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NON TRIVIAL OBJECT-FIXING ENDOFUNCTORS OF FULL SUBCATEGORIES OF FINITE SETS

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RÉSUMÉ. L'objet de cet article est l'existence d'endofoncteurs des sous-catégories pleines de Ens fixant les objets (les « fixobs ») mais non isomorphes à l'endofoncteur identité. Ces endofoncteurs sont appelés « NTF » (Non Trivial Fixobs).

Cet article présente des résultats partiels sur ce problème qui, dans sa forme générale ou dans le cas de la sous-catégorie pleine des ensembles infinis, reste à résoudre: nous résolvons complètement le problème dans le cas des sous-catégories pleines engendrées par des ensembles finis. Plus précisément, les NTF des sous-catégories pleines engendrées par un ensemble fini ou par deux ensembles finis de cardinaux différents sont totalement décrits, et il est établi que les sous-catégories pleines engendrées par au moins trois ensembles finis non vides de cardinaux différents ne possèdent pas de NTF. Comme il est expliqué à la section 1.3, le problème général de l'existence de NTF se pose naturellement dans le contexte de la recherche de quantificateurs non « standards ».

ABSTRACT. We investigate the question of the existence of endofunctors of full subcategories of Ens fixing all objects but not isomorphic to the identity endofunctor (we call such endofunctors *non trivial fixobs* or NTF). Some results are given on the general problem, which remains open, and in the case of Inf (the full subcategory of infinite sets). However, the problem is totally solved for full subcategories generated by finite sets. We give a full and effective description of all NTF of full subcategories generated by one or two finite sets, and we show that any full subcategory generated by at least three non empty non isomorphic finite sets has no NTF. The origin of this research is in the question of the existence of non standard quantifiers.

1. Introduction.

1.1. Trivial and non trivial fixobs. We shall use throughout $\not{0}, \not{1}, \not{2}, \dots, \not{n}$, *etc.* for $0, 1, 2, \dots, n$, *etc.* when it will please us to explicitely consider finite ordinals as the sets $\emptyset$, $\{0\}$, $\{0,1\}$, *etc.*; $\mathrm{End}_{\not{n}}$ will mean "the monoid of endomorphisms of $\not{n} = \{0, 1, \dots, n-1\}$", and similar remarks hold for $\mathfrak{S}_{\not{n}}$, $\mathfrak{A}_{\not{n}}$, and so on. A constant map with value a will be written $\ulcorner a \urcorner$. If $X, Y, Z, \dots$ are sets, $\langle X, Y, Z, \dots \rangle$ is the full subcategory of Ens (the category of sets and maps) generated by $X, Y, Z, \dots$; if $\alpha = \{\alpha_X, \alpha_Y, \alpha_Z, \dots\}$ is a family of automorphisms of $X, Y, Z, \dots$ (respectively), then $\widehat{\alpha}$ is the endofunctor of

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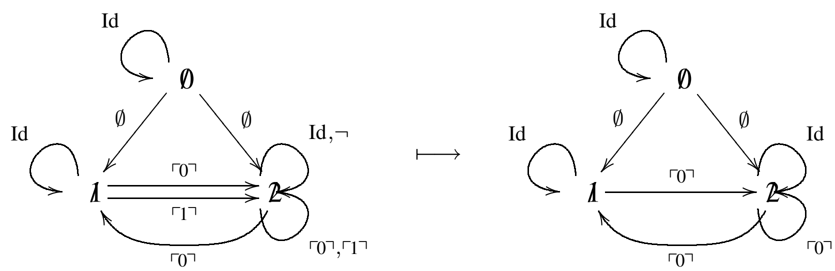
$\langle X, Y, Z, \dots \rangle$ defined by $\widehat{\alpha} : (f : X \to Y) \mapsto (\alpha_Y \circ f \circ \alpha_X^{-1})$, *i.e.* by the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle \alpha_X}\downarrow & & \downarrow{\scriptstyle \alpha_Y} \\ X & \xrightarrow[\widehat{\alpha}_f]{} & Y \end{array} \quad .$$

The cardinal of X will be noted $|\, X\,|$.

We call "fixob" of a category C an endofunctor F of C fixing each object, *i.e.* satisying $FX = X$ for each object X. A fixob F isomorphic to the identity endofunctor is canonically the same as the data of an isomorphism $\varphi = \{\varphi_X\}_X : \mathrm{Id}_C \to F$ and it is then given by $\widehat{\varphi}$; thus a fixob on C isomorphic to the identity just "reproduces" C in a category isomorphic to C; on the other hand, a fixob F not isomorphic to the identity, as a non inner endomorphism of group, "reorganizes" the structure of the arrows of the category. We shall speak of *Non Trivial Fixobs* (NTF), as opposed to trivial or inner fixobs for fixobs isomorphic to the identity. Note that an NTF can be an isomorphism.

A case of importance here is when C is $\langle \emptyset, 1, 2 \rangle$, and F is given by diagram (1.1). In this case, arrows are obtained through quotienting, but this is not always the case, as we will see in the next section.



(1.1)

1.2. The category of infinite sets possesses NTF. Let $\mathrm{Inf} = \mathrm{Ens}_\infty$ be the category of infinite sets and maps. Let us be given a functor $G : \mathrm{Inf} \to \mathrm{Ens}$ which does not increase cardinals, *i.e.* $|GX| \leq |X|$ for each infinite set X —then $\coprod_{GX} X$ is of cardinal $|X|$; we will write $\coprod_G X$ for $\coprod_{GX} X$. For $t \in GX$, we write j_t for the canonical injection $X \to \coprod_G X$ sending X "identically" on the t-th copy of X in $\coprod_G X$. Then, to each map $f : X \to Y$, we associate $\coprod_G f : \coprod_G X \to \coprod_G Y$ by mean of the rule "$\coprod_G f$ sends s in the t-th copy of X in $\coprod_G X$ to $f(s)$ in the $Gf(t)$-th copy of Y in $\coprod_G Y$" that is $f \mapsto \left(\coprod_G f : j_t(s) \mapsto j_{Gf(t)}(f(s))\right)$. This defines a functor $\coprod_G : \mathrm{Inf} \to \mathrm{Inf}$.

Let us also be given for each infinite set X a bijection $\alpha_X : \coprod_G X \to X$, through which $\widehat{\alpha} = \{\alpha_X\}_{X \text{ infinite}}$ yields a fixob $\widehat{\alpha} \coprod_G : \mathrm{Inf} \to \mathrm{Inf}$ given by

$$f : X \to Y \;\longmapsto\; \widehat{\alpha} \textstyle\coprod_G f = \alpha_Y \circ \coprod_G f \circ \alpha_X^{-1} \qquad \left(\begin{array}{ccc} \coprod_G X & \xrightarrow{\coprod_G f} & \coprod_G Y \\ {\scriptstyle \alpha_X}\downarrow & = & \downarrow{\scriptstyle \alpha_Y} \\ X & \xrightarrow{\widehat{\alpha}\coprod_G f} & Y \end{array} \right) .$$

In the particular case when $G = U$ for each X, with $|\, U\,| = u$, and $Gf = \mathrm{Id}_U$ for each f, $\widehat{\alpha} \coprod_G$ transforms maps with an image of cardinal k into maps with an image of cardinal ku. Such a $\widehat{\alpha}\coprod_G$ cannot be isomorphic to the identity endofunctor when $u > 1$.

In the particular case when G is the imbedding of Inf in Ens, $\widehat{\alpha}\coprod_G$ transforms maps with an image of cardinal k into maps with an image of cardinal $k \cdot k = k^2$. Such a $\widehat{\alpha}\coprod_G$ cannot be isomorphic to the identity endofunctor. Here Inf is "reorganized" via a fixob that "expands" ("unquotients") it rather than "quotients" it.

More generally, if $P \in \mathbb{N}[X]$, then $\widehat{\alpha}\coprod_P$ transforms maps with an image of finite cardinal k into maps with an image of cardinal $\gamma(k) = kP(k)$. Thus, $\{P\}_{P\in\mathbb{N}[X]}$ is an infinite family of NTF of Inf, no two of which are isomorphic. Another interesting possibility is with G given by $G(X) = \mathcal{P}_{\text{finite}}(X)$ (the set of finite subsets of X), $Gf : A \mapsto f(A)$; in this case, $\gamma(k) = k2^k$.

Still more generally, if from a family $\{G_n\}_{n\geq 1}$ we form the infinite series $\sum_{n\geq 1} G_n$, or some finite product $G_{n_1} \times \cdots \times G_{n_k}$, again we obtain a "cardinal reducing" functor Inf $\to$ Ens, and hence new NTF.

1.3. A word of motivation. The data of a set X virtually entails all what X generates in a model of a set theory (Zermelo-Fränkel, Gödel-Bernays, *etc.*), that is all what is "constructible" from X in the theory; for example, it entails the elements of the transfinite sequence $X, \mathcal{P}^1X, \ldots, \mathcal{P}^kX, \ldots, \cup_{k\in\mathbb{N}}\mathcal{P}^kX, \mathcal{P}^1\left(\cup_{k\in\mathbb{N}}\mathcal{P}^kX\right), \ldots$ Mathematical structures live as points of some terms of this sequence. For example, a topology on X may be seen as a point on $\mathcal{P}^2X$, the cartesian product $X \times X$ as a point in $\mathcal{P}^3X$, mappings, relations as points in some $\mathcal{P}^nX$, the free monoid X^* on X is a point in $\mathcal{P}^1\big(\cup_{k\in\mathbb{N}}\mathcal{P}^kX\big)$, *etc.*

The natural links between the structures on X are constrained by the natural links between the terms of this transfinite sequence. And so basically we have to know what are the *natural* links between the $\mathcal{P}^\gamma$'s, and moreover what "natural" means here.

For example, let us consider the links between $\mathcal{P}^1$ and $\mathcal{P}^2$. There are three known endofunctors P of Ens with $PX = \mathcal{P}^1X$: the direct image functor $\exists$, the inverse image functor C (contravariant), and $\forall$, defined respectively on $X \xrightarrow{f} Y$ by $A \mapsto f(A)$, $B \mapsto f^{-1}(B)$, and $A \mapsto Y\backslash f(X\backslash A)$ (the complement of the image of the complement). Now, the apparently simple question: "How is $\mathcal{P}X$ naturally imbedded in $\mathcal{P}^2X$?" has *at least* 14 meanings, for it means: "What are the natural transformations imbedding PX in QRX?", while P, Q and R may take their values in $\{\exists, C, \forall\}$ with the constraint that P and QR have the same variance. Here "at least" means that there might well be other functors L like $\exists, C$ and $\forall$ satisfying $LX = \mathcal{P}X$, so that the question may have more than 14 meanings (in the case with 14 meanings, *i.e.* with just $\{\exists, C, \forall\}$, a complete classification of the natural links between Id_{Ens}, $\mathcal{P}^1$ and $\mathcal{P}^2$ is stated in [DaGu1], with the details to be found in [DaGu2].

