where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x and $\lceil x \rceil$ denotes the least integer greater than or equal to x. The sequences $\boldsymbol{s}_{\alpha,\rho}$ and $\boldsymbol{s}'_{\alpha,\rho}$ are called *Sturmian*

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sequences if α is irrational, and *periodic balanced sequences* if α is rational. Furthermore, if $\alpha = \rho$, these sequences are called *characteristic* Sturmian sequences or *characteristic* periodic balanced sequences, according to whether or not α is irrational.

On the one hand, we observe that if α is not an integer then for all $n \geq 0$,

$$\lfloor \alpha \rfloor \leq s_{\alpha,\rho}(n) \leq \lfloor \alpha \rfloor + 1 \quad \text{and} \quad \lceil \alpha \rceil - 1 \leq s'_{\alpha,\rho}(n) \leq \lceil \alpha \rceil,$$

where $\lceil \alpha \rceil - 1 = \lfloor \alpha \rfloor$ and $\lceil \alpha \rceil = \lfloor \alpha \rfloor + 1$. On the other hand, if α is an integer, then $\boldsymbol{s}_{\alpha,\rho} = \boldsymbol{s}'_{\alpha,\rho}$ and $\alpha \leq s_{\alpha,\rho}(n) \leq \alpha + 1$ for all $n \geq 0$. Accordingly, the sequences $\boldsymbol{s}_{\alpha,\rho}$ and $\boldsymbol{s}'_{\alpha,\rho}$ take their values in the "alphabet" $\{k, k+1\}$ where $k = \lfloor \alpha \rfloor$. The classical definition of Sturmian sequences with values in $\{0, 1\}$ is thus obtained by subtracting $\lfloor \alpha \rfloor$ from each of the terms in the sequences $\boldsymbol{s}_{\alpha,\rho}$ and $\boldsymbol{s}'_{\alpha,\rho}$. Alternatively, one may restrict α to the interval $(0, 1)$. Hereafter, if the alphabet is not mentioned, it is understood that the sequences are over $\{0, 1\}$. We may also assume that $\rho \in [0, 1)$ or $\rho \in (0, 1]$ since $\boldsymbol{s}_{\alpha,\rho} = \boldsymbol{s}_{\alpha,\rho'}$ and $\boldsymbol{s}'_{\alpha,\rho} = \boldsymbol{s}'_{\alpha,\rho'}$ for any two real numbers ρ, ρ' such that $\rho - \rho'$ is an integer.

Example 2. Taking $\alpha = \rho = (3 - \sqrt{5})/2$, we obtain the well-known (binary) *Fibonacci sequence* $0100101001001010010100100101\cdots$.

Remark 3. Note that if α is irrational, then the (Sturmian) sequences $\boldsymbol{s}_{\alpha,\rho}$ and $\boldsymbol{s}'_{\alpha,\rho}$ are *aperiodic* (*i.e.*, not eventually periodic), whereas if α is rational, the sequences $\boldsymbol{s}_{\alpha,\rho}$ and $\boldsymbol{s}'_{\alpha,\rho}$ are (purely) periodic (see for instance [19, Lemma 2.14]). This justifies the use of "periodic" in the name of such sequences in the rational case. The reason for being called "balanced" is explained in Section 3.1.

Let T denote the *shift map* on sequences, defined as follows: if $\boldsymbol{s} := (s_n)_{n\geq 0}$, then $T(\boldsymbol{s}) = T((s_n)_{n\geq 0}) := (s_{n+1})_{n\geq 0}$. The main result in [8] reads as follows.

Theorem 4 (Bugeaud-Dubickas). *Let $b \geq 2$ be an integer and let ξ be an irrational number. Then the numbers $\{\xi b^n\}$, with $n \geq 0$, cannot all lie in an interval of length smaller than $1/b$. Furthermore, there exists a closed interval I of length $1/b$ containing the numbers $\{\xi b^n\}$ for all $n \geq 0$ if and only if the sequence of base b digits of the fractional part of ξ is a Sturmian sequence $\boldsymbol{s}$ on the alphabet $\{k, k+1\}$ for some $k \in \{0, 1, \ldots, b-2\}$. If this is the case, then ξ is transcendental, and the interval I is semi-open. It is open unless there exists an integer $j \geq 1$ such that $T^j(\boldsymbol{s})$ is a characteristic Sturmian sequence on the alphabet $\{k, k+1\}$.*

The purpose of this paper is to give a complete description of the minimal intervals containing all fractional parts $\{\xi 2^n\}$ for some positive real number ξ, and for all $n \geq 0$. More precisely, inspired by the definition of the lexicographic world (see Section 2.2), let us define a function F on $[0, 1]$ as follows.

Definition 5. For all $x \in [0, 1]$, let

$$S_x := \{\xi \in \mathbb{R} \mid \xi > 0,\ \forall n \geq 0,\ x \leq \{\xi 2^n\} < 1\}$$

and let $F : [0, 1] \to [0, 1]$ be the function defined by

$$F(x) = \begin{cases} \inf\{y \in [0, 1) \mid \exists \xi > 0,\ \forall n \geq 0,\ x \leq \{\xi 2^n\} \leq y\} & \text{if } S_x \neq \varnothing, \\ 1 & \text{if } S_x = \varnothing. \end{cases}$$

Remark 6. From Bugeaud-Dubickas's result for $b = 2$, we deduce the following two facts:

- For $x \in \left[\frac{1}{2}, 1\right]$, there is no irrational number $\xi > 0$ such that $x \le \{\xi 2^n\} < 1$ for all $n \ge 0$; nor is there a rational number $\xi > 0$ such that $x \le \{\xi 2^n\} < 1$ for all $n \ge 0$. (This can be seen by considering, for instance, the base 2 expansion of the fractional part of ξ for any rational number $\xi \ge x$.) Hence, $F(x) = 1$ for all $x \in \left[\frac{1}{2}, 1\right]$.

- If $\xi > 0$ is an irrational real number, then there exists a real number $x \in \left[0, \frac{1}{2}\right)$ such that all the fractional parts $\{\xi 2^n\}$ belong to the interval $\left[x, x + \frac{1}{2}\right]$ if and only if the base 2 expansion of the fractional part of ξ is a Sturmian sequence. Furthermore, for any such x, one has $F(x) = x + \frac{1}{2}$.

Note that it follows from our main theorem (see Theorem 7 later) that $0 \le F(x) < 1$ for $x \in \left[0, \frac{1}{2}\right)$.

Before stating our main theorem, let us note that the sequences $\boldsymbol{s}_{\alpha,\rho}$ and $\boldsymbol{s}'_{\alpha,\rho}$ (given in Definition 1) are said to have *slope* α and *intercept* ρ, in view of their geometric realization as approximations to the line $y = \alpha x + \rho$ (called *lower* and *upper mechanical words* in [19, Chapter 2]). From now on, we will assume that α and ρ are in the interval $[0, 1]$, in which case the sequences $\boldsymbol{s}_{\alpha,\rho}$ and $\boldsymbol{s}'_{\alpha,\rho}$ take their values in $\{0, 1\}$. If α is irrational, then we have $\boldsymbol{s}_{\alpha,\alpha} = \boldsymbol{s}'_{\alpha,\alpha}$, denoted by $\boldsymbol{c}_\alpha$. We also have $\boldsymbol{s}_{0,0} = \boldsymbol{s}'_{0,0} = 0^\infty$ and $\boldsymbol{s}_{1,1} = \boldsymbol{s}'_{1,1} = 1^\infty$, denoted by $\boldsymbol{c}_0$ and $\boldsymbol{c}_1$, respectively. In these cases, the sequence $\boldsymbol{c}_\alpha$ is the unique characteristic Sturmian sequence of slope α in $\{0, 1\}^{\mathbb{N}}$. Besides this, if $\alpha \in (0, 1)$ is rational, then the characteristic periodic balanced sequences of slope α, namely $\boldsymbol{s}_{\alpha,\alpha}$ and $\boldsymbol{s}'_{\alpha,\alpha}$, are distinct sequences in $\{0, 1\}^{\mathbb{N}}$ containing both 0's and 1's. More precisely, let us suppose that $\alpha = p/q \in (0, 1)$, with $\gcd(p, q) = 1$. Then by considering the prefix of length q of each of the sequences $\boldsymbol{s}_{p/q,0}$ and $\boldsymbol{s}'_{p/q,0}$, we find that there exists a unique word $w_{p,q}$ of length $q - 2$ such that

$$\boldsymbol{s}_{p/q,0} = (0w_{p,q}1)^\infty \quad \text{and} \quad \boldsymbol{s}'_{p/q,0} = (1w_{p,q}0)^\infty,$$

where v^∞ denotes the periodic sequence $vvvv\cdots$ for a given word v (see for instance [19, p. 59]). Hence, the two characteristic periodic balanced sequences of slope p/q in $\{0, 1\}^{\mathbb{N}}$ are given by

$$\boldsymbol{s}_{p/q,p/q} = T(\boldsymbol{s}_{p/q,0}) = (w_{p,q}10)^\infty \quad \text{and} \quad \boldsymbol{s}'_{p/q,p/q} = T(\boldsymbol{s}'_{p/q,0}) = (w_{p,q}01)^\infty.$$

The words $w_{p,q}$ are often referred to as *central words* in the literature; they hold a special place in the rich theory of Sturmian sequences (see, *e.g.*, [19, Chapter 2]). For instance, it follows from the work of de Luca and Mignosi [11, 12] that central words coincide with the palindromic prefixes of characteristic Sturmian sequences (see Section 3.3).

