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DUALITY FOR HARMONIC MIXED-NORM SPACES IN THE UNIT BALL OF $\mathbb{R}^n$

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RÉSUMÉ. Dans cet article, nous introduisons, en tant que généralisation de l'espace harmonique de Bergman, un type d'espace de fonctions harmoniques à norme mixte, dont nous étudions la dualité. En outre, nous donnons des conditions pour lesquelles certains opérateurs de Bergman sur les espaces harmoniques à norme mixte sont bornés.

ABSTRACT. In this paper, as a generalization of the harmonic Bergman space, a kind of mixed-norm space of harmonic functions is introduced and its duality is investigated. Moreover, the boundedness of certain Bergman type operators on the harmonic mixed-norm spaces is studied.

1. Introduction. Let B denote the unit ball of $\mathbb{R}^n$, $S = \partial B$ its boundary. By ν, we denote the Lebesgue measure on $\mathbb{R}^n$, normalized so that $\nu(B) = 1$, and σ the surface measure on S normalized so that $\sigma(S) = 1$.

As usual, for a measurable function f on B, the means $M_q(r, f)$ are defined by

$$M_q(r,f) = \left\{ \int_S |f(r\zeta)|^q d\sigma(\zeta) \right\}^{1/q}, \quad 0 < q < \infty,$$
$$M_\infty(r,f) = \sup_{\zeta \in S} |f(r\zeta)|.$$

A harmonic function is in the harmonic Hardy space $h^q(B)$, $0 < q \le \infty$, provided that $\sup_{0<r<1} M_q(r,f) < \infty$.

A positive continuous function φ on $[0, 1)$ is normal, if there exist $0 < a < b$ and $0 \le r_0 < 1$ such that

(i) $\varphi(r)/(1-r)^a$ is non-increasing in $[r_0, 1)$ and $\lim_{r\to 1} \varphi(r)/(1-r)^a = 0$;

(ii) $\varphi(r)/(1-r)^b$ is non-decreasing in $[r_0, 1)$ and $\lim_{r\to 1} \varphi(r)/(1-r)^b = \infty$.

Throughout this paper, such a, b will be called a non-increasing index and a non-decreasing index of φ respectively.

The functions $\{\varphi, \psi\}$ will be called a normal pair if φ is normal and for some non-decreasing index b, there exists $\beta > b$, such that

$$\varphi(r)\psi(r) = (1-r^2)^\beta, \quad 0 \le r < 1. \tag{1.1}$$

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β is called the index of the normal pair $\{\varphi, \psi\}$. If φ is normal then there exists ψ such that $\{\varphi, \psi\}$ is a normal pair. Note that if $\{\varphi, \psi\}$ is a normal pair then ψ is normal.

For $0 < p \leq \infty$, $1 \leq q \leq \infty$, and a normal function φ, let $L_{p,q}(\varphi)$ denote the space of (real, without loss of generality) measurable functions on B with

$$\|f\|_{p,q,\varphi} = \left\{ \int_0^1 r^{n-1}(1-r^2)^{-1}\varphi^p(r) M_q^p(r,f)\,dr \right\}^{1/p} < \infty, \quad 0 < p < \infty,$$
$$\|f\|_{\infty,q,\varphi} = \sup_{0<r<1} \varphi(r) M_q(r,f) < \infty.$$

Harmonic mixed-norm space $h_{p,q}(\varphi)$ consists of those functions in $L_{p,q}(\varphi)$ which are harmonic in B. It is easy to show that if $1 \leq p, q \leq \infty$ then $h_{p,q}(\varphi)$ is a Banach space under the norm $\| \cdot \|_{p,q,\varphi}$.

Let $L_h^p(d\nu_\alpha)$, $\alpha > -1$, $0 < p < \infty$, denote the so-called *weighted harmonic Bergman spaces*, that is the space of functions harmonic in B with

$$\int_B |f(x)|^p (1-|x|^2)^\alpha d\nu(x) < \infty,$$

then from the integral formula in polar coordinates

$$\int_B |f(x)|^p (1-|x|^2)^\alpha d\nu(x) = n \int_0^1 r^{n-1}(1-r^2)^\alpha M_p^p(r,f)\,dr,$$

we obtain

$$L_h^p(d\nu_\alpha) = h_{p,p}((1-r^2)^{(\alpha+1)/p}).$$

It means that the weighted harmonic Bergman spaces are exactly the harmonic mixed-norm spaces with $p = q$ and $\varphi(r) = (1-r^2)^{(\alpha+1)/p}$.

The main purpose of this paper is to investigate the duality in harmonic mixed-norm spaces. We find that the result in harmonic context is similar to its holomorphic counterpart.

Our main result is as follows.

Theorem 1.1. *Suppose that* $0 < p < \infty$, *and that* $1 \leq q < \infty$. *The dual of* $h_{p,q}(\varphi)$ *can be identified with* $h_{p',q'}(\psi)$. *The pairing is given by the following*

$$(f,g) = \int_B f(x)g(x)(1-|x|^2)^{\beta-1} d\nu(x), \tag{1.2}$$

where β *is the index of the normal pair* $\{\varphi, \psi\}$, $q' = q/(q-1)$ *and*

$$p' = \begin{cases} p/(p-1), & p > 1, \\ \infty, & 0 < p \leq 1. \end{cases}$$

More precisely,

(i) *If* $g \in h_{p',q'}(\psi)$ *and if we define* $\Lambda_g(f) = (f,g)$ *for* $f \in h_{p,q}(\varphi)$, *then* $\Lambda_g \in (h_{p,q}(\varphi))^*$ *and* $\|\Lambda_g\| \leq K\|g\|_{p',q',\psi}$.

(ii) *Conversely, for given* $\Lambda \in (h_{p,q}(\varphi))^*$, *there exists a unique* $g \in h_{p',q'}(\psi)$, *such that* $\Lambda(f) = (f,g)$ *for every* $f \in h_{p,q}(\varphi)$ *and* $\|g\|_{p',q',\psi} \leq K\|\Lambda\|$.

Note that Theorem 1.1 did not include the case when $p = \infty$. Instead, we consider the space

$$h^{(0)}_{\infty,q}(\varphi) = \{f \in h_{\infty,q}(\varphi) : \varphi(r)M_q(r,f) \to 0 \text{ as } r \to 1\}.$$

and obtain

Theorem 1.2. *If* $1 \le q \le \infty$, *then* $(h^{(0)}_{\infty,q}(\varphi))^* = h_{1,q'}(\psi)$, *the pairing is given by* (1.2). *The precise statement is similar to that of Theorem 1.1.*

We prove the result in the cases $1 < p < \infty$ and $0 < p \le 1$ with different methods. In the case $1 < p < \infty$, the proof is a standard one, somewhat a matter of routine, but in our new setting, we investigate even more general integral operators $P_{s,t}$ of Bergman type and obtain a necessary condition and a sufficient condition for the boundedness of the operators of that type. For the case $0 < p \le 1$, as a contrast to the previous case, we give a constructive proof based on the spherical-harmonics expansion of harmonic functions in the unit ball. We complete the proofs in Section 4. In Section 5, we present two well-known results as immediate corollaries of Theorem 1.1.