If Ens had an NTF L, then $\exists L$, CL and $\forall L$ would form a new ("non standard") triplet of functors with adjunctions relations "parallel" to the well known adjunction relations between $\exists, C$ and $\forall$ (when $\mathcal{P}$ is seen as ordered by inclusion), and that would mean a lot of non usual relations between the $\mathcal{P}^\gamma X$'s. Whence the question: "*Has* Ens *non trivial* fixobs*, and hence non standard quantifiers?*"

1.4. Results. This paper deals with the finite version of this problem. We give an effective procedure to list all NTF of any full subcategory $\langle X, Y\rangle$ of Fin $=$ Ens $_{\text{fin}}$ (the category of finite sets and maps); using the resulting description, we get that any full subcategory of Fin has no NTF if it contains three non empty non isomorphic objects X, Y, Z. Hence, Fin itself has no NTF (this is also proved separately when studying initial sections). Intermediate results, not depending on finiteness, are given in their full generality. Part of the results of this paper form the material of an explicit developement in the thesis of Farhan Ismail ([FI]).

2. NTF of initial sections of Ens.

2.1. A natural intrinsic criterion. An initial section of Ens is a full subcategory of Ens such that if it has an object X, it has all objects Y with $|Y| \leq |X|$. We shall write $\mathrm{Ens}_{<\kappa}$ for the full subcategory of Ens generated by all sets of cardinal strictly less than κ (the initial section bounded by κ); without loss of generality, we may restrict to the skeleton of $\mathrm{Ens}_{<\kappa}$, written $\mathrm{ens}_{<\kappa}$, whose objects are the cardinals that are elements of κ; we also write ens for the skeleton of Ens which is the full subcategory generated by all cardinals. Lower case greek letters μ, ν, $\ldots$ represent cardinals, while m, n, $\ldots$ represent only finite cardinals. We write $[\mu, \nu : (x_\iota)_{\iota\in\mu}]$ the map $\mu \to \nu$ given by $\iota \mapsto x_\iota$ for each $\iota \in \mu$, and (as above) simply $\ulcorner a \urcorner$ for a constant map with value a. F will be used only to mean a fixob.

Fixobs of initial sections preserve epis and monos as do all endofunctors of any full subcategory of Ens. Consequently, given a morphism $f : X \to Y$ of an initial section, $|\mathrm{Im}\, f| = |\mathrm{Im}\, Ff|$ because F transforms

$$\begin{tikzcd} & \lambda \arrow[dr, tail, "i"] & \\ X \arrow[ur, two heads, "p"] \arrow[rr, "f"] & & Y \end{tikzcd}$$

into

$$\begin{tikzcd} & \lambda \arrow[dr, tail, "Fi"] & \\ X \arrow[ur, two heads, "Fp"] \arrow[rr, "Ff"] & & Y \end{tikzcd} .$$

In particular, F transforms constant maps into constant maps. Let us remark here that "$|\mathrm{Im}\, f| = |\mathrm{Im}\, Ff|$ for fixobs of initial sections" implies that the examples of NTF of $\mathrm{Inf} = \mathrm{Ens}_\infty$ (see 1.0) cannot extend to NTF of Ens for they fail to send constant maps on constant maps.

If a fixob F of $\mathrm{ens}_{<\kappa}$ sends $1 \xrightarrow{\ulcorner x \urcorner} \mu$ ($\mu \in \kappa$) on $1 \xrightarrow{\ulcorner y \urcorner} \mu$, then, for any constant map $\lambda \xrightarrow{\ulcorner x \urcorner} \mu$ in the initial section $\mathrm{ens}_{<\kappa}$, $F\ulcorner x \urcorner = \ulcorner y \urcorner$, for

$$\begin{tikzcd} \lambda \arrow[r, "\ulcorner 0 \urcorner"] \arrow[rr, bend right, "\ulcorner x \urcorner"'] & 1 \arrow[r, "\ulcorner x \urcorner"] & \mu \end{tikzcd}$$

is transformed into

$$\begin{tikzcd} \lambda \arrow[r, "\ulcorner 0 \urcorner"] \arrow[rr, bend right, "F\ulcorner x \urcorner"'] & 1 \arrow[r, "\ulcorner y \urcorner"] & \mu \end{tikzcd} ,$$

i.e. a fixob F transforms all constant maps with a given value to constant maps with another given value, thus inducing an endomorphism of each object in the initial section.

Given a fixob $F : \mathrm{ens}_{<\kappa} \to \mathrm{ens}_{<\kappa}$, for each $\lambda < \kappa$, let $\widetilde{F}_\lambda : \lambda \to \lambda$ be the induced endomorphism at level λ, *i.e.* $F\ulcorner x_0 \urcorner = \ulcorner \widetilde{F}_\lambda x_0 \urcorner$: $F[1, \lambda : x_0] = [1, \lambda : \ulcorner \widetilde{F}_\lambda x_0 \urcorner]$. For $\mu, \nu < \kappa$,

$$\begin{tikzcd} 1 \arrow[r, "{[1,\mu:\iota]}"] \arrow[rr, bend right, "{[1,\nu:x_\iota]}"'] & \mu \arrow[r, "{[\mu,\nu:(x_\iota)_{\iota\in\mu}]}"] & \nu \end{tikzcd}$$

is transformed into

$$\begin{tikzcd} 1 \arrow[r, "{[1,\mu:\widetilde{F}_\mu \iota]}"] \arrow[rr, bend right, "{[1,\nu:\widetilde{F}_\nu x_\iota]}"'] & \mu \arrow[r, "{[\mu,\nu:(y_\iota)_{\iota\in\mu}]}"] & \nu \end{tikzcd} ,$$

which means $y_{\widetilde{F}_\mu \iota} = \widetilde{F}_\nu x_\iota$, *i.e.* that $\widetilde{F} = \{\widetilde{F}_\mu\}_{\mu\in\kappa}$ is a natural transformation:

$$\begin{tikzcd} \mu \arrow[r, "x"] \arrow[d, "\widetilde{F}_\mu"'] & \nu \arrow[d, "\widetilde{F}_\nu"] \\ \mu \arrow[r, "y"'] & \nu \end{tikzcd} .$$

It is immediate that, given fixobs R and H of an initial section, $\widetilde{R}\widetilde{H}$ is the natural transformation induced by RH, and that if G extends a fixob F of $\mathrm{ens}_{<\kappa}$ to a fixob of $\mathrm{ens}_{<\lambda}$, $(\kappa \leq \lambda)$, then for each $\iota \in \kappa$, $\widetilde{F}_\iota = \widetilde{G}_\iota$. Moreover, given a family of automorphisms $\alpha = \{\alpha_a\}_{a\in\kappa}$ of elements of κ, we immediately check that $\widetilde{\widetilde{\alpha}} = \alpha$. Hence:

A fixob F of ens *(resp.* $\mathrm{ens}_{<\kappa}$*) is isomorphic to the identity endofunctor if, and only if, the induced natural transformation $\widetilde{F}$ is a natural equivalence,* or equivalently, *if, and only if, it transforms bijectively each set of constant maps $\mathbb{1} \to \lambda$ into itself. One easily checks that this exactly means that a fixob F is isomorphic to the identity endofunctor if, and only if, it commutes with coproducts.* (2.1)

2.2. The identity naturally imbeds in fixobs of initial sections. The fixobs of $\mathrm{ens}_{<0}$ (the empty category), $\mathrm{ens}_{<1}$ and $\mathrm{ens}_{<2}$ can only be the identity endofunctor. Let now F be a fixob of $\mathrm{ens}_{<3}$. If $\widetilde{F}_{\mathbb{2}}$ is an isomorphism (*i.e.* the identity map or the negation, which we write $\neg$), then F is isomorphic to the identity endofunctor, as observed in (2.1) above, since $\widetilde{F}_{\mathbb{0}}$ and $\widetilde{F}_{\mathbb{1}}$ are obviously isomorphisms. If $\widetilde{F}_{\mathbb{2}} = \ulcorner 0 \urcorner$, then F must send all non isomorphisms (*i.e.* here the constant maps) on $\ulcorner 0 \urcorner$, and the negation (in this case, essentially all isomorphisms), on the identity

$$\left(\begin{tikzcd} \mathbb{1} \arrow[r, "\ulcorner x \urcorner"'] \arrow[d, "\widetilde{F}_{\mathbb{1}}"'] & \mathbb{2} \arrow[d, "\widetilde{F}_{\mathbb{2}} = \ulcorner 0 \urcorner"] \\ \mathbb{1} \arrow[r, "F\ulcorner x \urcorner"] & \mathbb{2} \end{tikzcd} , \quad \begin{tikzcd} \mathbb{1} \arrow[r, "\ulcorner x \urcorner"] \arrow[rr, bend right, "\ulcorner \neg x \urcorner"'] & \mathbb{2} \arrow[r, "\neg"] & \mathbb{2} \end{tikzcd} \overset{F}{\longmapsto} \begin{tikzcd} \mathbb{1} \arrow[r, "\ulcorner 0 \urcorner"] \arrow[rr, bend right, "\ulcorner 0 \urcorner"'] & \mathbb{2} \arrow[r, "F\neg"] & \mathbb{2} \end{tikzcd} \right) .$$

It is immediate that this defines a fixob of $\mathrm{ens}_{<3}$, which we write $R^{(0)}$. If $\widetilde{F}_{\mathbb{2}} = \ulcorner 1 \urcorner$, we have in a similar way $F = R^{(1)}$, with $R^{(1)}$ as in

$$R^{(i)},\ (i = 0, 1), \begin{cases} \text{isomorphisms} \mapsto \text{identities} \\ \text{non isomorphisms} \mapsto \text{constant maps } \ulcorner i \urcorner . \end{cases}$$

For instance, $R^{(0)}$ is depicted in (1.1) above.

Proposition 2.1. *$R^{(0)}$ and $R^{(1)}$ are isomorphic NTF and do not extend to fixobs of* $\mathrm{ens}_{<4}$.