Given a sequence $\boldsymbol{s} \in \{0, 1\}^{\mathbb{N}}$, let $r(\boldsymbol{s})$ denote the real number in $(0, 1)$ whose sequence of base 2 digits after the binary point is given by $\boldsymbol{s}$. Our main number-theoretical result reads as follows.

Theorem 7. *Let x be a real number in $[0, 1]$.*

(i) *If $x \ge \frac{1}{2}$, then $F(x) = 1$.*

(ii) *If $x = 0$, then $F(x) = 0$.*

(iii) *If* $x \in \left(0, \frac{1}{2}\right)$ *and if the base* 2 *expansion of* $2x$ *is given by a characteristic Sturmian sequence, then* $F(x) = x + \frac{1}{2}$*. Furthermore,* $F(x)$ *is the unique real number in* $[0,1]$ *that has a Sturmian base* 2 *expansion and satisfies* $x \le \{F(x)2^k\} \le F(x)$ *for all* $k \ge 0$.

(iv) *If* $x \in \left(0, \frac{1}{2}\right)$ *and if the base* 2 *expansion of* $2x$ *is given by a characteristic periodic balanced sequence of slope* $p/q \in (0,1)$*, with* $\gcd(p,q) = 1$*, then* $F(x)$ *is the rational number whose base* 2 *expansion is given by the periodic balanced sequence* $\boldsymbol{s}'_{p/q,0} = (1w_{p,q}0)^\infty$*. In this case,* $F(x) \le x + \frac{1}{2}$.

(v) *In all other cases,* $F(x)$ *can be explicitly computed: it is equal to the rational number whose base* 2 *expansion is given by a (unique) periodic balanced sequence* $\boldsymbol{s}'_{p/q,0} = (1w_{p,q}0)^\infty$ *where* p*,* q *are coprime integers, with* $0 < p < q$*, such that* $r((w_{p,q}01)^\infty) < 2x < r((w_{p,q}10)^\infty)$*. In these cases,* $F(x) < x + \frac{1}{2}$.

Moreover, in cases (iv) *and* (v)*,* $F(x)$ *is the unique real number in* $(0,1)$ *whose base* 2 *expansion is given by a periodic balanced sequence and which, for all* $k \ge 0$*, satisfies* $x \le \{F(x)2^k\} \le F(x)$.

Remark 8. It is known (see [13]) that real numbers having a Sturmian base 2 expansion are transcendental. As a consequence of Theorem 7, we deduce that if x is an algebraic real number in $\left[0, \frac{1}{2}\right)$, then $F(x)$ is rational.

2. The combinatorial approach

The main tool used by Bugeaud and Dubickas is *combinatorics on words*: real numbers are replaced by their base b expansion, and inequalities between real numbers are transformed into (lexicographic) inequalities between infinite sequences representing their base b expansions. We will establish a theorem of combinatorial flavour (Theorem 13), whose translation into a number-theoretical statement is exactly Theorem 7 above. A method for computing $F(x)$ in case (v) of Theorem 7 is given in Section 6.

2.1. Two combinatorial theorems

It happens that the case $b = 2$ of Bugeaud-Dubickas's theorem was already proved by Veerman in [25, 26]. The combinatorial result proved by Veerman, and by Bugeaud-Dubickas, is stated (and strengthened) in Theorems 9 and 10 below.

Theorem 9. *An aperiodic sequence* $\boldsymbol{s} := (s_n)_{n\ge0}$ *on* $\{0,1\}$ *is Sturmian if and only if there exists a sequence* $\boldsymbol{u} := (u_n)_{n\ge0}$ *on* $\{0,1\}$ *such that* $0\boldsymbol{u} \le T^k(\boldsymbol{s}) \le 1\boldsymbol{u}$ *for all* $k \ge 0$*. Moreover,* $\boldsymbol{u}$ *is the unique characteristic Sturmian sequence with the same slope as* $\boldsymbol{s}$*, and we have* $0\boldsymbol{u} = \inf\{T^k(\boldsymbol{s}),\ k \ge 0\}$ *and* $1\boldsymbol{u} = \sup\{T^k(\boldsymbol{s}),\ k \ge 0\}$.

Theorem 10. *An aperiodic sequence* $\boldsymbol{u}$ *on* $\{0,1\}$ *is a characteristic Sturmian sequence if and only if, for all* $k \ge 0$*,*

$$0\boldsymbol{u} < T^k(\boldsymbol{u}) < 1\boldsymbol{u}.$$

Furthermore, we have $0\boldsymbol{u} = \inf\{T^k(\boldsymbol{u}),\ k \ge 0\}$ *and* $1\boldsymbol{u} = \sup\{T^k(\boldsymbol{u}),\ k \ge 0\}$.

2.2. The lexicographic world

As discussed in [1], the results in Theorems 9 and 10 have been rediscovered several times since the work of Veerman in the mid-late 80's. One of the presentations of these statements is due to Gan [16]. It is based on the lexicographic(al) world, which seems to have been introduced in 2000, in a preprint version of [18].

For any two sequences $\boldsymbol{x}, \boldsymbol{y} \in \{0,1\}^{\mathbb{N}}$, define the set

$$\Sigma_{\boldsymbol{x},\boldsymbol{y}} := \{\boldsymbol{s} \in \{0,1\}^{\mathbb{N}} \mid \forall k \geq 0,\ \boldsymbol{x} \leq T^k(\boldsymbol{s}) \leq \boldsymbol{y}\},$$

where $\leq$ denotes the lexicographic order on $\{0,1\}^{\mathbb{N}}$ induced by $0 < 1$. The *lexicographic world* $\mathcal{L}$ is defined by

$$\mathcal{L} := \{(\mathbf{x}, \mathbf{y}) \in \{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}} \mid \Sigma_{\mathbf{x},\mathbf{y}} \neq \varnothing\}.$$

Moreover, by [16, Lemma 2.1], we have

$$\mathcal{L} = \{(\boldsymbol{u}, \boldsymbol{v}) \in \{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}} \mid \boldsymbol{v} \geq \phi(\boldsymbol{u})\},$$

where $\phi : \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$ is the map defined by

$$\phi(\boldsymbol{x}) := \inf\{\boldsymbol{y} \in \{0,1\}^{\mathbb{N}} \mid \Sigma_{\boldsymbol{x},\boldsymbol{y}} \neq \varnothing\}.$$

Trivially, $\phi(1\boldsymbol{x}) = 1^\infty = 111\cdots$ for any sequence $\boldsymbol{x} \in \{0,1\}^{\mathbb{N}}$.

In [16], Gan showed that for any sequence $\boldsymbol{u} \in \{0,1\}^{\mathbb{N}}$, the set $\Sigma_{0\boldsymbol{u},1\boldsymbol{u}}$ is not empty, *i.e.*, there exists a sequence $\boldsymbol{s} \in \{0,1\}^{\mathbb{N}}$ such that $0\boldsymbol{u} \leq T^k(\boldsymbol{s}) \leq 1\boldsymbol{u}$ for all $k \geq 0$ (see [16, Lemma 4.2]). Furthermore, the sequence $\phi(0\boldsymbol{u})$ has the foregoing property (by [16, Theorem 3.4]) and it is a Sturmian or periodic balanced sequence with the property that $T^k(\phi(0\boldsymbol{u})) \leq \phi(0\boldsymbol{u})$ for all $k \geq 0$ (see [16, Theorem 4.6]). Moreover, by [16, Lemma 5.4], the set $\Sigma_{0\boldsymbol{u},1\boldsymbol{u}}$ contains a unique Sturmian or periodic balanced sequence satisfying $T^k(\boldsymbol{s}) \leq \boldsymbol{s}$ for all $k \geq 1$. We deduce from these remarks that, for any sequence $\boldsymbol{s} \in \{0,1\}^{\mathbb{N}}$, if $\boldsymbol{s} = \phi(0\boldsymbol{u})$ for some sequence $\boldsymbol{u} \in \{0,1\}^{\mathbb{N}}$, then $\boldsymbol{s}$ is the unique Sturmian or periodic balanced sequence satisfying $0\boldsymbol{u} \leq T^k(\boldsymbol{s}) \leq 1\boldsymbol{u}$ and $T^k(\boldsymbol{s}) \leq \boldsymbol{s}$ for all $k \geq 0$. The converse of this statement also holds by [16, Corollary 5.6]. These observations establish Gan's main theorem (see below), which shows in particular that any element in the image of ϕ is a Sturmian or periodic balanced sequence in $\{0,1\}^{\mathbb{N}}$ (and such sequences are the lexicographically greatest amongst their shifts).