2. Preliminaries. We recall that a twice-continuously differentiable function f is harmonic on B if $\triangle f = 0$, where $\triangle = \partial^2/\partial x_1^2 + \cdots + \partial^2/\partial x_n^2$. A polynomial on $\mathbb{R}^n$ is homogeneous of degree k if it is a finite linear combination of monomials $x_1^{\theta_1} \dots x_n^{\theta_n}$, where $\theta_1, \dots, \theta_n$ are nonnegative integers such that $\theta_1 + \cdots + \theta_n = k$. Every harmonic function f on B can be decomposed as $f = \sum_{k=0}^{\infty} F_k$, where each F_k is a harmonic homogeneous polynomial of degree k, and the convergence is uniform on compact subsets of B. The space $\mathcal{H}_k(S)$ of restrictions to S of harmonic homogeneous polynomials of degree k, the so-called spherical harmonics of degree k, is an h_k-dimensional Hilbert space with respect to the usual inner product on $L^2(S,\sigma)$. Explicitly, the dimension

$$h_k = \binom{n+k-1}{n-1} - \binom{n+k-3}{n-1}. \tag{2.1}$$

Let $\{Y_{kj} : j = 1, \dots, h_k\}$ be an orthonormal basis in $\mathcal{H}_k(S)$. Then any harmonic function f on B can be represented as

$$f(x) = \sum_{k=0}^{\infty} \sum_{j=1}^{h_k} a_{kj} |x|^k Y_{kj}(\zeta), \tag{2.2}$$

where ζ is the vector of unit length parallels to x. In the sequel, we will use the simpler notation $\sum_{k,j} = \sum_{k=0}^{\infty} \sum_{j=1}^{h_k}$ for convenience. For each $\eta \in S$ the linear functional $u \mapsto u(\eta)$ on the space $\mathcal{H}_k(S)$ is uniquely represented by a $Z_k(\cdot,\eta) \in \mathcal{H}_k(S)$, called the zonal harmonic of degree k at η. Thus we have

$$\int_S Z_k(\zeta,\eta) u(\eta) d\sigma(\zeta) = u(\zeta), \tag{2.3}$$

for every $u \in \mathcal{H}_k(S)$. Then, by standard Hilbert space theory,

$$Z_k(\zeta,\eta) = \sum_{j=1}^{h_k} \langle Z_k(\cdot,\eta), Y_{kj} \rangle \cdot Y_{kj}(\zeta) = \sum_{j=1}^{h_k} Y_{kj}(\zeta) Y_{kj}(\eta) \tag{2.4}$$

for all $\zeta, \eta \in S$. We will also need the easy identity

$$Z_k(\eta, \eta) = h_k, \quad \forall \eta \in S. \tag{2.5}$$

The standard notation $P(x, \eta)$ always stands for the Poisson kernel for the unit ball

$$P(x, \eta) = \frac{1 - |x|^2}{|x - \eta|^n}$$

and $P[g]$ the Poisson integral of function g. For every $n \geq 2$,

$$P(x, \eta) = \sum_{k=0}^{\infty} |x|^k Z_k(\zeta, \eta) \tag{2.6}$$

for all $x \in B, \eta \in S$, the series converges absolutely and uniformly on $K \times S$ for every $K \subset B$. We refer to [1] for the above facts.

Lemma 2.1. *If $g \in \mathcal{H}_k(S)$, then $P[g](r\zeta) = r^k g(\zeta)$.*

Proof. The lemma follows from (2.3) and (2.6) immediately. □

If f is an harmonic function in B and has the spherical harmonics expansion (2.2), then for any $\alpha > 0$ we define its fractional derivative of order α by

$$f^{[\alpha]}(x) = \sum_{k,j} \frac{\Gamma(k + n/2 + \alpha)}{\Gamma(k + n/2)} a_{kj} r^k Y_{kj}(\zeta), \quad x = r\zeta. \tag{2.7}$$

In the sequel, we need the following result about the rate of growth of the means $M_q(r, f^{[\alpha]})$.

Lemma 2.2. [5, Theorem 7.7] Let f be harmonic in B and $1 \leq q \leq \infty$. Then for any $\alpha > 0$,

$$M_q(r^2, f^{[\alpha]}) \leq K(1 - r)^{-\alpha} M_q(r, f). \tag{2.8}$$

The following technical lemmas are also needed.

Lemma 2.3. [12, Lemma 6] *For $\gamma > 0$ and $\lambda > \gamma$, we have*

$$\int_0^1 \frac{(1 - r)^{\gamma - 1} dr}{(1 - \rho r)^{\lambda}} \leq \frac{K}{(1 - \rho)^{\lambda - \gamma}}. \tag{2.9}$$

Lemma 2.4. [9, Lemma 2.3] *Let φ be a normal function, a, b its non-increasing and non-decreasing index respectively. If $\lambda + \gamma > b > a > \lambda$, then there is a constant K such that for $0 \leq r < 1$ and $p > 0$,*

$$\int_0^1 \frac{\varphi^p(\rho) d\rho}{(1 - \rho)^{p\lambda + 1} (1 - r\rho)^{p\gamma}} \leq \frac{K \varphi^p(r)}{(1 - r)^{p(\lambda + \gamma)}}. \tag{2.10}$$

Lemma 2.5. [11, Lemma 3.3] *Suppose φ is normal and $g(r)$ non-decreasing. If $0 < p \le q$, then*

$$\left\{\int_0^1 \frac{\varphi^q(r)}{1-r} g^q(r)dr\right\}^{1/q} \le K\left\{\int_0^1 \frac{\varphi^p(r)}{1-r} g^p(r)dr\right\}^{1/p}. \tag{2.11}$$

Lemma 2.6. [8, Proposition 2.2] *For $x \in B$,*

$$\int_S \frac{d\sigma(\zeta)}{|x-\zeta|^{n-1+s}} \le \begin{cases} K, & s < 0, \\ K \log \dfrac{1}{1-|x|}, & s = 0, \\ K(1-|x|)^{-s}, & s > 0. \end{cases} \tag{2.12}$$

Lemma 2.7. *Suppose $0 < \delta < \lambda$, $x \in B$ and $\eta \in S$. Then*

$$\int_0^1 \frac{(1-t)^{\delta-1}}{|tx-\eta|^\lambda} dt \le \frac{K}{|x-\eta|^{\lambda-\delta}}, \tag{2.13}$$

with a constant K independent of x and η.

Proof. First note that if $|x-\eta| \ge 1$ then $|tx-\eta| \ge 1/2$ for any $0 < t \le 1$. In fact, $|tx-\eta| \ge 1-t \ge 1/2$ when $0 < t \le 1/2$; whereas $|tx-\eta| > |x-\eta| - (1-t) \ge 1/2$ if $1/2 < t \le 1$.

Thus

$$\int_0^1 \frac{(1-t)^{\delta-1}}{|tx-\eta|^\lambda} dt \le 2^\lambda \int_0^1 (1-t)^{\delta-1} dt = \frac{2^\lambda}{\delta} \le \frac{K}{2^{\lambda-\delta}} \le \frac{K}{|x-\eta|^{\lambda-\delta}}.$$

If $|x-\eta| < 1$, let $r = 1-|x-\eta|$. Then

$$\begin{aligned} 1-tr &= 1-t+t|x-\eta| < |tx-\eta| + |tx-\eta+\eta-t\eta| \\ &< 2|tx-\eta| + (1-t) < 3|tx-\eta|. \end{aligned}$$

Therefore

$$\int_0^1 \frac{(1-t)^{\delta-1}}{|tx-\eta|^\lambda} dt \le 3^\lambda \int_0^1 \frac{(1-t)^{\delta-1}}{(1-tr)^\lambda} dt \le \frac{K}{(1-r)^{\lambda-\delta}} = \frac{K}{|x-\eta|^{\lambda-\delta}}. \qquad \square$$

Lemma 2.8. *Suppose that $1 \le q < \infty$ and that f is harmonic in B, then*

$$M_\infty(r^2, f) \le 2^{1/q}(1-r)^{(1-n)/q} M_q(r, f), \quad 0 \le r < 1. \tag{2.14}$$

Proof. The function f is harmonic in B and $q \ge 1$, hence $|f_r|^q$ is subharmonic in B and continuous on $\bar{B}_n$, where $f_r(x) = f(rx)$. Thus we have

$$|f_r(x)|^q \le \int_S |f_r(\eta)|^q P(x,\eta) d\sigma(\eta). \tag{2.15}$$

Taking $x = r\zeta$ in (2.15) and the estimate

$$P(r\zeta, \eta) = \frac{1-r^2}{|r\zeta-\eta|^n} \le \frac{2}{(1-r)^{n-1}},$$

therefore yields (2.14). $\square$

The following two propositions can be proved in the same way as Propositions 2.3 and 2.4 in [11].

Proposition 2.9. *If* $0 < p < \infty,\ 1 \le q \le \infty$ *and* $f \in h_{p,q}(\varphi)$, *then*

$$\lim_{r\to 1} \|f_r - f\|_{p,q,\varphi} = 0, \tag{2.16}$$

where $f_r(x) = f(rx)$.