Proof. $\alpha = \{\mathrm{Id}_0, \mathrm{Id}_1, \neg\}$ is an isomorphism $R^{(0)} \to R^{(1)}$; both $R^{(0)}$ and $R^{(1)}$ are NTF by (2.1) since $R^{(i)}_{\mathbf{2}} = \ulcorner i \urcorner$, $(i = 0, 1)$. $R^{(0)}$ and $R^{(1)}$ being isomorphic, we now restrict to $R^{(0)}$. So, let F be a fixob of $\mathrm{ens}_{<4}$ extending $R^{(0)}$. Letting x vary through injections $\mathbf{2} \rightarrowtail \mathbf{3}$ in

$$\begin{tikzcd} \mathbf{2} \arrow[r, "x"] \arrow[d, "\widetilde{F}_{\mathbf{2}} = \ulcorner 0 \urcorner"] & \mathbf{3} \arrow[d, "\widetilde{F}_{\mathbf{3}}"] \\ \mathbf{2} \arrow[r, "Fx"'] & \mathbf{3} \end{tikzcd} ,$$

we see that $\widetilde{F}_{\mathbf{3}}$ must be $\ulcorner * \urcorner : \mathbf{3} \to \mathbf{3}$ for some element $* \in \mathbf{3}$, and that for any injection $\mathbf{2} \overset{x}{\rightarrowtail} \mathbf{3}$, $Fx(0) = *$. Upon composing these injections with the negation, and knowing that $F(\neg) = \mathrm{Id}_{\mathbf{2}}$ (since F extends $R^{(0)}$), we obtain

$$\begin{aligned} F[\mathbf{2}, \mathbf{3} : 0, 1] &= F[\mathbf{2}, \mathbf{3} : 1, 0] = [\mathbf{2}, \mathbf{3} : *, a] \\ F[\mathbf{2}, \mathbf{3} : 0, 2] &= F[\mathbf{2}, \mathbf{3} : 2, 0] = [\mathbf{2}, \mathbf{3} : *, b] \\ F[\mathbf{2}, \mathbf{3} : 1, 2] &= F[\mathbf{2}, \mathbf{3} : 2, 1] = [\mathbf{2}, \mathbf{3} : *, c] \end{aligned}$$

with $a, b, c \in \mathbf{3} - \{*\}$. Since F preserves the cardinal of the image, $* \notin \{a, b, c\} \subset \mathbf{3}$; thus $\{a, b, c\}$ is of cardinal 1 or 2. But upon evaluating F on commutative diagrams of the form

$$\begin{tikzcd} \mathbf{2} \arrow[r, tail] \arrow[rr, bend right] & \mathbf{3} \arrow[r, two heads] & \mathbf{2} \end{tikzcd} ,$$

we reach the contradictory conclusion that a, b, c must be different. The argument goes as below for $a \neq b$,

$$\left.\begin{array}{l} \begin{tikzcd} \mathbf{2} \arrow[r, tail, "{[\mathbf{2}, \mathbf{3}\,; 0,1]}"] \arrow[rr, bend right, "\ulcorner 0 \urcorner"'] & \mathbf{3} \arrow[r, two heads, "{f = [\mathbf{3}, \mathbf{2}\,; 0,0,1]}"] & \mathbf{2} \end{tikzcd} \longmapsto \begin{tikzcd} \mathbf{2} \arrow[r, tail, "{[\mathbf{2}, \mathbf{3}\,; *, a]}"] \arrow[rr, bend right, "\ulcorner 0 \urcorner"'] & \mathbf{3} \arrow[r, two heads, "Ff"] & \mathbf{2} \end{tikzcd} : Ff(a) = 0 \\ \begin{tikzcd} \mathbf{2} \arrow[r, tail, "{[\mathbf{2}, \mathbf{3}\,; 0,2]}"] \arrow[rr, bend right, "\mathrm{Id}"'] & \mathbf{3} \arrow[r, two heads, "{f = [\mathbf{3}, \mathbf{2}\,; 0,0,1]}"] & \mathbf{2} \end{tikzcd} \longmapsto \begin{tikzcd} \mathbf{2} \arrow[r, tail, "{[\mathbf{2}, \mathbf{3}\,; *, b]}"] \arrow[rr, bend right, "\mathrm{Id}"'] & \mathbf{3} \arrow[r, two heads, "Ff"] & \mathbf{2} \end{tikzcd} : Ff(b) = 1 \end{array}\right\} : a \neq b$$

and similarly for $b \neq c$ and $a \neq c$ (considering for example $\mathrm{Id}_{\mathbf{2}} = [\mathbf{3}, \mathbf{2}; 1, 0, 1] \circ [\mathbf{2}, \mathbf{3}; 1, 2]$ & $\ulcorner 1 \urcorner = [\mathbf{3}, \mathbf{2}; 1, 0, 1] \circ [\mathbf{2}, \mathbf{3}; 0, 2]$ for $b \neq c$ and $\mathrm{Id}_{\mathbf{2}} = [\mathbf{3}, \mathbf{2}; 0, 0, 1] \circ [\mathbf{2}, \mathbf{3}; 1, 2]$ & $\ulcorner 0 \urcorner = \mathbf{3}, \mathbf{2}; 0, 0, 1] \circ [\mathbf{2}, \mathbf{3}; 0, 1]$ for $a \neq c$). $\square$

Corollary 2.2. *If* $4 \leq \kappa$, *for any fixob F of* $\mathrm{ens}_{<\kappa}$ *or* Ens, *the components of $\widetilde{F}$ are injections, i.e. the identity endofunctor naturally imbeds in F. Moreover, F is isomorphic to the identity if, and only if, this imbedding is an isomorphism.*

Let F be such a fixob. Since F can neither extend $R^{(0)}$ nor $R^{(1)}$, $\widetilde{F}_{\mathbb{2}}$ is bijective; the result follows then from the commutative diagrams

$$\begin{tikzcd} \mathbb{2} \arrow[r, tail, "i"] \arrow[d, "\widetilde{F}_{\mathbb{2}}"'] & \alpha \arrow[d, "\widetilde{F}_{\alpha}"] \\ \mathbb{2} \arrow[r, tail, "Fi"'] & \alpha \end{tikzcd},$$

when i runs through all injections $\mathbb{2} \rightarrowtail \alpha$.

So, given a fixob F of $\text{ens}_{<\omega}$, the components of $\widetilde{F}$ are injective endomorphisms of finite sets, *i.e.* $\widetilde{F}$ is a natural equivalence.

Corollary 2.3. *For all* $n \geq 4$*, all fixobs of* $\text{ens}_{<\mathbb{n}}$ *are isomorphic to the identity endofunctor.*

Thus, the identity endofunctor of $\text{Ens}_{<\omega}$ is, up to an isomorphism, determined by its value on objects. We now go at the analysis of Id_C for C *any* full subcategory of Fin.

3. The endomorphisms of $\text{End}_{\mathbb{n}}$**.** The aim of this section is to fully describe endofunctors F of $\text{End}_{\mathbb{n}}$. Results whose proof does not require finiteness are given in their full generality. The endofunctors of $\text{End}_{\mathbb{n}}$ are characterized in Theorems 3.1 and 3.8. We begin with a few simple facts.

The identity endofunctor of any full subcategory of Ens *is the only endofunctor fixing all constant maps (i.e.* $F\ulcorner k \urcorner = \ulcorner k \urcorner$ *for all* $k \in X$*).* (3.1)

Indeed, for each $k \in X$ and each $f : X \to Y$, we have, $f \circ \ulcorner k \urcorner = \ulcorner fk \urcorner$; if F fixes all constant maps, then we have $Ff \circ \ulcorner k \urcorner = \ulcorner fk \urcorner$, which implies $Ff(k) = f(k)$ for all k, *i.e.* $Ff = f$; since f is an arbitrary map $X \to Y$, $F = \text{Id}$.

Observation (3.1) immediately implies that fixobs of any full subcategory of Ens are exactly those that are bijective on the set of constant maps; however, in general, a fixob is not determined by its value(s) on constant maps; for example, $F, G : \text{End}_{\mathbb{7}} \to \text{End}_{\mathbb{7}}$ given by

$$Ff = (0,1)(2,3) \text{ if } \mathfrak{S}_{\mathbb{7}} \ni f \text{ is odd, Id if it is even, and } u \text{ if } \S\mathbb{7} \not\ni f;$$
$$Gf = (0,1) \text{ if } \mathfrak{S}_{\mathbb{7}} \ni f \text{ is odd, Id if it is even, and } u \text{ if } \mathfrak{S}_{\mathbb{7}} \not\ni f;$$

where $u : \{0,1,2,3,4\} \mapsto 4, \{5,6\} \mapsto 6$, take the same unique value on constant maps but are not even isomorphic.

Given $h, k \in X$ *and* F *an endofunctor of* End_X*,* $F\ulcorner h \urcorner$ *and* $F\ulcorner k \urcorner$ *induce the same partition of* X *in fibers. Moreover,* $F\ulcorner k \urcorner = F\ulcorner h \urcorner$ *if, and only if,* $\text{Im}\, F\ulcorner k \urcorner = \text{Im}\, F\ulcorner h \urcorner$. (3.2)

Indeed, for all $k, h \in X$, $\ulcorner k \urcorner \circ \ulcorner h \urcorner = \ulcorner k \urcorner$, whence $F\ulcorner k \urcorner \circ F\ulcorner h \urcorner = F\ulcorner k \urcorner$. Thus, for all $h, k \in X$, $(\text{mod}\, F\ulcorner h \urcorner) \subset (\text{mod}\, F\ulcorner k \urcorner)$ (Notation: for a mapping g, $(\text{mod}\, g)$ is the equivalence partitioning the domain of g into fibers). Since this is for all h, k, $(\text{mod}\, F\ulcorner k \urcorner) = (\text{mod}\, F\ulcorner h \urcorner)$. Moreover, $F\ulcorner k \urcorner$ and $F\ulcorner h \urcorner$ are idempotent maps, and idempotents having the same fibers are equal if, and only if, they have the same image.

3.1. When F induces an automorphism of Aut_X. If F induces an automorphism of Aut_X, then F transforms constant endomaps of X into constant maps since, for all $f \in \mathrm{Aut}_X$,

$$\begin{tikzcd} X \arrow[r, "\ulcorner k \urcorner"] & X \\ X \arrow[from=1-1, "f"'] \arrow[ur, "\ulcorner k \urcorner"'] & \end{tikzcd}$$

is transformed into

$$\begin{tikzcd} X \arrow[r, "F\ulcorner k \urcorner"] & X \\ X \arrow[from=1-1, "Ff"'] \arrow[ur, "F\ulcorner k \urcorner"'] & \end{tikzcd},$$

and Ff could be any $g \in \mathrm{Aut}_X$. Let $\widetilde{F}$ be the endomorphism of X defined through $\ulcorner \widetilde{F}k \urcorner = F\ulcorner k \urcorner$. As for $\widetilde{F}$ in the case of a fixob F of an initial section, this is a natural transformation:

$$\begin{aligned} Ff(\widetilde{F}(x)) &= \left(Ff \circ \ulcorner \widetilde{F}(x) \urcorner\right)(x) = (Ff \circ F\ulcorner x \urcorner)(x) \\ &= F\,(f \circ \ulcorner x \urcorner)\,(x) = \left(F\ulcorner f(x) \urcorner\right)(x) \\ &= \ulcorner \widetilde{F}(f(x)) \urcorner(x) = \widetilde{F}(f(x)) \end{aligned}$$

i.e.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\widetilde{F}"'] & X \arrow[d, "\widetilde{F}"] \\ X \arrow[r, "Ff"] & X \end{tikzcd}$$

is commutative. In fact, $\widetilde{F}$ is bijective; indeed, for each $f \in \mathrm{Aut}_X$, the diagram

$$\begin{tikzcd} X \arrow[r, "\ulcorner k \urcorner"] \arrow[d, "f"'] & X \arrow[d, "f"] \\ X \arrow[r, "\ulcorner fk \urcorner"] & X \end{tikzcd}$$

is transformed into

$$\begin{tikzcd} X \arrow[r, "F\ulcorner k \urcorner \overset{\text{def}}{=} \ulcorner \widetilde{F}k \urcorner"] \arrow[d, "Ff"'] & X \arrow[d, "Ff"] \\ X \arrow[r, "F\ulcorner fk \urcorner \overset{\text{def}}{=} \ulcorner \widetilde{F}fk \urcorner"] & X \end{tikzcd}$$

whence $\ulcorner \widetilde{F}fk \urcorner = \ulcorner Ff\widetilde{F}k \urcorner$. If f runs through Aut_X, $Ff\widetilde{F}k$ runs through X for any fixed k, which establishes the surjectivity of $\widetilde{F}$; moreover, if $\widetilde{F}$ identifies k_1 and k_2, then $\widetilde{F}$ shall identify fk_1 and fk_2 for all $f \in \mathrm{Aut}_X$. Therefore, either $\widetilde{F}$ is injective or it is constant; if $|X| \leq 1$, $\widetilde{F}$ is clearly injective, and if not, $\widetilde{F}$ being already surjective, it cannot be constant, and thus it is injective. Therefore, we just showed that

An endofunctor F of End_X inducing an automorphism of Aut_X is isomorphic to the identity endofunctor. It is then expressible as the inner automorphism $\widetilde{F}$. (3.3)

As the converse is true, we have

Theorem 3.1. *An endofunctor F of* End_X *is isomorphic to the identity if, and only if, it induces an automorphism of* Aut_X.