Theorem 11. ([16, Theorem 1.1]) *For any sequence $\boldsymbol{s} \in \{0,1\}^{\mathbb{N}}$, the following conditions are equivalent:*

(i) *$\boldsymbol{s} = \phi(0\boldsymbol{u})$ for some sequence $\boldsymbol{u} \in \{0,1\}^{\mathbb{N}}$;*

(ii) *$\boldsymbol{s}$ is the unique Sturmian or periodic balanced sequence satisfying $0\boldsymbol{u} \leq T^k(\boldsymbol{s})$, $T^k(\boldsymbol{s}) \leq 1\boldsymbol{u}$ and $T^k(\boldsymbol{s}) \leq \boldsymbol{s}$ for all $k \geq 0$.*

Note. "Sturmian" in Gan's paper corresponds to what is called here (and classically) "Sturmian or periodic balanced".

Remark 12. It is well-known that the closure of the *shift-orbit* of a characteristic Sturmian sequence $\boldsymbol{s}$ (*i.e.*, the closure of $\{T^k(\boldsymbol{s}), k \geq 0\}$, denoted by $\overline{\mathcal{O}}(\boldsymbol{s})$) is precisely the set of all Sturmian sequences having the same slope as $\boldsymbol{s}$ (see for instance [19, Propositions 2.1.25 and 2.1.18], or [21]). In view of this fact, Gan's result can be strengthened using Theorem 9 as follows. If the sequence $\boldsymbol{s} := \phi(0\boldsymbol{u})$ is Sturmian, then $\boldsymbol{u}$ is the unique characteristic Sturmian sequence in $\overline{\mathcal{O}}(\boldsymbol{s})$, in which case $\boldsymbol{s} = 1\boldsymbol{u}$.

We will further strengthen Gan's result by describing $\phi(0\boldsymbol{u})$ for any given sequence $\boldsymbol{u} \in \{0,1\}^{\mathbb{N}}$. In particular, we will show that when $\boldsymbol{u}$ contains both 0's and 1's and is not a characteristic Sturmian sequence, there exists a unique pair of characteristic periodic balanced sequences $\boldsymbol{s}$ and $\boldsymbol{s}'$ of (rational) slope $p/q \in (0,1)$, with $\gcd(p,q) = 1$, such that $\boldsymbol{s}' \leq \boldsymbol{u} \leq \boldsymbol{s}$, in which case $\phi(0\boldsymbol{u}) = 1\boldsymbol{s}'$. Moreover, the sequences $\boldsymbol{s}$, $\boldsymbol{s}'$ can be explicitly determined in terms of $\boldsymbol{u}$.

With the same notation as in the introduction, our main combinatorial theorem reads as follows.

Theorem 13. *Let $\boldsymbol{u}$ be a sequence in $\{0,1\}^{\mathbb{N}}$.*

(i) $\phi(1\boldsymbol{u}) = 1^\infty$.

(ii) *If $\boldsymbol{u} \in \{0^\infty, 1^\infty\}$, then $\phi(0\boldsymbol{u}) = \boldsymbol{u}$.*

(iii) *If $\boldsymbol{u}$ is a characteristic Sturmian sequence, then $\phi(0\boldsymbol{u}) = 1\boldsymbol{u}$. Furthermore, $1\boldsymbol{u}$ is the unique Sturmian sequence in $\{0,1\}^{\mathbb{N}}$ satisfying $0\boldsymbol{u} \leq T^k(1\boldsymbol{u}) \leq 1\boldsymbol{u}$ for all $k \geq 0$.*

(iv) *If $\boldsymbol{u}$ is a characteristic periodic balanced sequence of rational slope p/q in $(0,1)$, with $\gcd(p,q) = 1$, then $\phi(0\boldsymbol{u}) = \boldsymbol{s}'_{p/q,0} = (1w_{p,q}0)^\infty$.*

(v) *If $\boldsymbol{u}$ does not take any of the forms given in parts* (ii)–(iv), *then there exists a unique pair of coprime integers p, q, with $0 < p < q$, such that*

$$(w_{p,q}01)^\infty < \boldsymbol{u} < (w_{p,q}10)^\infty,$$

in which case $\phi(0\boldsymbol{u}) = \boldsymbol{s}'_{p/q,0} = (1w_{p,q}0)^\infty$.

Moreover, in cases (iv) *and* (v), *$\phi(0\boldsymbol{u})$ is the unique periodic balanced sequence in $\{0,1\}^{\mathbb{N}}$ satisfying $0\boldsymbol{u} \leq T^k(\phi(0\boldsymbol{u})) \leq \phi(0\boldsymbol{u})$ for all $k \geq 0$.*

In the next section, we will recall some generalities about Sturmian and periodic balanced sequences. (For more on Sturmian sequences, the reader can consult, *e.g.*, [19, Chapter 2].) The proof of Theorem 13 is given in Section 4, and a corollary is stated in Section 5. Lastly, in Section 6, we show how to determine the "central word" $w_{p,q}$ such that $\phi(0\boldsymbol{u}) = (1w_{p,q}0)^\infty$ for any "generic" sequence $\boldsymbol{u}$ falling into case (v) of Theorem 13 above.

3. Sturmian and periodic balanced sequences

In what follows, we will use the following notation and terminology from combinatorics on words (see, *e.g.*, [19]). Let $w = x_1x_2\cdots x_m$ be a word over a finite non-empty alphabet $\mathcal{A}$ (where each x_i is a letter in $\mathcal{A}$). The *length* of w, denoted by $|w|$, is equal to m. The *empty word* is the unique word of length 0, denoted by ε. The number of occurrences of a letter x in w is denoted by $|w|_x$. The *reversal* of w is defined by $\tilde{w} = x_m\cdots x_2x_1$, and by convention $\varepsilon = \tilde{\varepsilon}$. If $w = \tilde{w}$, then w is called a *palindrome*. An integer $\ell \geq 1$ is said to be a *period* of w if, for all i, j with $1 \leq i, j \leq m$, $i \equiv j$

$(\bmod\ \ell)$ implies $x_i = x_j$. Note that any integer $\ell \geq |w|$ is a period of w with this definition. The word w is said to be *primitive* if it is not a power of a shorter word, *i.e.*, if $w = u^n$, then $n = 1$. A finite word z is said to be a *factor* of w if $z = x_i x_{i+1} \cdots x_j$ for some i, j, with $1 \leq i \leq j \leq m$. Similarly, a *factor* of a sequence $\boldsymbol{s} := s_0 s_1 s_2 s_3 \cdots$ is any finite word of the form $s_i s_{i+1} \cdots s_j$, with $i \leq j$.

Recall from the introduction that the *shift map* T is defined on sequences as follows: if $\boldsymbol{s} := (s_n)_{n \geq 0}$ then $T(\boldsymbol{s}) = T((s_n)_{n \geq 0}) := (s_{n+1})_{n \geq 0}$. This operator naturally extends to finite words as a *circular shift* by defining $T(xw) = wx$ for any letter x and finite word w.

Under the operation of concatenation, the set $\mathcal{A}^*$ of all finite words over $\mathcal{A}$ is a *free monoid* with identity element ε and set of generators $\mathcal{A}$. If x is a letter, then we use x^* to denote $\{x\}^*$, the set of all finite powers of x. From now on, all words and sequences will be over the alphabet $\{0, 1\}$.

3.1. Balanced sequences

All Sturmian sequences are "balanced" in the following sense (see for instance [22, 10, 5, 6, 19]).

Definition 14. A finite word or sequence w over $\{0, 1\}$ is said to be *balanced* if, for any two factors u and v of w, with $|u| = |v|$, we have $||u|_1 - |v|_1| \leq 1$ (or equivalently $||u|_0 - |v|_0| \leq 1$).