Proposition 2.10. *If* $f \in h_{\infty,q}(\varphi),\ 1 \le q \le \infty$ *then* $\lim_{r\to 1} \|f_r - f\|_{\infty,q,\varphi} = 0$ *if, and only if,* $f \in h^{(0)}_{\infty,q}(\varphi)$.

3. Bounded Projections. For $s, t > -n/2$, denote

$$Q_{s,t}(x,y) = \frac{2}{n}\sum_{k=0}^{\infty} \frac{\Gamma(k+n/2+s+1)}{\Gamma(k+n/2)\Gamma(s+1)} \frac{\Gamma(k+n/2+t)}{\Gamma(k+n/2)} |x|^k |y|^k Z_k(\zeta,\eta), \tag{3.1}$$

where ζ, η are the vectors of unit length parallel to x and y respectively. Now we define the following operator of Bergman type:

$$P_{s,t}f(x) = (1-|x|^2)^t \int_B Q_{s,t}(x,y)(1-|y|^2)^s f(y)\,d\nu(y). \tag{3.2}$$

Lemma 3.1. *Let* $n \ge 2$. *The kernel* $Q_{s,t}$ *satisfies the estimate*

$$|Q_{s,t}(x,y)| \le \frac{K}{\big||y|x-\eta\big|^{n+s+t}}, \tag{3.3}$$

where $\eta = y/|y|$.

Proof. For positive integers m, j we have

$$\frac{\Gamma(k+n/2+m+1)}{\Gamma(k+n/2)} = \frac{d^{m+1}(r^{k+n/2+m})}{dr^{m+1}}\bigg|_{r=1}$$

and

$$\frac{\Gamma(k+n/2+j)}{\Gamma(k+n/2)} = \frac{d^{j}(\rho^{k+n/2+j-1})}{d\rho^{j}}\bigg|_{\rho=1},$$

hence a comparison of (3.1) and (2.6) leads to

$$Q_{m,j}(x,y) = \frac{2}{n\Gamma(m+1)} \left\{ \frac{\partial^{m+1}}{\partial r^{m+1}} \frac{\partial^j}{\partial \rho^j} \left[r^{n/2+m}\rho^{n/2+j-1} P(r\rho|y|x,\eta) \right] \right\}_{r=1,\rho=1}. \tag{3.4}$$

Thus a similar argument to that in [4, pp. 34] shows that

$$|Q_{m,j}(x,y)| \le \frac{K}{\big||y|x-\eta\big|^{n+m+j}} \tag{3.5}$$

holds for all positive integers m, j.

For general $s, t > -n/2$, let $m > s$ and $j > t$ be positive integers. Noting that

$$Q_{s,t}(x,y) = \frac{\Gamma(m+1)\Gamma(j+1)}{\Gamma(m-s)\Gamma(s+1)\Gamma(j-t)} \times \int_0^1 \int_0^1 r^{n/2+s}(1-r)^{m-s-1}\rho^{n/2+t-1}(1-\rho)^{j-t-1}Q_{m,j}(r\rho x, y)d\rho dr,$$

and using the estimate (3.5) and Lemma 2.7, we obtain

$$\begin{aligned} |Q_{s,t}(x,y)| &\le K\int_0^1 (1-r)^{m-s-1}\int_0^1 (1-\rho)^{j-t-1}\,|Q_{m,j}(r\rho x,y)|\,d\rho dr \\ &\le K\int_0^1 (1-r)^{m-s-1}\int_0^1 \frac{(1-\rho)^{j-t-1}}{\big|r\rho|y|x-\eta\big|^{n+m+j}}d\rho dr \\ &\le K\int_0^1 \frac{(1-r)^{m-s-1}}{\big|r|y|x-\eta\big|^{n+m+t}}dr \le \frac{K}{\big||y|x-\eta\big|^{n+s+t}}. \end{aligned}$$

This completes the proof. □

Remark 3.2. Lemma 3.1 corresponds to Lemma 7.4 in [5], which gave the estimate

$$(3.6) \qquad |Q_{s,t}(x,y)| \le \frac{K}{\big||y|x-\eta\big|^{n+[s+t]}(1-|x|)^{\{s+t\}}} + \frac{K}{(1-|x||y|)^{1+s+t}},$$

where $[s+t]$ is the integer part of $s+t$, and $\{s+t\} = 1-[s+t]$. Clearly, the estimate (3.3) is sharper and more concise than (3.6). The proof given here is also simpler and more natural.

In a similar way, we also obtain the estimate for the derivative of $Q_{s,t}$.

Lemma 3.3. *Let ∇_x denote the gradient with respect to x. Then we have the estimate*

$$|\nabla_x Q_{s,t}(x,y)| \le \frac{K}{\big||y|x-\eta\big|^{n+s+t+1}}.$$

Lemma 3.4. *Suppose $1 \le q \le \infty$, $s, t > -n/2$ and $s+t > -1$. Then*

$$(3.7) \qquad M_q(r, P_{s,t}f) \le K(1-r^2)^t \int_0^1 \frac{\rho^{n-1}(1-\rho^2)^s}{(1-r\rho)^{s+t+1}} M_q(\rho, f)d\rho.$$

Proof. It follows from (3.2) and (3.3) that

$$\begin{aligned} (3.8) \qquad &|P_{s,t}f(r\zeta)| \\ &\le n(1-r^2)^t \int_0^1 \rho^{n-1}(1-\rho^2)^s \int_S |Q_{s,t}(r\zeta,\rho\eta)||f(\rho\eta)|d\sigma(\eta)d\rho \\ &\le K(1-r^2)^t \int_0^1 \rho^{n-1}(1-\rho^2)^s \int_S \frac{|f(\rho\eta)|}{|r\rho\zeta-\eta|^{n+s+t}}d\sigma(\eta)d\rho \end{aligned}$$

First assume $1 < q < \infty$, then by Hölder's inequality and Lemma 2.6 we have

$$\int_S \frac{|f(\rho\eta)|}{|r\rho\zeta-\eta|^{n+s+t}}d\sigma(\eta) \le \left\{\int_S \frac{|f(\rho\eta)|^q}{|r\rho\zeta-\eta|^{n+s+t}}d\sigma(\eta)\right\}^{1/q}\left\{\int_S \frac{d\sigma(\eta)}{|r\rho\zeta-\eta|^{n+s+t}}\right\}^{1/q'}$$
$$\le K(1-r\rho)^{-(s+t+1)/q'}\left\{\int_S \frac{|f(\rho\eta)|^q}{|r\rho\zeta-\eta|^{n+s+t}}d\sigma(\eta)\right\}^{1/q},$$

where $1/q + 1/q' = 1$.

Thus using Minkowski's inequality, Fubini's theorem and Lemma 2.6, we obtain

$$\begin{aligned}
&M_q(r, P_{s,t}f)\\
&\le K(1-r^2)^t \int_0^1 \rho^{n-1}(1-\rho^2)^s \left\{\int_S \left[\int_S \frac{|f(\rho\eta)|}{|r\rho\zeta-\eta|^{n+s+t}}d\sigma(\eta)\right]^q d\sigma(\zeta)\right\}^{1/q} d\rho\\
&\le K(1-r^2)^t \int_0^1 \frac{\rho^{n-1}(1-\rho^2)^s}{(1-r\rho)^{(s+t+1)/q'}} \left\{\int_S \left[\int_S \frac{|f(\rho\eta)|^q}{|r\rho\zeta-\eta|^{n+s+t}}d\sigma(\eta)\right] d\sigma(\zeta)\right\}^{1/q} d\rho\\
&\le K(1-r^2)^t \int_0^1 \frac{\rho^{n-1}(1-\rho^2)^s}{(1-r\rho)^{s+t+1}} M_q(\rho, f)d\rho.
\end{aligned}$$

When $q = 1$ or ∞, it is clear that the lemma follows from (3.8) directly. □

Proposition 3.5. *Suppose that $0 < p \le \infty$, $1 \le q \le \infty$. If $P_{s,t}$ is bounded on $L_{p,q}(\varphi)$, then $s > a-1$ and $t > -b$. Here a, b are any non-increasing index and non-decreasing index of φ respectively.*

Proof. We first assume that $0 < p < \infty$. Let $f_j(x) = |x|^{2j},\ j = 1, 2, \dots$, then $f_j \in L_{p,q}(\varphi)$ and

$$\|f_j\|_{p,q,\varphi} = \left\{\int_0^1 r^{n+2jp-1}(1-r^2)^{-1}\varphi^p(r)dr\right\}^{1/p}, \tag{3.9}$$

$j = 1, 2, \dots$ From the definition of normal function, we see that there exists a positive constant K such that $\varphi(r) \le K(1-r)^a$ for $0 \le r < 1$. Thus we have

$$\|f_j\|_{p,q,\varphi} \le K\left\{\frac{\Gamma(jp+n/2)\Gamma(ap)}{\Gamma(jp+n/2+ap)}\right\}^{1/p} \sim j^{-a}, \tag{3.10}$$

where the notation $A_j \sim B_j$ means that the ratio A_j/B_j has a positive finite limit as $j \to \infty$.