Proposition 3.2. *The mapping* $\widehat{(\ )} : \mathrm{Aut}_X \to \mathrm{Aut}_{\mathrm{End}_X}$ *given by* $\varphi \mapsto \widehat{\varphi}$ *is an isomorphism of groups.*

Proof. $\widehat{(\ )}$ is clearly a homomorphism of groups. (3.3) implies its surjectivity. If $\varphi \in \mathrm{Ker}\,\widehat{(\ )}$, then for each k, $\widehat{\varphi}\ulcorner k \urcorner = \ulcorner k \urcorner$, is, $\ulcorner \varphi k \urcorner = \ulcorner k \urcorner$ for all k, *i.e.* $\varphi = \mathrm{Id}_X$; thus $\mathrm{Ker}\,\widehat{(\ )} = \{\mathrm{Id}_X\}$, and $\widehat{(\ )}$ is injective. $\square$

Thus Aut_X imbeds in $\mathrm{End}_{\mathrm{End}_X}$: $\mathrm{Aut}_X \xrightarrow{\sim} \mathrm{Aut}_{\mathrm{End}_X} \hookrightarrow \mathrm{End}_{\mathrm{End}_X}$. Proposition 3.2 implies that all automorphisms of End_X are inner, and therefore induce inner automorphisms of Aut_X. Conversely, an inner automorphism $\widehat{\varphi}$ of Aut_X clearly extend to the inner automorphisms $\widehat{\varphi}$ of End_X.

Proposition 3.3. [1] *The automorphisms of* Aut_X *which extend to endomorphisms of* End_X *are those extending to automorphisms of* End_X*, and they are exactly the inner automorphisms;* [2] *moreover, an inner automorphism of* Aut_X *extends uniquely as an automorphism of* End_X.

Proof. [1] results from (3.3). It remains to prove [2]. For $|X| \leq 2$, $\mathrm{Aut}_{\mathrm{Aut}_X} = \{\mathrm{Id}_{\mathrm{Aut}_X}\}$, and there is nothing to prove. Thus, we need just check the unicity when $|X| \geq 3$. Let F induce the inner automorphisms $\widehat{\varphi}$ of Aut_X. Then $\widehat{\varphi^{-1}} \circ F$ induces the identity on Aut_X. It is sufficient to prove that the only endofunctor F inducing the identity on Aut_X is the identity endofunctor, for then $\widehat{\varphi^{-1}} \circ F = \mathrm{Id}_{\mathrm{End}_X}$, which implies $F = \widehat{\varphi}$. Thus let F induce the identity on Aut_X. Let $k \in X$ and ψ be an automorphism having only k as a fixed point. Then $F\ulcorner k \urcorner = F\psi \circ F\ulcorner k \urcorner$, *i.e.* $F\ulcorner k \urcorner = \psi \circ F\ulcorner k \urcorner$. This implies that $F\ulcorner k \urcorner = \ulcorner k \urcorner$, which implies in turn that $F = \mathrm{Id}_{\mathrm{End}_X}$ by (3.1). $\square$

Corollary 3.4. *A fixob F of an initial section is isomorphic to the identity if, and only if, it induces an automorphism of each* Aut_X.

Proof. We need just prove that if F induces an automorphism of each Aut_X, then it is isomorphic to the identity. Without loss of generality, we may suppose that F fixes each Aut_X for we may compose F with $\widehat{\widetilde{F}_X^{\,-1}}$. In this case, by (3.3), F fixes each End_X, and from that we deduce that F fixes all constant maps in the initial section since

$$\forall f \ : \quad \begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "\ulcorner h \urcorner"'] & Y \arrow[d, "\ulcorner h \urcorner"] \\ & Y \end{tikzcd} \quad \longmapsto \quad \begin{tikzcd} X \arrow[r, "Ff"] \arrow[dr, "F\ulcorner h \urcorner"'] & Y \arrow[d, "\ulcorner h \urcorner"] \\ & Y \end{tikzcd}$$

implies that $F\ulcorner h \urcorner = \ulcorner h \urcorner$, and thus, as observed in (3.1), that $F = \mathrm{Id}_{\mathrm{Ens}}$. $\square$

The following diagram sums up the situation, where $\mathrm{In}_{\mathrm{End}_X}$, $\mathrm{In}_{\mathrm{Aut}_X}$ are the "inner" endofunctors (*i.e.* the trivial fixobs):

$$\begin{tikzcd}
\mathrm{End}_{\mathrm{Aut}_X} & & \mathrm{End}_{\mathrm{End}_X} \arrow[ll] & \text{(Proposition 3.2)} \\
& \text{(Pullback)} & \mathrm{Aut}_{\mathrm{End}_X} \arrow[u, tail] & \mathrm{Aut}_X \arrow[l, "\widehat{()}"'] \arrow[dl, "\sim"] \\
\mathrm{Aut}_{\mathrm{Aut}_X} \arrow[uu, tail] & \mathrm{In}_{\mathrm{Aut}_X} \arrow[l, tail, "(3.3)"'] & \mathrm{In}_{\mathrm{End}_X} \arrow[l, "\sim"'] \arrow[u, "\sim"] &
\end{tikzcd}$$

(in fact, the four isomorphisms of this diagram are also obtained by Schreier in [Sch1]).

In the study of fixobs of initial sections, we associated with each fixob F of Ens a natural transformation also written $\widetilde{F}$, which imbeds, up to within an isomorphism, the identity endofunctor in F; this transformation $\widetilde{F}$ came from the value of F on constant maps which were transformed into constant maps, and coincide therefore with the transformation $\widetilde{F}_X$ (for a fixob F_X of End_X) when F_X induces an automorphism of Aut_X and is induced by a fixob of an initial section containing X.

3.2. When F kills $\mathfrak{S}_{\not{p}}$. We shall say that two endomorphisms f, g of X are *isoweighted* if they have the same "number" of fibers of any given cardinal. In particular, $\left|\complement \operatorname{Im} f\right| = \left|\complement \operatorname{Im} g\right|$ (the same number of empty fibers). One easily checks that f, g are isoweighted if, and only if, there exists $\varphi, \psi \in \mathrm{Aut}_X$ for which

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\psi} \\ X & \xrightarrow[g]{} & X \end{array}$$

is commutative (see Clifford [ClPr1] for similar considerations). Clearly, an endofunctor F of End_X preserves "isoweightedness", and if F kills Aut_X, F is constant on any class of isoweighted maps.

For any $f \in \mathrm{End}_X$, if $|\operatorname{Im} f| < |X|$, then f is isoweighted to an idempotent. (3.4)

To prove (3.4), we may restrict to the case where X is infinite since (3.4) is immediate for X finite. When X is infinite, it is then always possible to "construct" a partition $\{A_u\}_{u \in \operatorname{Im} f}$ of X with, for each $u \in \operatorname{Im} f$, $u \in A_u$ and $|A_u| = \left|f^{-1}(u)\right|$. Indeed, (a) since $|\operatorname{Im} f| < |X|$, $\complement \operatorname{Im} f = |X|$; choose a bijection $\psi : X \to \complement \operatorname{Im} f$ and create the partition of $\complement \operatorname{Im} f$: $\left\{C_u = \psi(f^{-1}(u))\right\}_{u \in \operatorname{Im} f}$; (b) for each $u \in \operatorname{Im} f$, choose $t_u \in C_u$ and set $B_u = (C_u \backslash t_u) \cup \{u\}$; also, set $T = \{t_u\}_{u \in \operatorname{Im} f}$; (c) because $|\operatorname{Im} f| < |X|$, one of the B_u is of cardinal $|X|$, say B_{u_0}; set

$$A_u = \begin{cases} B_u & \text{if } u \neq u_0 \\ B_u \cup T & \text{if } u = u_0\,. \end{cases}$$

Then, $g(t)$ = "*the unique u such that $t \in A_u$*" defines an idempotent g isoweighted to f.

Note that (3.4) is true without any restriction on $\operatorname{Im} f$ if X is finite; the restriction $|\operatorname{Im} f| < |X|$ is necessary if X is infinite (*e.g.* the shift $\omega \to \omega$ is isoweighted to no idempotent).

Since constant maps $X \to X$ are conjugate (and hence *a fortiori* isoweighted) to each other, if F kills Aut_X, then F assumes the same value on all constant maps. Let α_F be this value of F on any constant map; note that α_F is an idempotent in End_X.

Proposition 3.5. *Let F be an endofunctor of End_X killing Aut_X. Then, for all f such that $|\mathrm{Im}\, f| < |X|$, $F = \alpha_F$.*

Proof. The statement is vacuously true if $|X| \leq 1$. It is trivially true if $|\, X\,| = 2$ since then $|\mathrm{Im}\, f| < |\, X\,|$ means "constant". So we suppose $|\, X\,| \geq 3$.

—*Case 1:* $|\mathrm{Im}\, f|$ *is finite*. Let us suppose that there exists endomorphisms of X such that $1 < n = |\mathrm{Im}\, f| < |\, X\,|$, with $Ff \neq \alpha_F$. Let us consider the smallest n for which such an f exists. Since f isoweighted to an idempotent (by (3.4)), we suppose f idempotent. Clearly, one of the fibers $A_1, A_2, \ldots, A_n$ of f, say A_1, has at least two elements. Let $B_1, B_2, \ldots, B_n$ be a partition of X with $|\, B_i\,| = |\, A_i\,|$, $b_1 \in B_1 \cap A_1$, $b_2 \in B_2 \cap A_1$, $b_3 \in B_3, \ldots, b_n \in B_n$, and let $g(t) = b_k$ if $t \in B_k$. Then (1) f and g are isoweighted, whence $Ff = Fg$, and (2) $|\,\mathrm{Im}(f \circ g)\,| \leq n - 1$; so we have

$$\alpha_F = F(f \circ g) = Ff \circ Fg = Ff \circ Ff = Ff \qquad (\diamondsuit)$$

which is a contradiction as we supposed $Ff \neq \alpha_F$.