Recall from Remark 3 that Sturmian sequences are aperiodic. Morse, Hedlund, and Coven [22, 10] proved that the Sturmian sequences are precisely the aperiodic balanced sequences on two letters (also see [19, Theorem 2.1.3]). "Periodic balanced sequences" (as specified in Definition 1) are also balanced in the sense of the above definition (which justifies their name); moreover, they constitute the set of all periodic balanced sequences on two letters (see [19, Lemma 2.1.15] or [24]).

3.2. Characteristic Sturmian sequences

In [11], characteristic Sturmian sequences were characterized using iterated palindromic closure, defined as follows. The *palindromic (right-)closure* of a finite word w, denoted by $w^{(+)}$, is the (unique) shortest palindrome beginning with w. That is, if $w = uv$ where v is the longest palindromic suffix of w, then $w^{(+)} := uv\tilde{u}$. For example, $(011)^{(+)} = 0110$. The *iterated palindromic closure function*, denoted by Pal, is defined by iteration of the palindromic right-closure operator (see, *e.g.*, [17]). More precisely, Pal is defined recursively as follows: set $Pal(\varepsilon) = \varepsilon$, and for any word w and letter x, define $Pal(wx) := (Pal(w)x)^{(+)}$. For example,

$$Pal(011) = (Pal(01)1)^{(+)} = (0101)^{(+)} = 01010.$$

Note that Pal is injective; and moreover, it is clear from the definition that $Pal(w)$ is a prefix of $Pal(wx)$ for any word w and letter x. Hence, if v is a prefix of w, then $Pal(v)$ is a prefix of $Pal(w)$. The following theorem provides a combinatorial description of characteristic Sturmian sequences in terms of Pal.

Theorem 15. ([11]) *For any sequence* $\boldsymbol{s} \in \{0, 1\}^{\mathbb{N}}$, *the following properties are equivalent:*

(i) $\boldsymbol{s}$ *is a characteristic Sturmian sequence.*

(ii) *There exists a (unique) sequence*

$$\Delta := x_0x_1x_2x_3\ldots \in \{0,1\}^{\mathbb{N}} \setminus (\{0,1\}^*0^\infty \cup \{0,1\}^*1^\infty)$$

(i.e., not eventually constant), called the directive sequence *of* $\boldsymbol{s}$*, such that*

$$\boldsymbol{s} = \lim_{n\to\infty} Pal(x_0x_1x_2\cdots x_n) = Pal(\Delta).$$

Example 16. Recall from Example 2 that the (binary) Fibonacci sequence

$$\boldsymbol{f} = 01001010010\cdots$$

is the characteristic Sturmian sequence $\boldsymbol{c}_\alpha$ with $\alpha = (3-\sqrt{5})/2$; it has directive sequence $(01)^\infty$. That is,

$$\boldsymbol{f} = Pal(0101\cdots) = \underline{0}1\underline{0}0\underline{1}01\underline{0}010\cdots,$$

where the underlined letters indicate at which points palindromic closure is applied. The simple continued fraction expansion of $\alpha = (3-\sqrt{5})/2$ is $[0;2,1,1,1,\ldots]$. More generally, if $\alpha \in (0,1)$ is an irrational number with simple continued fraction expansion $[0; d_1+1, d_2, d_3, d_4, \ldots]$, where $d_1 \geq 0$ and $d_i \geq 1$ for $i > 1$, then $\boldsymbol{c}_\alpha = Pal(0^{d_1}1^{d_2}0^{d_3}1^{d_4}\cdots)$ (see [15, 7] and also [2, p. 206]).

3.3. Characteristic periodic balanced sequences

We will now recall some known combinatorial descriptions of the characteristic periodic balanced sequences in $\{0,1\}^{\mathbb{N}}$ (see Proposition 17 and Remark 18 below).

Let us first recall from the introduction that the characteristic balanced sequences of slopes 0 and 1 are $\boldsymbol{c}_0 = 0^\infty$ and $\boldsymbol{c}_1 = 1^\infty$, respectively. For all other rational slopes $p/q \in (0,1)$, with $\gcd(p,q) = 1$, there are exactly two characteristic periodic balanced sequences of slope p/q, given by

$$\boldsymbol{s}_{p/q,p/q} = T(\boldsymbol{s}_{p/q,0}) = (w_{p,q}10)^\infty \quad \text{and} \quad \boldsymbol{s}'_{p/q,p/q} = T(\boldsymbol{s}'_{p/q,0}) = (w_{p,q}01)^\infty,$$

where $w_{p,q}$ is a word of length $q-2$ in $\{0,1\}^*$. For example, with $p = 2$ and $q = 5$, we obtain the following two characteristic periodic balanced sequences of slope $2/5$:

$$\boldsymbol{s}_{2/5,2/5} = (01010)^\infty \quad \text{and} \quad \boldsymbol{s}'_{2/5,2/5} = (01001)^\infty,$$

where $w_{2,5} = 010$. Note that $w_{2,5}$ is a palindrome and $|w_{2,5}10|_1 = |w_{2,5}01|_1 = 2 = p$. More generally, one can verify that all words $w_{p,q}$ are palindromes and

$$|w_{p,q}10|_1 = |w_{p,q}01|_1 = p.$$

Furthermore, the words $w_{p,q}10$ and $w_{p,q}01$ (which have length q) are primitive since $\gcd(p,q) = 1$. Hereafter, the word $w_{p,q}$ will be called the *central word of slope* p/q; it is the unique central word of length $q-2$ containing $p-1$ occurrences of 1.

Note. The set of all central words of slope $p/q \in (0,1)$, where p and q are coprime integers, coincides with the family of "central words" in $\{0,1\}^*$ as defined in [19, Chapter 2] (in particular, see [19, Theorem 2.2.11 and Proposition 2.2.12]).

The following proposition collects together some equivalent definitions of central words. For many more, see the nice survey [3].

Proposition 17. *For any word* $w \in \{0,1\}^*$*, the following properties are equivalent:*

(i) w *is a central word;*

(ii) $0w1$ *and* $1w0$ *are balanced (see* [12]*);*

(iii) $w = Pal(v)$ *for some word* $v \in \{0,1\}^*$ *(see* [12, 11]*);*

(iv) w *has two periods* ℓ *and* m *such that* $\gcd(\ell, m) = 1$ *and* $|w| = \ell + m - 2$ *(see* [10, 12]*);*

(v) $w \in 0^* \cup 1^* \cup (P \cap P10P)$*, where* P *is the set of all palindromes in* $\{0,1\}^*$ *(see* [12]*);*

(vi) $w \in 0^* \cup 1^*$*, or there exists a unique pair of words* $w_1, w_2 \in \{0,1\}^*$ *such that* w *satisfies the equation* $w = w_1 01 w_2 = w_2 10 w_1$ *(see* [12, 11]*).*

Moreover, in (vi)*,* w_1 *and* w_2 *are central words,* $\ell_1 := |w_1| + 2$ *and* $\ell_2 := |w_2| + 2$ *are coprime periods of* w*, and* $\min\{\ell_1, \ell_2\}$ *is the minimal period of* w *(see* [9]*).*

Note. $P \cap (P10P) = P \cap (P01P)$.

Furthermore, by [19, Proposition 2.2.12], the central word $w_{p,q}$ of slope p/q is the central word with coprime periods ℓ and m, where $\ell + m = q$ and $mp \equiv 1 \pmod q$. For example, the central word $w_{2,5} = 010$ has coprime periods $\ell = 2$ and $m = 3$, where $2 + 3 = q$ and $mp = 6 \equiv 1 \pmod 5$. Also note that $w_{1,q} = 0^{q-2}$ and $w_{q-1,q} = 1^{q-2}$; in particular $w_{1,2} = \varepsilon$.

Remark 18. Let p and q be coprime integers, with $0 < p < q$. Then p/q has two distinct simple continued fraction expansions:

$$p/q = [0; d_1 + 1, \ldots, d_n, 1] \quad \text{and} \quad p/q = [0; d_1 + 1, \ldots, d_n + 1],$$

where $d_1 \geq 0$ and $d_i \geq 1$ for $i > 1$. It is known (see, *e.g.*, [3, Proposition 27]) that the word $v \in \{0,1\}^*$ such that $w_{p,q} = Pal(v)$ takes the form $v = 0^{d_1} 1^{d_2} 0^{d_3} \cdots x^{d_n}$, where $x = 0$ if n is odd, and $x = 1$ if n is even. For example, $2/5 = [0; 2, 1, 1] = [0; 2, 2]$ and $w_{2,5} = 010 = Pal(01)$. Moreover, as in the case of characteristic Sturmian sequences (see Theorem 15 and Example 16), the two characteristic periodic balanced sequences of slope p/q can be obtained by iterated palindromic closure. More precisely, with the above notation, we have

$$(w_{p,q} xy)^\infty = Pal(0^{d_1} 1^{d_2} \cdots x^{d_n + 1} y^\infty)$$

and

$$(w_{p,q} yx)^\infty = Pal(0^{d_1} 1^{d_2} \cdots x^{d_n} y x^\infty),$$

where $\{x, y\} = \{0, 1\}$. For example,

$$(w_{2,5} 10)^\infty = (01010)^\infty = Pal(0110^\infty)$$

and

$$(w_{2,5} 01)^\infty = (01001)^\infty = Pal(0101^\infty).$$

In [23], Pirillo proved that a word $w \in \{0,1\}^*$ is a palindromic prefix of some characteristic Sturmian sequence in $\{0,1\}^{\mathbb{N}}$, *i.e.*, $w = Pal(v)$ for some $v \in \{0,1\}^*$ (see Theorem 15), if and only if $w01$ is a circular shift of $w10$. From this fact and Proposition 17, we thus deduce the following result.