Using the fact that $\int_S Z_k(\zeta,\eta)d\sigma(\eta) = 0$ for $k \ge 1$, an elementary calculation yields

$$\int_B (1-|y|^2)^s Q_{s,t}(x,y)|y|^{2j}d\nu(y) = \frac{\Gamma(n/2+s+1)\Gamma(n/2+t)\Gamma(j+n/2)}{\Gamma(n/2)^2\Gamma(j+n/2+s+1)}. \tag{3.11}$$

For convenience we denote the above fraction by C_j. Then $P_{s,t}f_j(x) = C_j(1-|x|^2)^t$ and therefore

$$\|P_{s,t}f_j\|_{p,q,\varphi} = C_j\left\{\int_0^1 r^{n-1}(1-r^2)^{tp-1}\varphi^p(r)dr\right\}^{1/p}, \tag{3.12}$$

$j = 1, 2, \ldots$ Again from the definition of normal function, we find that $(1-r)^b \le K\varphi(r)$ for some positive constant K. Then (3.12) and the fact $\|P_{s,t}f_j\|_{p,q,\varphi} < \infty$ yield

$$\int_0^1 r^{n-1}(1-r^2)^{(t+b)p-1}dr < \infty,$$

which implies $t > -b$.

Noting that $C_j \sim j^{-s-1}$, a comparison of (3.10) and (3.12) leads to

$$\frac{\|P_{s,t}f_j\|_{p,q,\varphi}}{\|f_j\|_{p,q,\varphi}} \ge K \cdot j^{a-s-1} \tag{3.13}$$

holds for a positive constant K and any sufficiently large j. Thus the boundedness of $P_{s,t}$ on $L_{p,q}(\varphi)$ implies $s + 1 > a$.

When $p = \infty$, we also let $f_j(x) = |x|^{2j},\ j = 1, 2, \ldots$ In this case,

$$\|f_j\|_{\infty,q,\varphi} = \sup_{0<r<1} \varphi(r)r^{2j},$$

and

$$\|P_{s,t}f_j\|_{\infty,q,\varphi} = C_j \sup_{0<r<1} \varphi(r)(1-r^2)^t.$$

Just note that

$$\sup_{0<r<1}(1-r)^a r^{2j} = \left(\frac{a}{2j+a}\right)^a \left(\frac{2j}{2j+a}\right)^{2j} \sim j^{-a},$$

and then proceed along the same lines as in the previous case. □

Proposition 3.6. *Let $1 \le p \le \infty$, $1 \le q \le \infty$ and $s + 1 > b > a > -t$. Then $P_{s,t}$ is a bounded linear operator on $L_{p,q}(\varphi)$ and maps $L_{p,q}(\varphi)$ into $h_{p,q}(\varphi)$. Moreover, $P_{s,0}f = f$ for any $f \in h_{p,q}(\varphi)$.*

Proof. When $p = 1$, the first assertion of the proposition follows directly from Lemma 3.4.

We now assume that $1 < p < \infty$, $1 \le q \le \infty$ and write $s + 1 = \theta_1 + \theta_2 = \theta_3 + \theta_4$ such that (i) $\theta_i > 0,\ i = 1, 2, 3, 4$; (ii) $\theta_4 + a > \theta_2 > \max\{b,\ \theta_4 - t\}$. This can be done by taking $\theta_2 = (1+\varepsilon)b$, $\theta_4 = \theta_2 - (1-\varepsilon)a$ for a sufficiently small $\varepsilon > 0$. Then, using Lemmas 2.3, 3.4 and Hölder's inequality, we obtain

$$\begin{aligned} M_q(r, P_{s,t}f) &\le K(1-r^2)^t \left\{\int_0^1 \frac{(1-\rho^2)^{p'\theta_1-1}}{(1-r\rho)^{p'(\theta_3+t)}}d\rho\right\}^{1/p'} \\ &\qquad \times \left\{\int_0^1 \frac{\rho^{p(n-1)}(1-\rho^2)^{p\theta_2-1}}{(1-r\rho)^{p\theta_4}} M_q^p(\rho, f)d\rho\right\}^{1/p} \\ &\le \frac{K}{(1-r)^{\theta_3-\theta_1}} \left\{\int_0^1 \frac{\rho^{p(n-1)}(1-\rho^2)^{p\theta_2-1}}{(1-r\rho)^{p\theta_4}} M_q^p(\rho, f)d\rho\right\}^{1/p}. \end{aligned}$$

Thus, by Lemma 2.4 we have

$$\begin{aligned}\|P_{s,t}f\|_{p,q,\varphi}^p &\le K\int_0^1 \rho^{n-1}(1-\rho)^{p\theta_2-1}M_q^p(\rho,f)\\ &\qquad\times\int_0^1 \frac{\varphi^p(r)}{(1-r)^{p(\theta_3-\theta_1)+1}(1-r\rho)^{p\theta_4}}drd\rho\\ &\le K\int_0^1 \rho^{n-1}(1-\rho)^{-1}\varphi^p(\rho)M_q^p(\rho,f)d\rho \;=\; K\|f\|_{p,q,\varphi}^p.\end{aligned}$$

When $p=\infty$, let $\{\varphi,\psi\}$ be a normal pair with the index $s+1$. Then we write $(1-\rho^2)^s=(1-\rho^2)^{-1}\varphi(r)\psi(r)$ and hence

$$\begin{aligned}\|P_{s,t}f\|_{\infty,q,\varphi} &\le K\sup_{0<r<1}\varphi(r)\int_0^1 \frac{\rho^{n-1}(1-\rho^2)^{-1}\psi(\rho)}{(1-r\rho)^{s+1}}\varphi(\rho)M_q(\rho,f)d\rho\\ &\le K\|f\|_{\infty,q,\varphi}\sup_{0<r<1}\varphi(r)\int_0^1\frac{\psi(\rho)}{(1-\rho)(1-r\rho)^{s+1}}d\rho\le K\|f\|_{\infty,q,\varphi},\end{aligned}$$

where the last inequality follows from Lemma 2.4.

We now prove the second assertion of the proposition.

Let $f\in h_{p,q}(\varphi)$ has the expansion (2.2), then

$$f_t(x)=f(tx)=\sum_{k,j}a_{kj}(t|x|)^kY_{kj}(\zeta).$$

In view of (3.1), (2.4), and using the orthonormality of the system $\{Y_{kj}\colon j=1,\dots,h_k;\ k=0,1,\dots\}$, a direct computation implies

$$P_{s,0}f_t=f_t$$

for any $0<t<1$. Now

$$\|P_{s,0}f-f\|_{p,q,\varphi}\le\|P_{s,0}(f-f_t)\|_{p,q,\varphi}+\|f_t-f\|_{p,q,\varphi}\le K\|f_t-f\|_{p,q,\varphi},$$

letting $t\to1$, Proposition 2.9 gives the desired result for $1\le p<\infty$, $1\le q\le\infty$.