—*Case 2:* $|\mathrm{Im}\, f|$ *is infinite*. Since f is isoweighted to an idempotent (by (3.4)), we suppose f idempotent. Since $\mathrm{Im}\, f < |X|$, one of the fibers of f is of cardinal strictly larger than $|\mathrm{Im}\, f|$, say $X_0 = f^{-1}(x_0)$. Then, we choose a partition of X "isomorphic to the partition induced by (mod f)", with a system of representatives in X_0. Let g be the projection of this partition to its system of representants. $(1')$ $f \circ g$ is constant, $(2')$ f and g are isoweighted. Then we have again $(\diamondsuit)$. $\square$

Corollary 3.6. *The endofunctors of $\mathrm{End}_{\not{n}}$ killing $\mathfrak{S}_{\not{n}}$ are classified by the idempotents of $\mathrm{End}_{\not{n}}$: each such endofunctor of $\mathrm{End}_{\not{n}}$ is of the form*

$$F_h f = \begin{cases} \mathrm{Id}_{\not{n}} & \textit{if } f \in \mathfrak{S}_{\not{n}} \\ h & \textit{otherwise.} \end{cases}$$

for an idempotent $h \in \mathrm{End}_{\not{n}}$, and for each idempotent $h \in \mathrm{End}_{\not{n}}$, this defines an endofunctor of $\mathrm{End}_{\not{n}}$.

Corollary 3.7.

[1] *Let F be a fixob of $\langle X, Y\rangle$ with $|\, X\,| < |\, Y\,|$. Then, if F kills Aut_X and Aut_Y, it also kills End_X.*

[2] *Let F be a fixob of $\langle X, Y\rangle$ with $0 \neq |\, X\,| < |\, Y\,|$. Then, F cannot kill End_Y.*

[3] *Let F be a fixob of $\langle X, Y, Z\rangle$ with $0 \neq |\, X\,| < |\, Y\,| < |\, Z\,|$. Then, F cannot kill Aut_Y and Aut_Z.*

[4] *There are no fixobs of* Inf *killing all groups of automorphisms.*

Proof. [1] Let $i : X \to Y$ imbed X in Y, and let $X_0 = \mathrm{Im}\, i$. For each $g : X \to X$, let $\overline{g} : Y \to Y$ be an "extension" of g with

$$\mathrm{Im}\, g = i(\mathrm{Im}\, \overline{g}) \ : \ \begin{array}{ccc} X & \xrightarrow{i} & Y \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \overline{g}} \\ X & \xrightarrow{i} & Y \end{array} \ .$$

This diagram is transformed into

$$\begin{tikzcd} X \arrow[r, tail, "Fi"] \arrow[d, "Fg"'] & Y \arrow[d, "F\overline{g}=\alpha_Y"] \\ X \arrow[r, tail, "Fi"] & Y \end{tikzcd}$$

(we write α_Z for the value of F on all u's in End_Z with $|\,\mathrm{Im}\,u\,| < |\,Z\,|$). With $g = \mathrm{Id}_X$, we see that α_Y behaves on X_0 as Id_{X_0}, and with g a constant map, that it behaves on X_0 as α_X, if we identify X and X_0 via i. Therefore, $\alpha_X = \mathrm{Id}_X$, *i.e.* F kills End_Y.

[2] Let $\epsilon : \; Y \twoheadrightarrow X$ be a surjection. Let us suppose that a fixob F kills End_Y. Then

$$\begin{tikzcd} Y \arrow[r, two heads, "\epsilon"] \arrow[rr, bend right, "\epsilon\circ i"'] & X \arrow[r, tail, "i"] & Y \end{tikzcd}$$

is transformed into

$$\begin{tikzcd} Y \arrow[r, two heads, "F\epsilon"] \arrow[rr, bend right, "\mathrm{Id}_Y"'] & X \arrow[r, tail, "Fi"] & Y \end{tikzcd} ,$$

which implies that $Fi : \; X \rightarrowtail Y$ is also surjective, hence bijective. Contradiction!

[3] Let $0 < |\,X\,| < |\,Y\,| < |\,Z\,|$, with F killing Aut_Y and Aut_Z. Then, by (1) (applied to Y, Z), F kills End_Y, and by (2) (applied to X, Y), F cannot kill End_Y. Contradiction.

[4] This is an immediate consequence of (3). $\square$

3.3. When F induces a proper endomorphism of $\mathfrak{S}_{\not n}$.

3.3.1 The classification theorem. For each $\sigma \in \mathrm{Aut}_X$, we say that $u \in \mathrm{End}_X$ is *σ-idempotent* when $u = u \circ u = u \circ \sigma = \sigma \circ u$, *i.e.* when the following diagram is commutative,

$$\begin{tikzcd} & X \arrow[d, "\sigma"] \arrow[ddr, "u"] & \\ & X \arrow[dr, "u"] & \\ X \arrow[uur, "u"] \arrow[ur, "u"'] \arrow[rr, "u"'] & {[a]} & X \end{tikzcd}$$

i.e. when [a] u is idempotent, [b] u takes its values in the fixed points of σ, and [c] u "quotients" each orbit of σ to one point (*i.e.* the partition of X into σ-orbits is finer than the partition into u-fibers). These criteria correspond to triangles [a], [b] and [c] being commutative. Given a subgroup G of Aut_X, we say that u is G-idempotent when it is σ-idempotent for all σ in G; note that $\{\mathrm{Id}_X\}$-idempotent simply means idempotent, and that there are no Aut_X-idempotent.

Now, given an endomorphism Φ of $\mathfrak{S}_{\not n}$, and an $(\mathrm{Im}\,\Phi)$-idempotent λ in $\mathrm{End}_{\not n}$,

$$\Phi_\lambda f = \begin{cases} \Phi f & \text{if } f \in \mathfrak{S}_{\not n} \\ \lambda & \text{otherwise} \end{cases}$$

defines a functor $\Phi_\lambda : \mathrm{End}_{\mathfrak{n}} \to \mathrm{End}_{\mathfrak{n}}$. First, $\Phi_\lambda(\mathrm{Id}_n) = \Phi(\mathrm{Id}_n) = \mathrm{Id}_n$. Next, let $f, g \in \mathrm{End}_{\mathfrak{n}}$. There are four cases to consider when calculating $\Phi_\lambda(f \circ g)$, depending on f, g being or not isomorphisms. (α) *f and g are both isomorphisms*: then $f \circ g$ is an isomorphism, whence $\Phi_\lambda(f \circ g) = \Phi(f \circ g) = \Phi f \circ \Phi g = \Phi_\lambda f \circ \Phi_\lambda g$. ($\beta$) *Only f is an isomorphism*: then $f \circ g$ is not an isomorphism, whence $\Phi_\lambda(f \circ g) = \lambda$; on the other hand, $\Phi_\lambda f \circ \Phi_\lambda g = \Phi f \circ \lambda = \lambda$ (the last equality comes from the fact that λ takes its values in the fixed points of $\mathrm{Im}\,\Phi$). (γ) *Only g is an isomorphism*: then, as in (β), $\Phi_\lambda(f \circ g) = \lambda$, and $\Phi_\lambda f \circ \Phi_\lambda g = \lambda \circ \Phi g = \lambda$ (the last equality comes from the fact that λ quotients $\mathrm{Im}\,\Phi$-orbits to one point). (δ) *Neither f nor g are isomorphisms*: then, because n is finite, $f \circ g$ is not an isomorphism, and $\Phi_\lambda(f \circ g) = \lambda$; on the other hand, $\Phi_\lambda f \circ \Phi_\lambda g = \lambda \circ \lambda = \lambda$ (the last equality comes from λ being idempotent). Let us remark that finiteness is necessary for G-idempotence to allow for Φ_λ being a functor only when neither f nor g are isomorphisms, but it is then essential. For the remaining of the subsection, all sets will be finite. The aim of this subsection is the following theorem.

Theorem 3.8. [1] *The endofunctors F of* $\mathrm{End}_{\mathfrak{n}}$ *inducing a non invertible endomorphism Φ of $\mathfrak{S}_{\mathfrak{n}}$ are of the form*

$$Ff = \Phi_h f = \begin{cases} \Phi f & \text{if } f \text{ is an isomorphism} \\ h & \text{otherwise} \end{cases}$$

where h is a $\Phi(\mathfrak{S}_{\mathfrak{n}})$-idempotent; moreover, each $\Phi(\mathfrak{S}_{\mathfrak{n}})$-idempotent defines in this way an endofunctor of $\mathrm{End}_{\mathfrak{n}}$. *Therefore, NTF of $\langle \mathfrak{n} \rangle$ are classified by pairs (Φ, h) with Φ a non invertible endomorphism of $\mathfrak{S}_{\mathfrak{n}}$* (see Theorem 3.1) *and h a $\Phi(\mathfrak{S}_{\mathfrak{n}})$-idempotent.*

[2] *On the other hand, an endomorphism F of* $\mathrm{End}_{\mathfrak{n}}$ *inducing an automorphism Φ of $\mathfrak{S}_{\mathfrak{n}}$ is of the form $Ff = \widehat{\varphi} f = \varphi \circ f \circ \varphi^{-1}$ for a unique $\varphi \in \mathfrak{S}_{\mathfrak{n}}$* (this is Theorem 3.1).

Statement [2] is already contained in Theorem 3.1 and Proposition 3.3. The second part of statement [1] is proved above, and the first part has already been proved when F kills $\mathfrak{S}_{\mathfrak{n}}$ for it is then Corollary 3.6. So, for the remaining of this subsection, we may suppose, to prove [1], that F induces an endomorphism of $\mathfrak{S}_{\mathfrak{n}}$ with a proper kernel (we speak of a *proper endomorphism*).

Let Φ be an endomorphism of $\mathfrak{S}_{\mathfrak{n}}$. For $n \geq 5$, $\mathrm{Im}\,\Phi$ is generated by an involution, since $\mathfrak{A}_{\mathfrak{n}}$ is the only normal subgroup of $\mathfrak{S}_{\mathfrak{n}}$. The same is true for $n = 3$. The only singular case is with $n = 4$, since $\mathfrak{S}_{\mathfrak{4}}$ has two proper normal subgroups. The cases $n = 0, 1, 2$ are not concerned with this section since $\mathfrak{S}_{\mathfrak{n}}$ has no proper normal subgroup. So, we will consider separately the case $[n \geq 3,\ \ker\Phi = \mathfrak{A}_{\mathfrak{n}}]$, where $\mathrm{Im}\,\Phi$ is generated by an involution, and the case $[n = 4,\ \ker\Phi \neq \mathfrak{A}_{\mathfrak{4}}]$.