Proposition 19. *A word* $w \in \{0,1\}^*$ *is central if and only if* $w01$ *is a circular shift of* $w10$*.*

Consequently, for any two coprime integers p and q, with $0 < p < q$, the two characteristic periodic balanced sequences of slope p/q, namely $\boldsymbol{s}_{p/q,p/q} = (w_{p,q}10)^\infty$ and $\boldsymbol{s}'_{p/q,p/q} = (w_{p,q}01)^\infty$, are shifts of each other, and therefore they have the same set of factors.

Remark 20. Recall from Remark 12 that the closure of the shift-orbit of a characteristic Sturmian sequence $\boldsymbol{s}$ is precisely the set of all Sturmian sequences having the same slope as $\boldsymbol{s}$. Moreover, the (Sturmian) sequences in $\overline{\mathcal{O}}(\boldsymbol{s})$ are exactly the sequences that have the same set of factors as $\boldsymbol{s}$ (see for instance [19, Propositions 2.1.25 and 2.1.18], or [21]). Likewise, the shift-orbit of a characteristic periodic balanced sequence $\boldsymbol{u}$ consists of all the periodic balanced sequences with the same set of factors (and also the same slope) as $\boldsymbol{u}$. However, in contrast to the aperiodic case, we deduce from Proposition 19 that, if $\boldsymbol{u}$ is a characteristic periodic balanced sequence containing both 0's and 1's, then $\overline{\mathcal{O}}(\boldsymbol{u})$ contains exactly two distinct characteristic periodic balanced sequences, which take the form $(w01)^\infty$ and $(w10)^\infty$, where w is the central word having the same slope as $\boldsymbol{u}$.

The following useful result is due to de Luca [11]; in particular, see [11, Remark 1 and Proposition 9] and also [9, Lemma 5].

Proposition 21. *Let w be a central word in $\{0,1\}^*$. If $w = w_1 01 w_2 = w_2 10 w_1$, where w_1 and w_2 are (central) words, then*

$$Pal(w0) = (w0)^{(+)} = w_2 10 w_1 01 w_2 \quad \textit{and} \quad Pal(w1) = (w1)^{(+)} = w_1 01 w_2 10 w_1.$$

We end this section with a result (Corollary 23 below) that will be particularly useful in the proof of our main combinatorial theorem. Let us first recall that a non-empty finite word v over an alphabet $\mathcal{A}$ is said to be a *Lyndon word* (resp. *anti-Lyndon word*) if v is a primitive word that is lexicographically less (resp. lexicographically greater) than all of its circular shifts with respect to a given total order on $\mathcal{A}$.

Proposition 22. ([4, Theorem 3.2 and Corollary 3.1]) *A non-empty finite word $v \in \{0,1\}^*$ is a balanced Lyndon word (resp. balanced anti-Lyndon word) with respect to the lexicographic order if and only if $v = 0w1$ (resp. $v = 1w0$) for some central word $w \in \{0,1\}^*$.*

As a direct consequence of the above proposition, we have the following result.

Corollary 23. *For any central word $w \in \{0,1\}^*$, the (primitive) words $0w1$ and $1w0$ are the lexicographically least and greatest words amongst their circular shifts.*

4. Proof of Theorem 13

Assertions (i) and (ii) are straightforward (see [16, Lemma 2.4]). In order to prove the other assertions, we first recall the inequalities that are equivalent to $\boldsymbol{s} = \phi(0\boldsymbol{u})$ from Theorem 11:

$$0\boldsymbol{u} \le T^k(\boldsymbol{s}) \le 1\boldsymbol{u} \quad \text{and} \quad T^k(\boldsymbol{s}) \le \boldsymbol{s}, \quad \text{for all } k \ge 0. \tag{1}$$

Assertion (iii) is a consequence of Gan's result (Theorem 11) together with Theorem 9 (see Remark 12).

We will now prove assertions (iv) and (v). Suppose that $\boldsymbol{u}$ is not a characteristic Sturmian sequence and that $\boldsymbol{u}$ contains both 0's and 1's. Then we know from Theorem 11 that $\boldsymbol{s} := \phi(0\boldsymbol{u})$ is a periodic balanced sequence satisfying the inequalities in (1). Indeed, $\boldsymbol{s}$ cannot be Sturmian, for otherwise $\boldsymbol{u}$ would be a characteristic Sturmian sequence by Remark 12.

By Remark 20, $\overline{\mathcal{O}}(\boldsymbol{s})$ contains exactly two distinct characteristic periodic balanced sequences, given by

$$\boldsymbol{s}_{01} := (w01)^\infty \quad \text{and} \quad \boldsymbol{s}_{10} := (w10)^\infty,$$

where $w \in \{0,1\}^*$ is the central word with the same slope as $\boldsymbol{s}$. We now deduce from Corollary 23 that the lexicographically least sequence in $\overline{\mathcal{O}}(\boldsymbol{s})$ is

$$(0w1)^\infty = 0(w10)^\infty = 0\boldsymbol{s}_{10}$$

and the lexicographically greatest sequence in $\overline{\mathcal{O}}(\boldsymbol{s})$ is

$$(1w0)^\infty = 1(w01)^\infty = 1\boldsymbol{s}_{01}.$$

Hence,

$$0\boldsymbol{s}_{10} \leq T^k(\boldsymbol{s}) \leq 1\boldsymbol{s}_{01}, \quad \text{for all } k \geq 0. \tag{2}$$

Moreover, since $\boldsymbol{s}$ is the lexicographically greatest sequence in its shift-orbit (by the second inequality in (1)), we have $\boldsymbol{s} = (1w0)^\infty = 1\boldsymbol{s}_{01}$.

We will now show that $\boldsymbol{s}_{01} \leq \boldsymbol{u} \leq \boldsymbol{s}_{10}$. Since $0\boldsymbol{s}_{10}$ and $1\boldsymbol{s}_{01}$ are the lexicographically least and greatest elements in $\overline{\mathcal{O}}(\boldsymbol{s})$, the inequalities in (1) imply that

$$0\boldsymbol{u} \leq 0\boldsymbol{s}_{10} \quad \text{and} \quad 1\boldsymbol{s}_{01} \leq 1\boldsymbol{u}.$$

Hence $\boldsymbol{s}_{01} \leq \boldsymbol{u} \leq \boldsymbol{s}_{10}$; that is, $(w01)^\infty \leq \boldsymbol{u} \leq (w10)^\infty$.

Furthermore, we note that there does not exist another central word z such that $(z01)^\infty \leq \boldsymbol{u} \leq (z10)^\infty$. For if so, then the set

$$[0\boldsymbol{u}, 1\boldsymbol{u}] := \{\boldsymbol{s} \in \{0,1\}^{\mathbb{N}} \mid 0\boldsymbol{u} \leq \boldsymbol{s} \leq 1\boldsymbol{u}\}$$

would contain the periodic balanced sequences

$$0(z10)^\infty = (0z1)^\infty \quad \text{and} \quad 1(z01)^\infty = (1z0)^\infty,$$

and hence all of the shifts of the characteristic periodic balanced sequence $(z01)^\infty$, since the former two sequences are the lexicographically least and greatest sequences in the shift-orbit of $(z01)^\infty$ (by Proposition 19 and Corollary 23). But by [16, Lemma 5.4], the set $[0\boldsymbol{u}, 1\boldsymbol{u}]$ contains a unique periodic balanced shift-orbit. Therefore, since $\overline{\mathcal{O}}((w01)^\infty) \subseteq [0\boldsymbol{u}, 1\boldsymbol{u}]$, we must have $z = w$. We have thus established the following lemma.