Now assume that $f\in h_{\infty,q}(\varphi)$. Let $\{\varphi,\psi\}$ be a normal pair with the index $s+1$ and $g\in h_{1,q'}(\psi)$. Then, from Lebesgue's dominated convergence theorem we have

$$\lim_{r\to1}\int_B f_r(x)g(x)(1-|x|^2)^s d\nu(x)=\int_B f(x)g(x)(1-|x|^2)^s d\nu(x).$$

For any given $y\in B$, let $g(x)=Q_{s,0}(y,x)$. It is clear that $g\in h_{1,q'}(\psi)$. Thus

$$\begin{aligned}\int_B f(x)Q_{s,0}(y,x)(1-|x|^2)^s d\nu(y) &= \lim_{r\to1}\int_B f_r(x)Q_{s,0}(y,x)(1-|x|^2)^s d\nu(x)\\ &=\lim_{r\to1}f(ry)=f(y),\end{aligned}$$

that is, $P_{s,0}f=f$. $\square$

Remark 3.7. Since $a < b$, Propositions 3.5 and 3.6 only give a necessary condition and a sufficient condition for the boundedness of $P_{s,t}$ separately. But if we consider a normal function φ_0 of the form

$$\varphi_0(r) = (1-r)^\alpha \left(\log \frac{K_1}{1-r}\right)^{\theta_1} \left(\log^{(2)} \frac{K_2}{1-r}\right)^{\theta_2} \dots \left(\log^{(m)} \frac{K_m}{1-r}\right)^{\theta_m},$$

where for $j = 1, \dots, m$, we let $\log^{(j)} =$ the j-fold composition of log with itself, θ_j a positive real number and K_j sufficiently large so that $\log^{(j)}(K_j/(1-r))$ is positive and continuous in $[0, 1)$, then a necessary and sufficient condition is obtained: $P_{s,t}$ is a bounded operator on $L_{p,q}(\varphi_0)$ if, and only if, $s+1 > \alpha > -t$. To see this, we first note that for any sufficiently small $\varepsilon > 0$, $a = \alpha - \varepsilon$ is a non-increasing index of φ_0 and $b = \alpha + \varepsilon$ a non-decreasing index. Then, from Propositions 3.5, we have $s+1 > \alpha - \varepsilon$, $t > -\alpha - \varepsilon$ and hence $s+1 \geq \alpha \geq -t$. But in view of (3.10) and (3.12), $P_{s,t}$ is unbounded on $L_{p,q}(\varphi_0)$ if $s+1 = \alpha$ or $t = -\alpha$. Thus the boundedness of $P_{s,t}$ implies $s+1 > \alpha > -t$. On the other hand, when $s+1 > \alpha > -t$, we can take $a = \alpha - \varepsilon$, $b = \alpha + \varepsilon$ for a sufficiently small $\varepsilon > 0$ such that $s+1 > a > b > -t$, then the boundedness of $P_{s,t}$ follows from Proposition 3.6.

More particularly, letting $p = q$ and taking $\varphi(r) = (1-r^2)^{(\alpha+1)/p}$, we find that for $1 \leq p < \infty$, $\alpha > -1$, $P_{s,t}$ is bounded on $L^p(\nu_\alpha)$ if, and only if, $(s+1)p > \alpha+1 > -tp$. This is the corresponding real-variables version of Proposition 6 in [3].

Proposition 3.8. *For every $s > -n/2$, The projection $P_{s,0}$ is a bounded operator from $L^\infty(B)$ onto $\mathcal{B}_h(B)$. Here $\mathcal{B}_h(B)$ is the harmonic Bloch space on B, which consists of those functions f harmonic in B with*

$$\|f\|_{\mathcal{B}_h} := |f(0)| + \sup_{x\in B}(1-|x|^2)|\nabla f(x)| < \infty.$$

Proof. Let $\tilde{f} \in L^\infty(B)$ and $f = P_{s,0}\tilde{f}$. Differentiating under the integral sign, we obtain

$$\nabla f(x) = \int_B \nabla_x Q_{s,0}(x,y)(1-|y|^2)^s \tilde{f}(y) d\nu(y).$$

By Lemmas 3.3, 2.6 and 2.3, there is a constant $K > 0$ such that

$$\int_B |\nabla_x Q_{s,0}(x,y)|(1-|y|^2)^s d\nu(y) \leq K \int_0^1 (1-\rho^2)^s \int_S \frac{d\sigma(\eta)}{|\rho x - \eta|^{n+s+1}} d\rho \leq \frac{K}{1-|x|^2},$$

and then

$$(1-|x|^2)|\nabla f(x)| \leq (1-|x|^2) \int_B |\nabla_x Q_{s,0}(x,y)|(1-|y|^2)^s |\tilde{f}(y)| d\nu(y) \leq K\|\tilde{f}\|_\infty$$

for any $x \in B$. Since $|f(0)| \leq \|\tilde{f}\|_\infty$, we see that $P_{s,0}$ is a bounded operator from $L^\infty(B)$ into $\mathcal{B}_h(B)$. Next we show that $P_{s,0}$ is onto.

Suppose $f \in \mathcal{B}_h$. Let $\tilde{f}(x) = \frac{1}{s+1}(1-|x|^2)[Rf(x) + (n/2+s+1)f(x)]$, where $Rf(x) = x \cdot \nabla f(x)$ denotes the radial derivative of f. Clearly $|Rf| \leq |\nabla f|$ for any

$f \in C^1(B)$. From the definition of the Bloch space, it is easy to find that $|f(x)| \leq 2\|f\|_{\mathcal{B}_h} \log \frac{2}{1-|x|^2}$ on B whenever $f \in \mathcal{B}_h(B)$. This implies $(1-|x|^2)|f(x)| \leq 2\|f\|_{\mathcal{B}_h}$. Therefore $\tilde{f} \in L^\infty(B)$ and $\|\tilde{f}\|_\infty \leq K\|f\|_{\mathcal{B}_h}$ for some constant $K > 0$. Now it is enough to show that

$$P_{s,0}\tilde{f}(x) = f(x). \tag{3.14}$$

In fact, writing

$$Rf(x) + (n/2 + s + 1)f(x) = \sum_{k,j}(k + n/2 + s + 1)a_{kj}|x|^k Y_{kj}(\zeta)$$

and using the orthonormality of the system $\{Y_{kj} : \ j = 1, \dots, h_k;\ k = 0, 1, \dots\}$, a direct computation gives (3.14). □

4. Proof of Theorem 1.1.

Lemma 4.1. *Let $1 < p < \infty$, $1 \leq q < \infty$. p', q' are the conjugates of p and q respectively. Λ is a bounded linear functional on $L_{p,q}(\varphi)$ if, and only if, it can be represented by*

$$\Lambda(f) = \int_B f(x)g(x)(1 - |x|^2)^{-1}\varphi^p(|x|)d\nu(x), \tag{4.1}$$

where g is a uniquely determined function of $L_{p',q'}(\varphi^{\frac{p}{p'}})$, and $\|\Lambda\| = \|g\|_{p',q',\varphi^{\frac{p}{p'}}}$.

Proof. Let $X_1 = S$, $d\mu_1 = d\sigma$ and $X_2 = [0, 1]$, $d\mu_2 = nr^{n-1}(1 - r^2)^{-1}\varphi^p(r)dr$, then the lemma follows immediately from Theorem 1 in [2]. □

Proof of Theorem 1.1 in the case where $1 < p < \infty$. Let $f \in h_{p,q}(\varphi)$ and $g \in h_{p',q'}(\psi)$. Hölder's inequality implies that

$$\begin{aligned}|(f, g)| &\leq n \int_0^1 r^{n-1}(1 - r^2)^{-1}\varphi(r)\psi(r)M_q(r, f)M_{q'}(r, g)dr \\ &\leq K\|f\|_{p,q,\varphi}\|g\|_{p',q',\psi}.\end{aligned}$$

This shows that every $g \in h_{p',q'}(\psi)$ defines a bounded linear functional Λ_g on $h_{p,q}(\varphi)$, by the formula $\Lambda_g(f) = (f, g)$, and $\|\Lambda_g\| \leq K\|g\|_{p',q',\psi}$.

Conversely, when $1 < p < \infty,\ 1 \leq q < \infty$, let $\Lambda \in \big(h_{p,q}(\varphi)\big)^*$. By Hahn-Banach Theorem, Λ extends to be a bounded linear functional on $L_{p,q}(\varphi)$. Thus from Lemma 4.1, there exists $G \in L_{p',q'}(\varphi^{\frac{p}{p'}})$ such that for any $f \in L_{p,q}(\varphi)$,

$$\Lambda(f) = \int_B f(x)G(x)(1 - |x|^2)^{-1}\varphi^p(|x|)d\nu(x) \tag{4.2}$$

and $\|\Lambda\| = \|G\|_{p',q',\varphi^{p/p'}}$.