3.3.2 Case $[n \geq 3,\ \ker\Phi = \mathfrak{A}_{\mathfrak{n}}]$. In these cases $\Phi(\mathfrak{S}_{\mathfrak{n}})$ is generated by an involution, say $\Phi(\mathfrak{S}_{\mathfrak{n}}) = \langle \tau \rangle$. We first observe that

$$\textit{For all } k \in n,\ F\ulcorner k \urcorner \textit{ is a single } \langle \tau \rangle\textit{-idempotent mapping.} \tag{3.5}$$

Indeed, (1) $F\ulcorner k \urcorner$ is an idempotent taking its values in the fixed points of τ, for if $0, i, k$ are different, $(0, i) \circ \ulcorner k \urcorner = \ulcorner k \urcorner$ is transformed into $\tau \circ F\ulcorner k \urcorner = F\ulcorner k \urcorner$. (2) $F\ulcorner k \urcorner$ is the same for all k: for $n = 3$, $\tau \circ F\ulcorner k\urcorner = F\ulcorner k\urcorner$ implies that $F\ulcorner k\urcorner = \ulcorner h\urcorner$, with h the unique fixed point of τ, while for $n \geq 4$, with $0, i, j, k$ different,

$$\begin{tikzcd} \not{n} \arrow[r, "\ulcorner k\urcorner"] \arrow[rr, bend right, "\ulcorner j\urcorner"'] & \not{n} \arrow[r, "(0,i)(j,k)"] & \not{n} \end{tikzcd}$$

is transformed into

$$\begin{tikzcd} \not{n} \arrow[r, "F\ulcorner k\urcorner"] \arrow[rr, bend right, "F\ulcorner j\urcorner"'] & \not{n} \arrow[r, "\mathrm{Id}_n"] & \not{n} \end{tikzcd} .$$

(3) $F\ulcorner k\urcorner$ quotients each $\langle\tau\rangle$-orbit to one point for, i, j, k being different, $\ulcorner k\urcorner \circ (i, j) = \ulcorner k\urcorner$ is transformed into $F\ulcorner k\urcorner \circ \tau = F\ulcorner k\urcorner$ (this proves (3.5)). As before, we write α_F for the single value of F on constant maps.

Next, we observe that if $n \geq 4$, two isoweighted maps f, g, which are not isomorphisms, may be "connected" by two even permutations

$$\begin{tikzcd} \not{n} \arrow[r, "f"'] \arrow[d, "\varphi"'] & \not{n} \arrow[d, "\psi"] \\ \not{n} \arrow[r, "g"] & \not{n} \end{tikzcd} ,$$

which implies that F is constant on each isoweighted of non isomorphisms. From this observation, it follows that *for* $n \geq 4$, F *is constant on* $\mathrm{End}_{\not{n}} - \mathfrak{S}_{\not{n}}$, *with value* α_F. This is proved by induction on $|\operatorname{Im} f|$ ($f \in \mathrm{End}_{\not{n}} - \mathfrak{S}_{\not{n}}$). Without loss of generality, we may suppose Ff idempotent (see (3.4)). For $|\operatorname{Im} f| = 1$ (ie f constant), this has been proved above. Let f have $k + 1 < n$ elements in its image. Then select g isoweighted to f (hence $|\operatorname{Im} g| = |\operatorname{Im} f|$) with $|\operatorname{Im} g| = k + 1$ in such a way that $g \circ f$ has an image of cardinal at most k (this is always possible if $k + 1 < n$). If we suppose that F takes the value α_F on maps u with $|\operatorname{Im} u| \leq k$, then we have ($\diamondsuit$) (see the proof of Proposition 3.5), which completes the proof.

For $n = 3$, F is also constant on $\mathrm{End}_{\not{3}} - \mathfrak{S}_{\not{3}}$. Let

$$\Phi = \begin{pmatrix} (0,1) & (0,2) \\ \sigma_k & \sigma_k \end{pmatrix},$$

with σ_k a transposition fixing some $k \in \not{3}$. $F\ulcorner h\urcorner = \ulcorner k\urcorner$ for all $h \in \not{3}$ (see (2) in the proof of (3.5) above). This implies that for any f (bijective or not) Ff fixes k since F transforms

$$\begin{tikzcd} \not{3} \arrow[r, "\ulcorner h\urcorner"] \arrow[rr, bend right, "\ulcorner fh\urcorner"'] & \not{3} \arrow[r, "f"] & \not{3} \end{tikzcd}$$

into

$$\begin{tikzcd} \not{3} \arrow[r, "\ulcorner k\urcorner"] \arrow[rr, bend right, "\ulcorner k\urcorner"'] & \not{3} \arrow[r, "Ff"] & \not{3} \end{tikzcd} .$$

Now,

$$\pi = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 2 \end{pmatrix}$$

being idempotent, $F\pi$ is idempotent; it follows that $F\pi$ is (1) either conjugate to π itself, or (2) a constant map, or (3) $\mathrm{Id}_{\mathbf{3}}$ — which are the only idempotents of $\mathrm{End}_{\mathbf{3}}$. (1) is impossible; indeed, we may suppose without loss of generality that F fixes π; then,

$$\mu = \begin{pmatrix} 0 & 1 & 2 \\ 2 & 1 & 2 \end{pmatrix}$$

being conjugate to π by the cyclic transposition $\rho = (0,1,2)$ (whose image by F is $\mathrm{Id}_{\mathbf{3}}$), we would have $F\pi = F\mu = \pi$, and thus,

$$\begin{tikzcd} \mathbf{3} \arrow[r, "\pi"] \arrow[rr, bend right, "\ulcorner 2 \urcorner"'] & \mathbf{3} \arrow[r, "\mu"] & \mathbf{3} \end{tikzcd} \overset{F}{\longmapsto} \begin{tikzcd} \mathbf{3} \arrow[r, "\pi"] \arrow[rr, bend right, "\ulcorner k \urcorner"'] & \mathbf{3} \arrow[r, "\pi"] & \mathbf{3} \end{tikzcd} \quad \left(= \begin{tikzcd} \mathbf{3} \arrow[r, "\pi"] \arrow[r, bend right, "\ulcorner k \urcorner"'] & \mathbf{3} \end{tikzcd} \right) \qquad (*)$$

which is impossible, the right-hand side being not commutative. (3) is also impossible, for then F would send conjugates of π to $\mathrm{Id}_{\mathbf{3}}$, and we would have, for $\nu = \rho^{-1} \circ \mu \circ \rho$, $\mathrm{Id}_{\mathbf{3}} = \mathrm{Id}_{\mathbf{3}} \circ \mathrm{Id}_{\mathbf{3}} = F\nu \circ F\pi = F(\nu \circ \mu) = F\ulcorner 0 \urcorner = \ulcorner k \urcorner$. Thus, we have (2), *i.e.* $F\pi = \ulcorner k \urcorner$ (see $(*)$ above). As all f with $|\operatorname{Im} f| = 2$ differ from a conjugate of π by ρ or ρ^{-1}, $Ff = \ulcorner k \urcorner$ for all these f. This completes the proof that Ff *is constant on* $\mathrm{End}_{\mathbf{3}} - \mathfrak{S}_{\mathbf{3}}$. Thus

$$\textit{For } n \geq 3, \textit{ if } \Phi(\mathfrak{S}_{\mathbf{n}}) \simeq \mathfrak{S}_{\mathbf{2}}, \ F \textit{ is constant on } \mathrm{End}_{\mathbf{n}} - \mathfrak{S}_{\mathbf{n}}, \textit{ with value } \alpha_F. \qquad (3.6)$$

And (3.6) implies directly Theorem 3.8 when $\ker \Phi = \mathfrak{A}_{\mathbf{n}}$.

3.3.3 Case $[\mathbf{n} = \mathbf{4}, \ker \Phi = \mathfrak{K}_{\mathbf{4}}]$. The normal subgroups of $\mathfrak{S}_{\mathbf{4}}$ are

$$\mathfrak{K}_{\mathbf{4}} = \{\mathrm{Id}, (0,1)(2,3), (0,2)(1,3), (0,3)(1,2)\} \text{ and } \mathfrak{A}_{\mathbf{4}};$$

$\mathfrak{K}_{\mathbf{4}}$ is isomorphic to the Klein four-group. It is the kernel of 24 endomorphisms of $\mathfrak{S}_{\mathbf{4}}$, which are easily seen as follows. If Ψ is the endomorphism of $\mathfrak{S}_{\mathbf{4}}$ given by

$$\Psi = \begin{pmatrix} (0,1) & (0,2) & (0,3) \\ (0,1) & (0,2) & (1,2) \end{pmatrix},$$

whose kernel is $\mathfrak{K}_{\mathbf{4}}$, then Ψ maps $\mathfrak{S}_{\mathbf{4}}$ onto $\mathfrak{S}_{\mathbf{3}}$; when Ψ is followed by any of the six automorphisms of $\mathfrak{S}_{\mathbf{3}}$, and then by any of the the four embeddings of $\mathfrak{S}_{\mathbf{3}}$ in $\mathfrak{S}_{\mathbf{4}}$, we get the 24 endomorphisms of $\mathfrak{S}_{\mathbf{4}}$ with kernel $\mathfrak{K}_{\mathbf{4}}$.

$$\begin{tikzcd} \mathfrak{S}_{\mathbf{4}} \arrow[r, "\Psi"] & \mathfrak{S}_{\mathbf{3}} \arrow[r, "\sim", "\text{6 choices}"'] & \mathfrak{S}_{\mathbf{3}} \arrow[r, tail, "\text{4 choices}"'] & \mathfrak{S}_{\mathbf{4}} \end{tikzcd}$$

It amounts also to considering $\widehat{\varphi} \circ \Psi$ when φ runs through $\mathfrak{S}_{\mathbf{4}}$. So, when looking at endofunctors of $\mathrm{End}_{\mathbf{4}}$, up to within an isomorphism of functors, inducing an endomorphism of $\mathfrak{S}_{\mathbf{4}}$ with kernel $\mathfrak{K}_{\mathbf{4}}$ we may suppose, without loss of generality, that F induces Ψ. On the other hand, Ψ is induced by

$$\Psi_{\ulcorner 3 \urcorner} f = \begin{cases} \Psi f & \text{if } f \in \mathfrak{S}_{\mathbf{4}} \\ \ulcorner 3 \urcorner & \text{otherwise.} \end{cases}$$

Thus, Theorem 3.8, when the kernel of the induced endomorphisms of $\mathfrak{S}_n$ is not $\mathfrak{A}_n$ (the case is with $n = 4$), is proved if we can prove that $\Psi_{\ulcorner 3 \urcorner}$ is the only endofunctor of End_4 inducing Ψ.