Lemma 24. *Suppose that $\boldsymbol{u}$ is a sequence in $\{0,1\}^{\mathbb{N}} \setminus \{0^\infty, 1^\infty\}$ that is not characteristic Sturmian. Then there exists a unique central word $w \in \{0,1\}^*$ such that $(w01)^\infty \leq \boldsymbol{u} \leq (w10)^\infty$. Moreover, $\phi(0\boldsymbol{u}) = (1w0)^\infty$.*

Assertions (iv) and (v) are direct consequences of the above lemma, and the last statement in the theorem follows from Theorem 11.

5. A corollary

By Theorem 15, the set of all characteristic Sturmian sequences in $\{0,1\}^{\mathbb{N}}$ is given by

$$\mathcal{S} = \{\boldsymbol{u} \in \{0,1\}^{\mathbb{N}} \mid \exists \boldsymbol{v} \in \{0,1\}^{\mathbb{N}} \setminus (\{0,1\}^* 0^\infty \cup \{0,1\}^* 1^\infty)\,,\ \boldsymbol{u} = Pal(\boldsymbol{v})\}.$$

And it follows from Proposition 17 that the set of all characteristic periodic balanced sequences in $\{0,1\}^{\mathbb{N}}$ is given by

$$\mathcal{P} = \{0^\infty, 1^\infty\} \cup \mathcal{P}_{01} \cup \mathcal{P}_{10},$$

where

$$\mathcal{P}_{01} := \{\boldsymbol{u} \in \{0,1\}^{\mathbb{N}} \mid \exists v \in \{0,1\}^*,\ \boldsymbol{u} = (Pal(v)01)^\infty\}$$

and

$$\mathcal{P}_{10} := \{\boldsymbol{u} \in \{0,1\}^{\mathbb{N}} \mid \exists v \in \{0,1\}^*,\ \boldsymbol{u} = (Pal(v)10)^\infty\}.$$

Given a characteristic periodic balanced sequence in $\{0,1\}^{\mathbb{N}}$ of the form

$$\boldsymbol{s} := (Pal(v)xy)^\infty,$$

where $v \in \{0,1\}^*$ and $\{x,y\} = \{0,1\}$, we let $\bar{\boldsymbol{s}}$ denote the other characteristic periodic balanced sequence in the shift-orbit of $\boldsymbol{s}$, *i.e.*, $\bar{\boldsymbol{s}} := (Pal(v)yx)^\infty$ (see Remark 20).

As an immediate consequence of Theorem 13, we obtain the following description of the lexicographic world.

Corollary 25. *We have* $\mathcal{L} = \{(01^\infty, 1^\infty)\} \cup \mathcal{L}_0 \cup \mathcal{L}_1 \cup \mathcal{L}_{01} \cup \mathcal{L}_{10} \cup \mathcal{L}_*$*, where*

$$\begin{cases} \mathcal{L}_0 := \{(0^\infty, \boldsymbol{v}) \mid \boldsymbol{v} \in \{0,1\}^{\mathbb{N}}\}, \\ \mathcal{L}_1 = \{(1\boldsymbol{u}, 1^\infty) \mid \boldsymbol{u} \in \{0,1\}^{\mathbb{N}}\}, \\ \mathcal{L}_{01} = \{(0\boldsymbol{u}, \boldsymbol{v}) \in 0(\mathcal{S} \cup \mathcal{P}_{01}) \times \{0,1\}^{\mathbb{N}} \mid \boldsymbol{v} \geq 1\boldsymbol{u}\}, \\ \mathcal{L}_{10} = \{(0\boldsymbol{u}, \boldsymbol{v}) \in 0\mathcal{P}_{10} \times \{0,1\}^{\mathbb{N}} \mid \boldsymbol{v} \geq 1\bar{\boldsymbol{u}}\}, \\ \mathcal{L}_* = \{(0\boldsymbol{u}, \boldsymbol{v}) \in (\{0,1\}^{\mathbb{N}} \setminus 0(\mathcal{S} \cup \mathcal{P})) \times \{0,1\}^{\mathbb{N}} \mid \exists \boldsymbol{s} \in \mathcal{P}_{01}, \boldsymbol{s} \leq \boldsymbol{u} \leq \bar{\boldsymbol{s}}, \boldsymbol{v} \geq 1\boldsymbol{s}\}. \end{cases}$$

6. How to determine $\phi(0\boldsymbol{u})$ for a generic sequence $\boldsymbol{u}$

The following theorem provides a method for determining the central word w such that $\phi(0\boldsymbol{u}) = (1w0)^\infty$ for any "generic" sequence $\boldsymbol{u} \in \{0,1\}^{\mathbb{N}}$ falling into case (v) of Theorem 13. Hereafter, a prefix of $\boldsymbol{u}$ that is a central word is called a *central prefix* of $\boldsymbol{u}$.

Theorem 26. *Suppose $\boldsymbol{u}$ is a sequence in $\{0,1\}^{\mathbb{N}} \setminus \{0^\infty, 1^\infty\}$ that is neither a characteristic Sturmian sequence nor a characteristic periodic balanced sequence. Let v be the longest central prefix of $\boldsymbol{u}$. Then v is finite ($v \neq \varepsilon$), and $\phi(0\boldsymbol{u})$ is determined as follows:*

(i) *If $v = 1^k$ for some $k \geq 1$, then $\phi(0\boldsymbol{u}) = (1^k0)^\infty = (1w_{p,q}0)^\infty$, where $p = k$ and $q = k+1$.*

(ii) *If $v = 0^k$ for some $k \geq 1$, then $\phi(0\boldsymbol{u}) = (10^k)^\infty = (1w_{p,q}0)^\infty$, where $p = 1$ and $q = k+1$.*

(iii) *Suppose that v contains both 0's and 1's. Let v_1 and v_2 be the unique pair of central words such that $v = v_1 01 v_2 = v_2 10 v_1$, where $\ell_1 := |v_1| + 2$ and $\ell_2 := |v_2| + 2$ are coprime periods of v. Consider the prefix of length $2|v| + 4$ of $\boldsymbol{u}$, namely the prefix $vxyz$, where $x, y \in \{0, 1\}$ and $|z| = |v| + 2$.*

(a) *If either $xy = 01$ and $z > v01$, or $xy = 10$ and $z < v10$, then*

$$\phi(0\boldsymbol{u}) = (1v0)^\infty = (1w_{p,q}0)^\infty,$$

where $p = |v|_1 + 1$ and $q = |v| + 2 = \ell_1 + \ell_2$.

(b) *If either $xy = 01$ and $z < v01$, or $xy = 00$, then*

$$\phi(0\boldsymbol{u}) = (1v_2 0)^\infty = (1w_{p,q}0)^\infty,$$

where $p = |v_2|_1 + 1$ and $q = \ell_2$.

(c) *If either $xy = 10$ and $z > v10$, or $xy = 11$, then*

$$\phi(0\boldsymbol{u}) = (1v_1 0)^\infty = (1w_{p,q}0)^\infty,$$

where $p = |v_1|_1 + 1$ and $q = \ell_1$.

Note. In assertion (iii), it cannot happen that $z = vxy$ when $x \neq y$. For instance, if $xy = 01$ and $z = v01$, then $\boldsymbol{u}$ would begin with the word

$$v01v01 = v_2 10 v_1 01 v_2 10 v_1 01,$$

where the prefix $v_2 10 v_1 01 v_2 = v01v_2$ is a central word, by Propositions 17 and 21. But then $\boldsymbol{u}$ has a central prefix longer than v; thus $z \neq v01$. Similarly, if $xy = 10$, then $z \neq v10$.

The following lemma is needed for the proof of Theorem 26.

Lemma 27. *Suppose that v is a central word in $\{0,1\}^* \setminus (0^* \cup 1^*)$. Let v_1 and v_2 be the unique pair of central words such that v satisfies the equation $v = v_1 01 v_2 = v_2 10 v_1$. Then $v01v_2$ (resp. $v10v_1$) is a prefix of the characteristic periodic balanced sequence $(v_2 10)^\infty$ (resp. $(v_1 01)^\infty$).*

Note. By Propositions 17 and 21, the words $v01v_2$ and $v10v_1$ are central words since $v01v_2 = Pal(v0)$ and $v10v_1 = Pal(v1)$.