Let $\tilde{G}(x) = \varphi(|x|)^{p-1}\psi(|x|)^{-1}G(x)$ and $g = P_{\beta-1,0}\tilde{G}$, where β is the index of the pair $\{\varphi, \psi\}$. Then Proposition 3.6 shows that

$$\|g\|_{p',q',\psi} \leq K\|\tilde{G}\|_{p',q',\psi} \leq K\|G\|_{p',q',\varphi^{p/p'}} = K\|\Lambda\|, \tag{4.3}$$

so $g \in h_{p',q'}(\psi)$. It is easy to verify that $\Lambda(f) = (f, g)$ for any $f \in h_{p,q}(\varphi)$.

Next, we consider the space $h_{p,\infty}(\varphi)$. From (2.14), we find that $h_{p,q}(\varphi_q) \subset h_{p,\infty}(\varphi)$ and

$$\|f\|_{p,\infty,\varphi} \le 2^{1/q}\|f\|_{p,q,\varphi_q} \tag{4.4}$$

for all $f \in h_{p,q}(\varphi_q)$, where $\varphi_q(r) = \varphi(r)(1-r)^{(1-n)/q}$. So if $\Lambda \in (h_{p,\infty}(\varphi))^*$ then $\Lambda \in (h_{p,q}(\varphi_q))^*$. Therefore, there exists $g \in h_{p',q'}(\psi_q)$, $\psi_q(r) = \psi(r)(1-r)^{(n-1)/q}$, such that $\Lambda(f) = (f, g)$ for all $f \in h_{p,q}(\varphi_q)$, and

$$\|g\|_{p',q',\psi_q} \le K \sup\left\{ \frac{|\Lambda(f)|}{\|f\|_{p,q,\varphi_q}} : f \in h_{p,q}(\varphi_q) \right\}. \tag{4.5}$$

Looking a little more carefully at (4.3) and the proof of Proposition 3.6 we can see that the constant K is independent of q. Combining (4.4) and (4.5), we obtain

$$\|g\|_{p',q',\psi_q} \le K \cdot 2^{1/q} \sup\left\{ \frac{|\Lambda(f)|}{\|f\|_{p,\infty,\varphi}} : f \in h_{p,\infty}(\varphi) \right\} = K \cdot 2^{1/q}\|\Lambda\|.$$

Thus by Fatou's lemma,

$$\begin{aligned}
&\int_0^1 (1-r^2)^{-1}\psi^{p'}(r)M_1^{p'}(r,g)dr \\
&\qquad = \int_0^1 (1-r^2)^{-1}\psi^{p'}(r) \lim_{q\to\infty} \left[(1-r)^{(n-1)/q} M_{q'}(r,g)\right]^{p'} dr \\
&\qquad \le \liminf_{q\to\infty} \int_0^1 (1-r^2)^{-1} \left[\psi(r)(1-r)^{(n-1)/q}\right]^{p'} M_{q'}^{p'}(r,g)dr \\
&\qquad = \liminf_{q\to\infty} \|g\|_{p',q',\psi_q}^{p'} \le K^{p'} \lim_{q\to\infty} 2^{1/q}\|\Lambda\|^{p'} \le K\|\Lambda\|^{p'}.
\end{aligned}$$

This means that $g \in h_{p',1}(\psi)$ and $\|g\|_{p',1,\psi} \le K\|\Lambda\|$. □

To prove Theorem 1.1 in the case $0 < p \le 1$, we need the following

Lemma 4.2. *If* $\Lambda \in (h_{p,q}(\varphi))^*$, $0 < p \le \infty$, $1 \le q < \infty$, *then*

$$F(x) = \sum_{k,j} r^k \Lambda(P[Y_{kj}])Y_{kj}(\zeta) \tag{4.6}$$

is harmonic in B, *where* $x = r\zeta$.

Proof. By (2.4) and (2.5), we note that $|Y_{kj}(\zeta)| \le h_k^{1/2}$, $j = 1, \dots, h_k$; $k = 0, 1, \dots$ Then by Lemma 2.1,

$$M_q(r, P[Y_{kj}]) \le \left(\int_S |Y_{kj}(\zeta)|^q d\sigma(\zeta)\right)^{1/q} \le h_k^{1/2} \tag{4.7}$$

and hence

$$\|P[Y_{kj}]\|_{p,q,\varphi} = \left\{\int_0^1 r^{n-1}(1-r)^{-1}\varphi^p(r)M_q^p(r, P[Y_{kj}])dr\right\}^{1/p} \le Kh_k^{1/2}. \tag{4.8}$$

Again by (2.4) and (2.5), we find that $\sum_{j=1}^{h_k} |Y_{kj}(\zeta)| \le h_k$, $k = 0, 1, \dots$ Combining this with (4.8), we have

$$\left|\sum_{j=1}^{h_k} \Lambda(P[Y_{kj}])Y_{kj}(\zeta)\right| \le \sum_{j=1}^{h_k} \|\Lambda\| \, \|P[Y_{kj}]\|_{p,q,\varphi} \, |Y_{kj}(\zeta)|$$
$$\le K\|\Lambda\| h_k^{\frac{3}{2}} = O(k^{3(n-2)/2}).$$

This shows that the series (4.6) converges uniformly on any compact subset of B and hence F is harmonic in B. □

Proof of Theorem 1.1 in the case $0 < p \le 1$**.** (i) For any $f \in h_{p,q}(\varphi)$ and any $g \in h_{\infty,q'}(\psi)$, Hölder's inequality and Lemma 2.5 give

$$|(f,g)| \le K \int_0^1 r^{n-1}(1-r^2)^{-1}\varphi(r)\psi(r)M_q(r,f)M_{q'}(r,g)dr$$
$$\le K\|g\|_{\infty,q',\psi} \left\{\int_0^1 r^{n-1}(1-r^2)^{-1}\varphi(r)M_q(r,f)dr\right\}$$
$$\le K\|g\|_{\infty,q',\psi}\|f\|_{p,q,\varphi},$$

so if we define $\Lambda_g(f) = (f,g)$, then $\Lambda_g \in \big(h_{p,q}(\varphi)\big)^*$, and $\|\Lambda_g\| \le K\|g\|_{\infty,q',\psi}$.

(ii) Conversely, given $\Lambda \in \big(h_{p,q}(\varphi)\big)^*$, we define

$$g(x) = \frac{2}{n\Gamma(\beta)} \sum_{k,j} \frac{\Gamma(k+n/2+\beta)}{\Gamma(k+n/2)} r^k \Lambda(P[Y_{kj}])Y_{kj}(\zeta), \tag{4.9}$$

where $x = r\zeta$ and β is the index of the normal pair $\{\varphi, \psi\}$. By Lemma 4.2, g is harmonic in B. To prove $g \in h_{\infty,q'}(\psi)$, we note that

$$M_q(r^2, g) = \sup\left\{\left|\int_S g(r^2\zeta)u(\zeta)d\sigma(\zeta)\right| : u \in L^q(S,\sigma),\ \|u\|_q = 1\right\}. \tag{4.10}$$

Let $u \in L^q(S,\sigma)$, then its Poisson integral $P[u] \in h^q(B)$ and $\|P[u]\|_q = \|u\|_q$ (*cf.* Theorem 6.12 in [1]). Then by Lemma 2.2,

$$M_q(r\rho, P[u]^{[\beta]}) \le K(1-r\rho)^{-\beta}\|P[u]\|_q \le K(1-r\rho)^{-\beta}, \tag{4.11}$$

and Lemma 2.4 yields

$$\left\|P[u]_{r^2}^{[\beta]}\right\|_{p,q,\varphi} \le \int_0^1 (1-\rho)^{-1}\varphi^p(\rho)M_q^p(r^2\rho, P[u]^{[\beta]})d\rho \tag{4.12}$$
$$\le K\int_0^1 \frac{\varphi^p(\rho)d\rho}{(1-\rho)(1-r\rho)^{p\beta}} \le \frac{K}{\psi^p(r)}.$$