So, let F be an endofunctor of End_4 inducing Ψ on $\mathfrak{S}_4$. We observe that

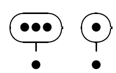
(1) if f, g are isoweighted with $Ff = \ulcorner 3 \urcorner$, then $F = \ulcorner 3 \urcorner$, since $g = \psi \circ f \circ \varphi^{-1}$ is transformed into $Fg = \Psi(\psi) \circ \ulcorner 3 \urcorner \circ \Psi\varphi^{-1}$ with $\Psi\varphi$, $\Psi\psi$ fixing 3;

(2) $F\ulcorner k \urcorner = \ulcorner 3 \urcorner$ for all k; since $\ulcorner k \urcorner$ is isoweighted to $\ulcorner 0 \urcorner$, it is sufficient (by (1)) to prove that $F\ulcorner 0 \urcorner = \ulcorner 3 \urcorner$; this in turn comes from $Fx\ulcorner 0 \urcorner = F((1,2) \circ \ulcorner 0 \urcorner) = \Psi((1,2)) \circ F\ulcorner 0 \urcorner = (1,2) \circ F\ulcorner 0 \urcorner$ and $F\ulcorner 0 \urcorner = F((2,3) \circ \ulcorner 0 \urcorner) = \Psi((2,3)) \circ F\ulcorner 0 \urcorner = (0,1) \circ F\ulcorner 0 \urcorner$;

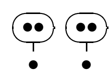
(3) for all f in End_4, Ff fixes 3 for

$$\begin{tikzcd} 4 \arrow[r, "\ulcorner 0 \urcorner"] \arrow[rr, bend right, "\ulcorner f0 \urcorner"'] & 4 \arrow[r, "f"] & 4 \arrow[r, mapsto, "F"] & 4 \arrow[r, "\ulcorner 3 \urcorner"] \arrow[rr, bend right, "\ulcorner 3 \urcorner"'] & 4 \arrow[r, "Ff"] & 4 \end{tikzcd} .$$

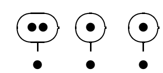
To prove that $Ff = \ulcorner 3 \urcorner$ for all non injective f, it is sufficient to prove, in view of observations (1) and (2), that in each isoweighted class of a non injective and non constant map f, there is a g with $Fg = \ulcorner 3 \urcorner$.The isoweighted classes of such f are:



(A)
One three-point fiber,
One one-point fiber.



(B)
Two two-point fibers.



(C)
One two-point fiber,
Two one-point fibers.

We can prove, through various compositions and conjugations (see below) that

$$F\begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 3 \end{pmatrix} = \ulcorner 3 \urcorner \qquad F\begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 3 & 3 \end{pmatrix} = \ulcorner 3 \urcorner \qquad F\begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \end{pmatrix} = \ulcorner 3 \urcorner$$

Here are the details.

$\diamond$ Case (A) – Let f be

$$f = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 3 \end{pmatrix}.$$

Letting σ run through $\mathfrak{S}_3 \subset \mathfrak{S}_4$, we have $Ff = F(f \circ \sigma) = Ff \circ F\sigma = Ff \circ \sigma$, whence $Ff(\{0,1,2\})$ is a singleton. On the other hand, $Ff = F((1,2) \circ f) = (1,2) \circ Ff$, and similarly $Ff = (0,2) \circ Ff$, whence Ff never takes a value in $\{0,1,2\}$. We conclude that $F = \ulcorner 3 \urcorner$.

$\diamond$ Case (B) – Let f be

$$f = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 3 & 3 \end{pmatrix}.$$

$$Ff = F\left[\begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 1 & 3 & 3 \end{pmatrix}\right] = \ulcorner 3 \urcorner \circ F\begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 1 & 3 & 3 \end{pmatrix} = \ulcorner 3 \urcorner$$

$\diamond$ Case (C) – Let f be

$$f = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \end{pmatrix}.$$

C.1. $Ff = F(f \circ (0,1)) = Ff \circ (0,1)$, whence Ff is of the form

$$\begin{pmatrix} 0 & 1 & 2 & 3 \\ k & k & t & 3 \end{pmatrix}$$

(recall that it fixes 3 by observation (3)). As $\ulcorner 0 \urcorner = f^3$, we have $\ulcorner 3 \urcorner = (Ff)^3$, and therefore $k \notin \{0,1\}$, $t \neq 2$

C.2. In fact $k \neq 2$. Indeed, with $k = 2$

– $t \in \{0,1\}$ would be impossible because

$$(Ff)^3 = \begin{pmatrix} 0 & \cdots \\ 2 & \cdots \end{pmatrix};$$

– $t = 3$ would be impossible because, then

$$Ff = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 2 & 2 & 3 & 3 \end{pmatrix},$$

and we would have, by case (B), $\ulcorner 3 \urcorner = F\begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 2 & 1 & 2 \end{pmatrix}$, that is

$$\ulcorner 3 \urcorner = F((0,2)f(0,2)(1,2)f(1,2)) = \begin{pmatrix} 0 & \cdots \\ 0 & \cdots \end{pmatrix};$$

C.3. It remains to eliminate the two following cases

$$M_1 = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 3 & 3 & 1 & 3 \end{pmatrix} \quad \text{and} \quad M_2 = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 3 & 3 & 0 & 3 \end{pmatrix}$$

(*i.e.* with $k = 3$, $t = 0, 1$).

– If $Ff = M_1$, then, by case (A),

$$\ulcorner 3 \urcorner = F\begin{pmatrix} 0 & 1 & 2 & 3 \\ 2 & 0 & 0 & 0 \end{pmatrix},$$

that is $\ulcorner 3 \urcorner = F((1,2)f(1,2)(0,2)f(0,2)) = \begin{pmatrix} 0 & \cdots \\ 2 & \cdots \end{pmatrix}$;

– If $Ff = M_2$, then, by case (B),

$$\ulcorner 3 \urcorner = F\begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 0 & 1 \end{pmatrix},$$

that is $\ulcorner 3 \urcorner = F((0,1)f(0,1)(0,2)f(0,2)) = \begin{pmatrix} 0 & \cdots \\ 1 & \cdots \end{pmatrix}$.

This completes the proof of Theorem 3.8 as the only remaining case is

$$\begin{pmatrix} 0 & 1 & 2 & 3 \\ 3 & 3 & 3 & 3 \end{pmatrix}.$$

Remark 3.9. It would be interesting to connect Theorem 3.8 with the theorem of [ScTe2] on page 2580.

4. Fixobs of full subcategories of Fin. The core of this section is the description of all fixobs $\mathbb{F}$ of the full subcategory $\langle \not m, \not n\rangle$ of Ens generated by $\not m, \not n$ with $m < n$. Given such an $\mathbb{F}$, it induces endofunctors of $\mathrm{End}_{\not m}$ and $\mathrm{End}_{\not n}$; if this restriction to $\mathrm{End}_{\not m}$ (resp. $\mathrm{End}_{\not n}$) is not isomorphic to the identity endofunctor, we write $\alpha_{\not m}$ (resp. $\alpha_{\not n}$) its constant value on $\mathrm{End}_{\not m} - \mathfrak{S}_{\not m}$(resp. $\mathrm{End}_{\not n} - \mathfrak{S}_{\not n}$).

4.1. Characterizing the trivial fixobs of $\langle \not m, \not n\rangle$ and $\langle \not m, \not n, \not p, \ldots\rangle$.

4.1.1 A first characterization of the trivial fixobs of $\langle \not m, \not n\rangle$. Since the monoid of fixobs of $\langle \not 0, \not n\rangle$ is canonically isomorphic to the monoid of fixobs of $\langle \not n\rangle = \mathrm{End}_{\not n}$, we assume $0 < m < n$.

Theorem 4.1. *A fixob $\mathbb{F}$ of $\langle \not m, \not n\rangle$ is isomorphic to* $\mathrm{Id}_{\langle \not m, \not n\rangle}$ *if (and only if) it induces an automorphism of* $\mathfrak{S}_{\not n}$.

Proof. We first suppose that $\mathbb{F}$ induces automorphisms of both $\mathfrak{S}_{\not m}$ and $\mathfrak{S}_{\not n}$. In this case, from Theorem 3.1, $\mathbb{F}$ induces endofunctors of $\mathrm{End}_{\not m}$ and $\mathrm{End}_{\not n}$ both isomorphic to the identities on $\mathrm{End}_{\not m}$ and $\mathrm{End}_{\not n}$. Therefore, composing $\mathbb{F}$ if necessary with an endofunctor of $\langle \not m, \not n\rangle$ isomorphic to the identity, we may assume in full generality that $\mathbb{F}$ fixes both $\mathrm{End}_{\not m}$ and $\mathrm{End}_{\not n}$. Then $\mathbb{F}$ fixes all constant maps $\not m \to \not n$ or $\not n \to \not m$ because, for all possible f and k we have:

$$\begin{tikzcd}[column sep=small] \not m \arrow[rr, "\ulcorner k\urcorner"] \arrow[dr, "f"'] & & \not n \\ & \not n \arrow[ur, "\ulcorner k\urcorner"'] & \end{tikzcd} \longmapsto \begin{tikzcd}[column sep=small] \not m \arrow[rr, "\mathbb{F}\ulcorner k\urcorner"] \arrow[dr, "\mathbb{F}f"'] & & \not n \\ & \not n \arrow[ur, "\ulcorner k\urcorner"'] & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd}[column sep=small] \not n \arrow[rr, "\ulcorner k\urcorner"] \arrow[dr, "f"'] & & \not m \\ & \not m \arrow[ur, "\ulcorner k\urcorner"'] & \end{tikzcd} \longmapsto \begin{tikzcd}[column sep=small] \not n \arrow[rr, "\mathbb{F}\ulcorner k\urcorner"] \arrow[dr, "\mathbb{F}f"'] & & \not m \\ & \not m \arrow[ur, "\ulcorner k\urcorner"'] & \end{tikzcd}.$$

Finally, this implies that $\mathbb{F}$ fixes all maps by (3.1); therefore, at the outset, $\mathbb{F}$ was isomorphic to the identity endofunctor on $\langle \not m, \not n\rangle$.

Let us now suppose that $\mathbb{F}$ induces an automorphism of $\mathfrak{S}_{\not n}$, without any assumption on its behaviour on $\mathfrak{S}_{\not m}$. We prove that $\mathbb{F}$ induces then an automorphism of $\mathfrak{S}_{\not m}$; is sufficient to prove that $\mathbb{F}$ is injective on $\mathfrak{S}_{\not m}$.

Let $\varphi_1, \varphi_2 \in \mathfrak{S}_{\not m}$ with $\mathbb{F}\varphi_1 = \mathbb{F}\varphi_2$, and let $\mu_1, \ldots, \mu_{(n-m)!}, \lambda_1, \ldots, \lambda_{(n-m)!}$ be the extensions of φ_1 and φ_2 (respectively) to $\not n$. We have the $2 \times (n-m)!$ commutative diagrams

$$\begin{tikzcd} \not m \arrow[r, tail, "\mathbb{F}i"] \arrow[d, "\mathbb{F}\varphi_1 = \mathbb{F}\varphi_2"'] & \not n \arrow[d, "\mathbb{F}\mu_k, \mathbb{F}\lambda_k"] \\ \not m \arrow[r, tail, "\mathbb{F}i"'] & \not n \end{tikzcd} \qquad (k = 1, 2, \ldots, (n-m)!).$$

Because of the injectivity of $\mathbb{F}i$, there must be repetition in the finite sequence

$$\mathbb{F}\mu_1, \mathbb{F}\mu_2, \ldots, \mathbb{F}\mu_{(n-m)!}, \mathbb{F}\lambda_1, \mathbb{F}\lambda_2, \ldots, \mathbb{F}\lambda_{(n-m)!}.$$

The $\mathbb{F}\mu_i$ (resp. $\mathbb{F}\lambda_j$) are different since F induces an automorphism of $\mathfrak{S}_{\not n}$; therefore, there are indices p, q such $\mathbb{F}\mu_p = \mathbb{F}\lambda_q$, and this implies $\mu_p = \lambda_q$, which in turn implies $\varphi_1 = \varphi_2$. This completes the proof. $\square$

4.1.2 A second characterization of the trivial fixobs of $\langle \boldsymbol{m}, \boldsymbol{n} \rangle$. Let $\mathbb{F}$ be a fixob of $\langle \boldsymbol{m}, \boldsymbol{n} \rangle$ not isomorphic to the identity. By the preceeding section, $\mathbb{F}$ does not induce an automorphism of $\mathfrak{S}_{\boldsymbol{n}}$. There are possible cases:

(1-a) $\mathbb{F}$ kills $\mathfrak{S}_{\boldsymbol{n}}$,
(1-b) $\operatorname{Ker} \mathbb{F}_{\boldsymbol{n}} = \mathfrak{A}_{\boldsymbol{n}}$ (in which case $\mathbb{F}\mathfrak{S}_{\boldsymbol{n}}$ is of order 2, say $\mathbb{F}\mathfrak{S}_{\boldsymbol{n}} = \langle \tau \rangle$),
(2) $\boldsymbol{n} = \boldsymbol{4}$ with $\mathbb{F}_{\boldsymbol{4}}$ isomorphic to the Klein four-group (in which case $\mathbb{F}\mathfrak{S}_{\boldsymbol{4}} \simeq \mathfrak{S}_{\boldsymbol{3}}$).