Proof of Lemma 27. We prove only that $(v_2 10)^\infty$ begins with the central word $Pal(v0) = v01v_2$, since the proof of the other case is very similar. By Proposition 17, $\ell_1 := |v_1| + 2$ and $\ell_2 := |v_2| + 2$ are coprime periods of v, where $|v| = \ell_1 + \ell_2 - 2$. In particular, since $\ell_2 = |v_2 10|$ is a period of v with $\ell_2 < |v|$, there exists an integer $k \geq 1$ such that $v = (v_2 10)^k v_2'$, where v_2' is a (possibly empty) prefix of $v_2 10$, in which case $v_1 = (v_2 10)^{k-1} v_2'$ since $v = v_2 10 v_1$. Moreover, since v is a palindrome, $\tilde{v}_2'$ is a prefix of v, and therefore $v_2' = \tilde{v}_2'$, *i.e.*, v_2' is a palindrome. Furthermore, $v_2'01$ is a prefix of v since its reversal $10v_2'$ is a suffix of v. We will now show that the central word $Pal(v0) = v01v_2$ is a prefix of the characteristic periodic balanced sequence $(v_2 10)^\infty$ by considering five different cases according to the length of the palindrome v_2'.

Case 1: If $v_2' = v_2 10$, then $v = (v_2 10)^{k+1}$ and $v_1 = (v_2 10)^k$. Since v is a palindrome, $v_2 10$ is a palindrome and we have $v = (v_2 10)^{k+1} = (01 v_2)^{k+1}$. Therefore,

$$(v_2 10)^\infty = (v_2 10)^{k+1} \underbrace{01 v_2}_{v_2 10} (v_2 10)^\infty = v 01 v_2 (v_2 10)^\infty.$$

Thus, the central word $Pal(v0) = v01v_2$ is a prefix of $(v_2 10)^\infty$.

Case 2: If $v_2' = v_2 1$, then since v_2' and v_2 are palindromes, we have $v_2 1 = 1 v_2$, and hence v_2 is a power of 1; in particular, $v_2 = 1^{\ell_2 - 2}$. Therefore

$$\begin{aligned} (v_2 10)^\infty &= (v_2 10)^k v_2 10 v_2 10 (v_2 10)^\infty \\ &= \underbrace{(v_2 10)^k v_2 1}_{v} 0 \underbrace{1 1^{\ell_2 - 2}}_{1 v_2} 0 (v_2 10)^\infty \\ &= v 01 v_2 0 (v_2 10)^\infty \\ &= Pal(v0) 0 (v_2 10)^\infty. \end{aligned}$$

Case 3: If $v_2' = v_2$, then $v = (v_2 10)^k v_2$, and therefore $v_1 = (v_2 10)^{k-1} v_2$. But this implies that $\ell_1 = k \ell_2$, which is impossible since ℓ_1 and ℓ_2 are coprime integers greater than 1.

Case 4: If $v_2 = v_2' 0$, then since v_2 and v_2' are palindromes, we have $v_2' 0 = 0 v_2'$. Therefore v_2' (and hence v_2) is a power of 0; in particular, $v_2 = 0^{\ell_2 - 2}$. Thus

$$\begin{aligned} (v_2 10)^\infty &= (v_2 10)^k v_2 10 v_2 10 (v_2 10)^\infty \\ &= \underbrace{(v_2 10)^k v_2'}_{v} 01 \underbrace{v_2 0}_{0 v_2} 10 (v_2 10)^\infty \\ &= v 01 v_2 010 (v_2 10)^\infty \\ &= Pal(v0) 010 (v_2 10)^\infty. \end{aligned}$$

Note that we cannot have $v_2 = v_2' 1$ because $v_2' 01$ and v_2 are both prefixes of v.

Case 5: If $|v_2'| \le |v_2| - 2$, then $v_2 = v_2' 01 v_2''$ for some (possibly empty) word v_2'' in $\{0,1\}^*$, in which case $v = (v_2' 01 v_2'' 10)^k v_2'$. (Note that neither $v_2' 1$ nor $v_2' 00$ is a prefix of v_2 because $v_2' 01$ and v_2 are both prefixes of v.) Since v is a palindrome that begins with the palindrome $v_2 = v_2' 01 v_2''$ and therefore ends with $\tilde{v}_2 = v_2 = \tilde{v}_2'' 10 v_2'$, we see that $\tilde{v}_2'' 10 v_2' = v_2'' 10 v_2'$. Hence v_2'' is a palindrome. Moreover, v_2'' is a central word since v_2'' is a palindromic prefix (and also a palindromic suffix) of the central word v_2 and any palindromic prefix (or suffix) of a central word is central (see [12] or [19, Corollary 2.2.10]). Thus, by Proposition 17, v_2 satisfies the equation $v_2 = v_2'' 10 v_2' = v_2' 01 v_2''$.

Hence, we have

$$\begin{aligned}
(v_2 10)^\infty &= (v_2 10)^k \underbrace{v_2' 01 v_2'' 10}_{v_2 10} v_2 10 (v_2 10)^\infty \\
&= (v_2 10)^k v_2' 01 \underbrace{v_2'' 10 v_2' 01}_{v_2 01} v_2'' 10 (v_2 10)^\infty \\
&= v 01 v_2 01 v_2'' 10 (v_2 10)^\infty \\
&= Pal(v0) 01 v_2'' 10 (v_2 10)^\infty .
\end{aligned}$$

In all of the above cases (with the exception of the impossible case (3)), we have shown that the central word $Pal(v0) = v01v_2$ is a prefix of $(v_2 10)^\infty$, as required. $\square$

Proof of Theorem 26. Suppose that $\boldsymbol{u}$ is a sequence in $\{0,1\}^{\mathbb{N}} \setminus \{0^\infty, 1^\infty\}$ that is neither a characteristic Sturmian sequence nor a characteristic periodic balanced sequence. Then the longest central prefix of $\boldsymbol{u}$, say v, is non-empty since it could (at the very least) be a letter. Furthermore, v is finite; otherwise, if v were infinite, then $\boldsymbol{u}$ would be either a characteristic Sturmian sequence or a characteristic periodic balanced sequence (see Theorem 15 and Remark 18).

We know from Theorem 13 (or Lemma 24) that there exists a unique central word $w \in \{0,1\}^*$ such that $(w01)^\infty < \boldsymbol{u} < (w10)^\infty$, in which case $\phi(0\boldsymbol{u})$ is equal to the periodic balanced sequence $(1w0)^\infty$. We will show how to determine w in terms of the longest central prefix v. Note that w is either empty or a (palindromic) prefix of v, by the maximality of v.

First suppose that $v = x^k$ for some $x \in \{0,1\}$ and $k \geq 1$. Then by the maximality of v as a central prefix of $\boldsymbol{u}$, it follows that $\boldsymbol{u}$ begins with $x^k y = vy$, where $y \in \{0,1\}$ and $y \neq x$. Further, the prefix of length $2k+1$ of $\boldsymbol{u}$ takes the form $x^k y u$, where $|u| = k$ and $|u|_x \leq k-1$; otherwise $\boldsymbol{u}$ would begin with $x^k y x^k = Pal(x^k y)$, contradicting the fact that v $(= Pal(x^k))$ is the longest central prefix of $\boldsymbol{u}$. If $x = 1$, then we easily see that

$$(1^{k-1}01)^\infty < \boldsymbol{u} \, (= 1^k 0 u \cdots) < (1^k 0)^\infty,$$

where the latter inequality follows from the fact that $|u| = k$ and $u < 1^k$ (since u contains at most $k-1$ occurrences of the letter 1). Hence, it follows from Lemma 24 that $\phi(0\boldsymbol{u}) = (1^k 0)^\infty = (1w_{p,q}0)^\infty$, where $p = k$ and $q = k+1$. Similarly, if $x = 0$, we have

$$(0^k 1)^\infty < \boldsymbol{u} \, (= 0^k 1 u \cdots) < (0^{k-1}10)^\infty,$$

where the first inequality follows from the fact that $|u| = k$ and $0^k < u$ (since u contains at most $k-1$ occurrences of the letter 0). Therefore, by Lemma 24 again, $\phi(0\boldsymbol{u}) = (10^k)^\infty = (1w_{p,q}0)^\infty$, where $p = 1$ and $q = k+1$. We have thus proved assertions (i) and (ii) of the theorem.

Now suppose that the longest central prefix v of $\boldsymbol{u}$ contains both 0's and 1's. Then, by Proposition 17, there exists a unique pair of central words $v_1, v_2 \in \{0,1\}^*$ such that $v = v_1 01 v_2 = v_2 10 v_1$, where $\ell_1 := |v_1| + 2$ and $\ell_2 := |v_2| + 2$ are coprime periods of v, and $\min\{\ell_1, \ell_2\}$ is the minimal period of v. Consider the prefix of length $2|v| + 4$ of

$\boldsymbol{u}$, namely the prefix $vxyz$, where $x, y \in \{0,1\}$ and $|z| = |v| + 2$. We will now prove each of the cases (a), (b), and (c) of assertion (iii).