If $P[u](x) = \sum_{k,j} r^k b_{kj} Y_{kj}(\zeta)$, then by the continuity of Λ, we have

$$\text{(4.13)} \quad \left|\int_S g(r^2\zeta)u(\zeta)d\sigma(\zeta)\right| = \left|\int_S g(r\eta)P[u](r\eta)d\sigma(\zeta)\right|$$
$$= \left|\frac{2}{n\Gamma(\beta)}\sum_{k,j}\frac{\Gamma(k+n/2+\beta)}{\Gamma(k+n/2)}b_{kj}r^{2k}\Lambda(P[Y_{kj}])\right|$$
$$= \left|\frac{2}{n\Gamma(\beta)}\Lambda\left(P[u]_{r^2}^{[\beta]}\right)\right| \leq K\|\Lambda\|\left\|P[u]_{r^2}^{[\beta]}\right\|_{p,q,\varphi}.$$

Combining (4.10), (4.12) and (4.13) gives $g \in h_{\infty,q'}(\psi)$ and $\|g\|_{\infty,q',\psi} \leq K\|\Lambda\|$.

For any $c(x) = \sum_{k,j} r^k c_{kj}Y_{kj}(\zeta)$ and $d(x) = \sum_{k,j} r^k d_{kj}Y_{kj}(\zeta)$, a direct computation shows that

$$(c,d) = \int_B c(x)d(x)(1-|x|^2)^{\beta-1}d\nu(x)$$
$$= n/2\Gamma(\beta)\sum_{k,j}\frac{\Gamma(k+n/2)}{\Gamma(k+n/2+\beta)}c_{kj}d_{kj}.$$

Thus, for $f(x) = \sum_{k,j} r^k a_{kj}Y_{kj}(\zeta) \in h_{p,q}(\varphi)$,

$$\text{(4.14)} \qquad (f_r, g) = \sum_{k,j} r^k a_{kj}\Lambda(P[Y_{kj}]) = \Lambda(f_r).$$

Letting $r \to 1$, the continuity of Λ and Proposition 2.9 gives

$$\Lambda(f) = (f,g).$$

To finish the proof, we will need to show the unique of g. In fact, if g defines the zero functional, then on applying

$$\text{(4.15)} \qquad (f,g) = n/2\Gamma(\beta)\sum_{k,j}\frac{\Gamma(k+n/2)}{\Gamma(k+n/2+\beta)}a_{kj}b_{kj}$$

with $P[Y_{kj}]$ for f, we find that g is identically zero. The theorem is proved. □

Proof of Theorem 1.2. Let $f \in h^{(0)}_{\infty,q}(\varphi)$ and $g \in h_{1,q'}(\psi)$, then

$$|(f,g)| \leq K\int_0^1 (1-r)^{-1}\varphi(r)\psi(r)M_q(r,f)M_{q'}(r,g)dr \leq K\|f\|_{\infty,q,\varphi}\|g\|_{1,q',\psi}.$$

This means that every $g \in h_{1,q'}(\psi)$ defines a bounded linear functional on $h^{(0)}_{\infty,q}(\varphi)$ by the formula

$$\Lambda_g(f) = (f,g), \quad f \in h^{(0)}_{\infty,q}(\varphi),$$

and $\|\Lambda_g\| \leq K\|g\|_{1,q',\psi}$.

Conversely, for given $\Lambda \in (h^{(0)}_{\infty,q}(\varphi))^*$, we define g as in the proof of Theorem 1.1 in the case $0 < p \leq 1$. By the continuity of Λ and Proposition 2.10, we have $\Lambda(f) = (f, g)$ for every $f \in h^{(0)}_{\infty,q}(\varphi)$. We now prove $g \in h_{1,q'}(\psi)$. Clearly,

$$\|g_r\|_{1,q',\psi} = \sup\left\{ \frac{|F(g_r)|}{\|F\|_{(h_{1,q'}(\psi))^*}} : F \in (h_{1,q'}(\psi))^*,\ F \neq 0 \right\}. \tag{4.16}$$

By Theorem 1.1, $(h_{1,q'}(\psi))^* = h_{\infty,q}(\varphi)$. This means for any $F \in (h_{1,q'}(\psi))^*$, there exists a unique $f \in h_{\infty,q}(\varphi)$, such that $F(g_r) = (g_r, f)$ for $g_r \in h_{1,q'}(\varphi)$ and $\|f\|_{\infty,q,\varphi} \leq K\,\|F\|_{(h_{1,q'}(\psi))^*}$. Thus (4.16) can be written as

$$\|g_r\|_{1,q',\psi} \leq K \sup\left\{ \frac{|(g_r, f)|}{\|f\|_{\infty,q,\varphi}} : f \in h_{\infty,q}(\varphi),\ f \neq 0 \right\}.$$

Now for any $f \in h_{\infty,q}(\varphi)$,

$$|(g_r, f)| = |\Lambda(f_r)| \leq \|\Lambda\|\,\|f_r\|_{\infty,q,\varphi} \leq \|\Lambda\|\,\|f\|_{\infty,q,\varphi}.$$

This implies that $\|g_r\|_{1,q',\psi} \leq K\,\|\Lambda\|$ for any $0 < r < 1$, so $g \in h_{1,q'}(\psi)$ and $\|g\|_{1,q',\psi} \leq K\,\|\Lambda\|$. □

5. Applications. As applications, we easily obtain the following two well-known results.

Corollary 5.1. *Let $1 < p < \infty$ and $\alpha > -1$, then the dual of the weighted harmonic Bergman space $L^p_h(d\nu_\alpha)$ may be identified with the space $L^{p'}_h(d\nu_\alpha)$, where $p' = p/(p-1)$ and the pairing is given by*

$$(f, g) = \int_B f(x)g(x)(1-|x|^2)^\alpha d\nu(x). \tag{5.1}$$

Proof. Taking $\varphi(r) = (1-r^2)^{(\alpha+1)/p}$ and $\psi(r) = (1-r^2)^{(\alpha+1)/p'}$ in Theorem 1.1 implies the desired result. □

Corollary 5.2. *Let $\alpha > -1$, then the dual of $L^1_h(d\nu_\alpha)$ may be identified with the space $\mathcal{B}_h(B)$, the pairing is given by (5.1).*

Proof. Let $g \in \mathcal{B}_h(B)$, then by the proof of Proposition 3.8, $g = P_{\alpha,0}\tilde{g}$, where $\tilde{g}(x) = \frac{1}{\alpha+1}(1-|x|^2)[Rg(x) + (n/2+\alpha+1)g(x)] \in L^\infty(B)$ and $\|\tilde{g}\|_\infty \leq K\|g\|_{\mathcal{B}_h}$ for some constant $K > 0$. Then for any $f \in L^1_h(d\nu_\alpha)$, Fubini's theorem and Proposition 3.6 give

$$\begin{aligned}
|(f,g)| &= \left| \int_B f(x) \left[\int_B Q_{\alpha,0}(x,y)(1-|y|^2)^\alpha \tilde{g}(y)d\nu(y) \right] (1-|x|^2)^\alpha d\nu(x) \right| \\
&= \left| \int_B \tilde{g}(y) \left[\int_B Q_{\alpha,0}(y,x)(1-|x|^2)^\alpha f(x)d\nu(x) \right] (1-|y|^2)^\alpha d\nu(y) \right| \\
&\leq \|\tilde{g}\|_\infty \|f\|_{L^1_h(d\nu_\alpha)} \leq K\|g\|_{\mathcal{B}_h}\|f\|_{L^1_h(d\nu_\alpha)}.
\end{aligned}$$

This shows that every $g \in \mathcal{B}_h(B)$ defines a bounded linear functional Λ_g on $L^1_h(d\nu_\alpha)$, by the formula $\Lambda_g(f) = (f, g)$.

We now prove the converse.