Case (a): Let us first suppose that $\boldsymbol{u}$ begins with $v01z$, where $|z| = |v01|$ and $z > v01$. Then it is easy to see that

$$(v01)^\infty < \boldsymbol{u} < (v10)^\infty.$$

Consequently, by Lemma 24, we have $\phi(0\boldsymbol{u}) = (1v0)^\infty$. Moreover, $v = w_{p,q}$, where $p = |v|_1 + 1$ and $q = |v| + 2 = \ell_1 + \ell_2$. Similarly, if $\boldsymbol{u}$ begins with $v10z$, where $|z| = |v10|$ and $z < v10$, then $(v01)^\infty < \boldsymbol{u} < (v10)^\infty$, and therefore $\phi(0\boldsymbol{u}) = (1v0)^\infty$ by Lemma 24.

Case (b): In this case, either $\boldsymbol{u}$ begins with $v00$ or $\boldsymbol{u}$ begins with $v01z$, where $|z| = |v01|$ and $z < v01$. Since $v_2 10$ is a prefix of v (which in turn is a prefix of $\boldsymbol{u}$), we have $(v_2 01)^\infty < \boldsymbol{u}$. Furthermore, by Lemma 27, the characteristic periodic balanced sequence $(v_2 10)^\infty$ begins with the central word $Pal(v0) = v01v_2$. Thus, if $\boldsymbol{u}$ begins with $v00$, then $\boldsymbol{u} < (v_2 10)^\infty$.

Alternatively, if $\boldsymbol{u}$ begins with $v01z$, where $|z| = |v01|$ and $z < v01$, then we will show that $\boldsymbol{u} < (v_2 10)^\infty$ by considering the prefix of length $|v| + \ell_2$ of $\boldsymbol{u}$, namely $v01z_2$, where $|z_2| = |v_2|$. We first note that $z_2 \leq v_2$ since v_2 is a prefix of v and z_2 is a prefix of z, where z and v satisfy $z < v01$. Furthermore, $z_2 < v_2$ (*i.e.*, $z_2 \neq v_2$). Otherwise, if $z_2 = v_2$, then $\boldsymbol{u}$ would begin with the central word $Pal(v0) = v01v_2$. But then $\boldsymbol{u}$ would have a central prefix that is longer than v; a contradiction. Therefore $z_2 < v_2$, and hence $\boldsymbol{u} < (v_2 10)^\infty$ since $(v_2 10)^\infty$ begins with $v01v_2$, where $v_2 > z_2$, as shown above.

Case (c): This case is symmetric to case (b). □

Example 28. The following examples demonstrate the computation of $\phi(0\boldsymbol{u})$ for sequences $\boldsymbol{u}$ in $\{0,1\}^{\mathbb{N}}$ that are neither characteristic Sturmian nor periodic and balanced. Where appropriate, the longest central prefix of the sequence is highlighted in boldface.

(1) The following two general facts can easily be deduced from the proofs of parts (i) and (ii) of Theorem 26:

(a) $\phi(0\boldsymbol{u}) = (1^k 0)^\infty$ for any sequence $\boldsymbol{u}$ having a prefix of the form $1^k 0v$, where $k \geq 1$, $|v| = k$ and $|v|_1 \leq k - 1$.

(b) $\phi(0\boldsymbol{u}) = (10^k)^\infty$ for any sequence $\boldsymbol{u}$ having a prefix of the form $0^k 1v$, where $k \geq 1$, $|v| = k$ and $|v|_0 \leq k - 1$.

(2) By part (iii)(a) of Theorem 26, $\phi(0\boldsymbol{u}) = (10100100)^\infty = (1w_{3,8}0)^\infty$ for any sequence $\boldsymbol{u}$ beginning with

$$Pal(010)011 = w_{3,8}011 = \mathbf{010010}011.$$

(3) Let $\boldsymbol{u}$ be the (non-characteristic) Sturmian sequence

$$1\boldsymbol{f} = \mathbf{101}001010010010100101001001 01\cdots,$$

where $\boldsymbol{f}$ is the (binary) Fibonacci sequence (see Examples 2 and 16). Then the longest central prefix of $\boldsymbol{u}$ is $w_{3,5} = 101 = Pal(10)$ and $\boldsymbol{u}$ begins with $w_{3,5}00$. Therefore, by part (iii)(b) of Theorem 26, we have $\phi(0\boldsymbol{u}) = \phi(01\boldsymbol{f}) = (10)^\infty = (1w_{1,2}0)^\infty$.

(4) By part (iii)(c) of Theorem 26, $\phi(0\boldsymbol{u}) = (10100)^\infty = (1w_{2,5}0)^\infty$ for any sequence $\boldsymbol{u}$ beginning with

$$Pal(010)101 = w_{3,8}101 = \mathbf{010010}101.$$

(5) By parts (iii)(b) and (iii)(c) of Theorem 26, $\phi(\boldsymbol{x}) = (10)^\infty = (1w_{1,2}0)^\infty$ for any sequence $\boldsymbol{x}$ beginning with 0011 or 0100. In particular, $\phi(0\boldsymbol{t}) = (10)^\infty$ for the *Thue-Morse sequence* $\boldsymbol{t}$, which is the fixed point beginning with 0 of the morphism defined by $0 \mapsto 01$ and $1 \mapsto 10$, that is

$$\boldsymbol{t} = \mathbf{0}110100110010110\cdots.$$

Also note that $\phi(\boldsymbol{t}) = (110)^\infty = (1w_{2,3}0)^\infty$.

(6) Recall that the central word $w_{p,q}$ of slope $p/q \in (0,1)$, where $\gcd(p,q) = 1$, has length $q-2$ and contains $p-1$ occurrences of 1 (and $q-p-1$ occurrences of 0). We observe that if $p > 2q$ (*i.e.*, if $w_{p,q}$ contains more 1's than 0's), then $w_{p,q}$ begins with 1; otherwise, if $p < 2q$, then $w_{p,q}$ begins with 0. Hence, we deduce the following general facts from part (iii)(a) of Theorem 26:

(a) If $p > 2q$, then $\phi(0\boldsymbol{u}) = (1w_{p,q}0)^\infty$ for any sequence $\boldsymbol{u}$ beginning with $w_{p,q}100$.

(b) If $p < 2q$, then $\phi(0\boldsymbol{u}) = (1w_{p,q}0)^\infty$ for any sequence $\boldsymbol{u}$ beginning with $w_{p,q}011$.

Remark 29. To determine the longest central prefix of a sequence $\boldsymbol{u} \in \{0,1\}^{\mathbb{N}}$ (which is neither a characteristic Sturmian sequence nor a characteristic periodic balanced sequence), possibly the easiest way is to check each palindromic prefix of $\boldsymbol{u}$ (in order of increasing length) to see if it is equal to $Pal(u)$ for some $u \in \{0,1\}^*$, until there are no more palindromic prefixes or until one reaches a palindromic prefix that is not in the image of Pal.

Note. Theorem 26 also provides a method for computing $F(x)$ in case (v) of Theorem 7. For example, $F\left(\frac{1}{4}\right) = \frac{2}{3}$ since the base 2 expansion of $\frac{1}{4}$ is $01000\cdots$ (or $00111\cdots$) and we have $\phi(01000\cdots) = (10)^\infty = \phi(00111\cdots)$, where $(10)^\infty$ is the base 2 expansion of $2/3$ (see part (1) of Example 28 above).

As a more complicated example, let us consider for instance the computation of $F\left(\frac{1}{2\pi}\right)$. The base 2 expansion of the fractional part of $\frac{1}{2\pi}$ $(= 0.15915\ldots)$ begins as follows:

$$\left(\frac{1}{2\pi}\right)_2 = 0\,\underbrace{\mathbf{01010}}_{Pal(011)}\,00101111100110\cdots$$

and we have

$$(\underline{010}01)^\infty < \left(2\cdot\frac{1}{2\pi}\right)_2 < (\underline{010}10)^\infty$$

where $r((1\underline{010}0)^\infty) = 20/31$. Thus $F\left(\frac{1}{2\pi}\right) = \frac{20}{31}$, *i.e.*, the minimal interval containing all the fractional parts $\left\{\frac{1}{2\pi}\cdot 2^k\right\}$, with $k \geq 0$, is $\left[\frac{1}{2\pi}, \frac{20}{31}\right]$.

7. Larger bases

What precedes uses essentially base 2 expansions. One may ask what happens with base b expansions, where $b \geq 3$, or what can be said about the intervals containing all $\{\xi b^n\}$ for some ξ. The result of Bugeaud and Dubickas in [8] recalled at the beginning (see Theorem 4) implies that Sturmian sequences (with values on an alphabet $\{k, k+1\}$ for some $k \in \{0, 1, \ldots, b-2\}$) should again play a fundamental rôle.

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