Let $\Lambda \in (L^1_h(d\nu_\alpha))^*$. Recall that $L^1_h(d\nu_\alpha) = h_{1,1}((1-r^2)^{\alpha+1})$ and take $\varphi(r) = (1-r^2)^{\alpha+1}$, $\psi(r) = 1 - r^2$ in Theorem 1.1, then we have

$$(L^1_h(d\nu_\alpha))^* = h_{\infty,\infty}(1-r).$$

That means, there exists a unique $\tilde{g} \in h_{\infty,\infty}(1-r)$, such that

$$\Lambda(f) = \int_B f(x)\tilde{g}(x)(1-|x|^2)^{\alpha+1} d\nu(x)$$

for any $f \in L^1_h(d\nu_\alpha)$. Here note that the index of $\{\varphi, \psi\}$ is $\alpha + 2$.

Let

$$g(x) = \int_B \tilde{g}(y) Q_{\alpha,0}(x,y)(1-|y|^2)^{\alpha+1} d\nu(y).$$

By Lemma 3.3,

$$\begin{aligned}
|\nabla g(x)| &= \int_B |\tilde{g}(y)||\nabla_x Q_{\alpha,0}(x,y)|(1-|y|^2)^{\alpha+1} d\nu(y) \\
&\leq K \|\tilde{g}\|_{\infty,\infty,1-r} \int_0^1 \rho^{n-1}(1-\rho)^\alpha \left[\int_{S_n} \frac{d\sigma(\eta)}{|\rho x - \eta|^{n+1+\alpha}}\right] d\rho \\
&\leq K \|\tilde{g}\|_{\infty,\infty,1-r} \int_0^1 \frac{(1-\rho)^\alpha}{(1-r\rho)^{\alpha+2}} d\rho \leq \frac{K}{1-r} \|\tilde{g}\|_{\infty,\infty,1-r},
\end{aligned}$$

that is $g \in \mathcal{B}_h(B)$, and

$$\begin{aligned}
(f,g) &= \int_B f(x)(1-|x|^2)^\alpha \left\{\int_B \tilde{g}(y) Q_{\alpha,0}(x,y)(1-|y|^2)^{\alpha+1} d\nu(y)\right\} d\nu(x) \\
&= \int_B \tilde{g}(y)(1-|y|^2)^{\alpha+1} \left\{\int_B f(x) Q_{\alpha,0}(x,y)(1-|x|^2)^\alpha d\nu(x)\right\} d\nu(y) \\
&= \int_B f(y)\tilde{g}(y)(1-|y|^2)^{\alpha+1} d\nu(y) = \Lambda(f)
\end{aligned}$$

This completes the proof. □

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Résumé substantiel en français. Dans cet article, nous introduisons, en tant que généralisation de l'espace harmonique de Bergman, un type d'espace de fonctions harmoniques à norme mixte, dont nous étudions la dualité. En outre, nous donnons des conditions pour lesquelles certains opérateurs de type Bergman sur les espaces harmoniques à norme mixte sont bornés.

Une fonction positive continue φ sur $[0,1)$ est dite normale, s'il existe $0 < a < b$ et $0 \leq r_0 < 1$ tels que :

(i) $\varphi(r)/(1-r)^a$ est non-croissante dans $[r_0, 1)$ et $\lim_{r\to 1} \varphi(r)/(1-r)^a = 0$;

(ii) $\varphi(r)/(1-r)^b$ est non-décroissante dans $[r_0, 1)$ et $\lim_{r\to 1} \varphi(r)/(1-r)^b = \infty$.

Les fonctions $\{\varphi, \psi\}$ constituent une paire normale si φ est normale et, pour un b comme plus haut, il existe $\beta > b$ tel que

$$\varphi(r)\psi(r) = (1 - r^2)^\beta, \quad 0 \leq r < 1. \tag{0.1}$$

β est appelé l'indice de la paire normale $\{\varphi, \psi\}$.

Pour $0 < p \leq \infty$, $1 \leq q \leq \infty$, et une fonction normale φ, soit $L_{p,q}(\varphi)$ l'espace des fonctions mesurables (réelles, sans perte de généralité) telles que

$$\|f\|_{p,q,\varphi} = \left\{ \int_0^1 r^{n-1}(1 - r^2)^{-1}\varphi^p(r)M_q^p(r, f)dr \right\}^{1/p} < \infty,$$

où la moyenne intégrale $M_q(r, f)$ est définie comme d'habitude et $\|f\|_{\infty,q,\varphi}$ a le sens évident. L'espace harmonique à norme mixte $h_{p,q}(\varphi)$ est formé par les fonctions de $L_{p,q}(\varphi)$ qui sont harmoniques sur B. Si $1 \leq p, q \leq \infty$, alors $h_{p,q}(\varphi)$ est évidemment un espace de Banach pour la norme $\| \cdot \|_{p,q,\varphi}$.

De même que pour son correspondant holomorphe, notre résultat principal est comme suit.

Théorème 1.1. *Supposons que $0 < p < \infty$, $1 \leq q < \infty$. Le dual de $h_{p,q}(\varphi)$ peut être identifié à $h_{p',q'}(\psi)$. La dualité est donnée par*

$$(f, g) = \int_B f(x)g(x)(1 - |x|^2)^{\beta-1}d\nu(x),$$

où β est l'indice de la paire normale $\{\varphi, \psi\}$, $q' = q/(q - 1)$ et

$$p' = \begin{cases} p/(p-1), & p > 1, \\ \infty, & 0 < p \leq 1. \end{cases}$$

Plus précisément,

(i) *Si $g \in h_{p',q'}(\psi)$ et si on définit $\Lambda_g(f) = (f, g)$ pour $f \in h_{p,q}(\varphi)$, alors $\Lambda_g \in (h_{p,q}(\varphi))^*$ et $\|\Lambda_g\| \leq K\|g\|_{p',q',\psi}$.*

(ii) *Réciproquement, à un $\Lambda \in (h_{p,q}(\varphi))^*$ donné correspond un unique $g \in h_{p',q'}(\psi)$ tel que $\Lambda(f) = (f, g)$ pour tout $f \in h_{p,q}(\varphi)$ et $\|g\|_{p',q',\psi} \leq K\|\Lambda\|$.*

Nous prouvons le résultat avec des méthodes différentes dans le cas $1 < p < \infty$ et $0 < p \leq 1$.

Dans le cas $1 < p < \infty$, la preuve est standard, et même routinière, mais, dans notre nouveau contecte, nous étudions des opérateurs intégraux encore plus généraux $P_{s,t}$ de type Bergman

$$P_{s,t}f(x) = (1 - |x|^2)^t \int_B Q_{s,t}(x, y)(1 - |y|^2)^s f(y)d\nu(y),$$

où s, $t > -n/2$ et

$$Q_{s,t}(x, y) = \frac{2}{n}\sum_{k=0}^{\infty} \frac{\Gamma(k + n/2 + s + 1)}{\Gamma(k + n/2)\Gamma(s + 1)} \frac{\Gamma(k + n/2 + t)}{\Gamma(k + n/2)}|x|^k|y|^k Z_k(\zeta, \eta),$$

où $Z_k(\zeta, \eta)$ est l'harmonique dite de zone de degré k. Nous obtenons une condition nécessaire et une condition suffisante pour que $P_{s,t}$ soit borné sur $L_{p,q}(\varphi)$.

Proposition 3.5. *Supposons que $0 < p \leq \infty$, $1 \leq q \leq \infty$. Si $P_{s,t}$ est borné sur $L_{p,q}(\varphi)$, alors $s > a-1$ et $t > -b$. Ici a, b sont respectivement un indice non-croissant quelconque et un indice non-décroissant quelconque de φ.*

Proposition 3.6. *Soit $1 \leq p \leq \infty$, $1 \leq q \leq \infty$ and $s+1 > b > a > -t$. Alors $P_{s,t}$ est un opérateur linéaire borné sur $L_{p,q}(\varphi)$ et applique $L_{p,q}(\varphi)$ dans $h_{p,q}(\varphi)$. En outre, $P_{s,0}f = f$ pour tout $f \in h_{p,q}(\varphi)$.*

Ce dernier résultat conduit au résultat de dualité de façon standard.

Dans le cas $0 < p \leq 1$, et contrairement au cas précédent, nous donnons une preuve constructive basée sur l'expansion sphérique-harmonique des fonctions harmoniques sur la boule unité.

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