We will consider (1-a) and (1-b) together by allowing the involution τ to be $\mathrm{Id}_{\boldsymbol{n}}$. In what follows, we consider separately cases $\boldsymbol{m} = \boldsymbol{1}$ and $m > 1$. The plan is as follows:

4.1.2.A. Case $\langle \boldsymbol{1}, \boldsymbol{n} \rangle$.
A.1: $\mathbb{F}\mathfrak{S}_{\boldsymbol{n}} = \langle \tau \rangle$ for some involution $\tau \in \mathfrak{S}_{\boldsymbol{n}}$;
A.2: $\mathbb{F}\mathfrak{S}_{\boldsymbol{n}} \simeq \mathfrak{S}_{\boldsymbol{3}}$;
A.3: Conclusion

4.1.2.B. Case $\langle \boldsymbol{m}, \boldsymbol{n} \rangle$ $(m > 1)$.
B.1: $\boldsymbol{n} = \boldsymbol{m} \cup \{\boldsymbol{m}\}$ (*i.e.* $n = m + 1$);
B.2: $\boldsymbol{m} = \boldsymbol{3}$, $\boldsymbol{n} = \boldsymbol{4}$ with $\mathbb{F}\mathfrak{S}_{\boldsymbol{4}} \simeq \mathfrak{S}_{\boldsymbol{3}}$;
B.3: $\boldsymbol{n}$ and $\boldsymbol{m}$ differ by at least 2 elements.

4.1.2.A. We first consider the case $\langle \boldsymbol{1}, \boldsymbol{n} \rangle$. If $\boldsymbol{n} = \boldsymbol{2}$, we are in the case $\langle \boldsymbol{1}, \boldsymbol{2} \rangle$, which is canonically isomorphic to $\mathrm{Ens}_{<3}$, whose non trivial fixobs are $R^{(i)}$, $(i = 0, 1)$, as given in section 2. So let $n \geq 3$.

••– *A.1. Let us first suppose that* $\mathbb{F}\mathfrak{S}_{\boldsymbol{n}} = \langle \tau \rangle$ for some involution τ (possibly $\mathrm{Id}_{\boldsymbol{n}}$), with $\mathbb{F}f = \alpha_{\boldsymbol{n}}$ a fixed τ-idempotent map when $f \in \mathrm{End}_{\boldsymbol{n}} - \mathfrak{S}_{\boldsymbol{n}}$. Choosing two elements i, j of $\boldsymbol{n}$ different from 0, we have

$$\begin{tikzcd} & \boldsymbol{n} \\ \boldsymbol{1} \arrow[ur, "\ulcorner i \urcorner"] \arrow[r, "\ulcorner 0 \urcorner"'] & \boldsymbol{n} \arrow[u, "{(0,i,j)}"'] \end{tikzcd} \quad \longmapsto \quad \begin{tikzcd} & \boldsymbol{n} \\ \boldsymbol{1} \arrow[ur, "\mathbb{F}\ulcorner i \urcorner"] \arrow[r, "\mathbb{F}\ulcorner 0 \urcorner"'] & \boldsymbol{n} \arrow[u, equal] \end{tikzcd}$$

which proves that $\mathbb{F}$ is constant on $\hom(\boldsymbol{1}, \boldsymbol{n})$, say with value $\ulcorner i_0 \urcorner : \boldsymbol{1} \to \boldsymbol{n}$. For any $f \in \mathrm{End}_{\boldsymbol{n}} - \mathfrak{S}_{\boldsymbol{n}}$ we then have

$$\begin{tikzcd} & \boldsymbol{n} \\ \boldsymbol{1} \arrow[ur, "\ulcorner f(0) \urcorner"] \arrow[r, "\ulcorner 0 \urcorner"'] & \boldsymbol{n} \arrow[u, "f"'] \end{tikzcd} \quad \longmapsto \quad \begin{tikzcd} & \boldsymbol{n} \\ \boldsymbol{1} \arrow[ur, "\ulcorner i_0 \urcorner"] \arrow[r, "\ulcorner i_0 \urcorner"'] & \boldsymbol{n} \arrow[u, "\mathbb{F}f = \alpha_{\boldsymbol{n}}"'] \end{tikzcd}$$

which proves that i_0 is a fixed point of the idempotent $\alpha_{\boldsymbol{n}}$. Finally,

$$\begin{tikzcd} \boldsymbol{n} \arrow[r] \arrow[rr, bend left, "\ulcorner 0 \urcorner"] & \boldsymbol{1} \arrow[r, "\ulcorner 0 \urcorner"] & \boldsymbol{n} \end{tikzcd} \quad \longmapsto \quad \begin{tikzcd} \boldsymbol{n} \arrow[r] \arrow[rr, bend left, "\alpha_{\boldsymbol{n}}"] & \boldsymbol{1} \arrow[r, "\ulcorner i_0 \urcorner"] & \boldsymbol{n} \end{tikzcd}$$

proves that $\alpha_{\boldsymbol{n}}$ is constant, and hence that $\alpha_{\boldsymbol{n}} = \ulcorner i_0 \urcorner$. Thus, when $\mathbb{F}\mathfrak{S}_{\boldsymbol{n}} = \langle \tau \rangle$ for involution τ (possibly $\mathrm{Id}_{\mathbf{n}}$), $\mathbb{F}$ is given (for some $i_0 \in \mathbf{n}$) by

$$\mathbb{F}f = \begin{cases} \ulcorner i_0 \urcorner : \mathbf{1} \to \mathbf{n} & \text{if } f : \mathbf{1} \to \mathbf{n} \\ \ulcorner i_0 \urcorner : \mathbf{n} \to \mathbf{n} & \text{if } f \in \mathrm{End}_{\mathbf{n}} - \mathfrak{S}_{\mathbf{n}} \\ \tau^s & \text{if } f \in \mathfrak{S}_{\mathbf{n}}, \text{ with } \mathrm{sgn}\, f = s. \end{cases}$$

••– *A.2. Let us next suppose that* $\mathbb{F}\mathfrak{S}_{\mathbf{n}} \simeq \mathfrak{S}_{\mathbf{3}}$, *i.e.* that we are in the case $\langle \mathbf{1}, \mathbf{4} \rangle$. If $\mathbb{F}|_{\mathfrak{S}_{\mathbf{4}}}$ is described by

$$\Phi = \begin{pmatrix} (0,1) & (0,2) & (0,3) \\ (b_0,b_1) & (b_0,b_2) & (b_1,b_2) \end{pmatrix}$$

then, for any non isomorphism $f : \mathbf{4} \to \mathbf{4}$, with b_3 the unique element of $\mathbf{4} - \{b_0, b_1, b_2\}$, we have (by Theorem 3.8)

$$\begin{tikzcd} \mathbf{1} \arrow[r, "\ulcorner 0 \urcorner"] \arrow[rr, bend left, "\ulcorner f_0 \urcorner"] & \mathbf{4} \arrow[r, "f"] & \mathbf{4} \end{tikzcd} \longmapsto \begin{tikzcd} \mathbf{1} \arrow[r, "\mathbb{F}\ulcorner 0 \urcorner"] \arrow[rr, bend left, "\mathbb{F}\ulcorner f_0 \urcorner"] & \mathbf{4} \arrow[r, "\ulcorner b_3 \urcorner"] & \mathbf{4} \end{tikzcd}$$

which proves that, on $\mathrm{End}_{\mathbf{4}} - \mathfrak{S}_{\mathbf{4}}$, $\mathbb{F}f$ is constant with value the only possible $\mathbb{F}\mathfrak{S}_{\mathbf{4}}$-idempotent map, namely $\ulcorner b_3 \urcorner$.

••– *A.3. Conclusion.* Collecting all this information, we have

Theorem 4.2. *Let $\mathbb{F}$ be a non trivial fixob of $\langle \mathbf{1}, \mathbf{n} \rangle$ inducing a non invertible endomorphism Φ of $\mathfrak{S}_{\mathbf{n}}$. Then $\mathbb{F}$ is of the form*

$$\mathbb{F}f = \begin{cases} \ulcorner i \urcorner & \text{if } f : \mathbf{1} \to \mathbf{n} \\ \ulcorner i \urcorner & \text{if } f : \mathbf{n} \to \mathbf{n} \text{ is not an isomorphism} \\ \Phi f & \text{if } f : \mathbf{n} \to \mathbf{n} \text{ is an isomorphism} \end{cases}$$

where $\ulcorner i \urcorner$ is $\Phi\mathfrak{S}_{\mathbf{n}}$-idempotent. In other words, the non trivial fixobs of $\langle \mathbf{1}, \mathbf{n} \rangle$ are classified by the $\Phi\mathfrak{S}_{\mathbf{n}}$-idempotent constant maps $\mathbf{n} \to \mathbf{n}$, with Φ being a non invertible endomorphism of $\mathfrak{S}_{\mathbf{n}}$.

In the proof of this theorem, the involution τ may be $\mathrm{Id}_{\mathbf{n}}$. Therefore, this theorem entails the case $\langle \mathbf{1}, \mathbf{2} \rangle$, with $\tau = \mathrm{Id}_{\mathbf{2}}$, *i.e.* it entails the case $\mathrm{Ens}_{<3}$.

4.1.2.B. We next consider the case $\langle \mathbf{m}, \mathbf{n} \rangle$ with $m > 1$. For technical reasons, we treat separately the cases $\langle \mathbf{m}, \mathbf{m} \cup \{\mathbf{m}\} \rangle$ and $\langle \mathbf{m}, \mathbf{n} \rangle$ with $n \geq m+2$. Before going into the details, we state the result: there are no NTF of $\langle \mathbf{m}, \mathbf{n} \rangle$ inducing an automorphism of $\mathfrak{S}_{\mathbf{m}}$ if $m > 1$; in other words, the cases $\langle \mathbf{1}, \mathbf{n} \rangle$ are the only cases where non trivial fixobs exist that induce an automorphism of $\mathfrak{S}_{\mathbf{m}}$. Stated positively, this is the following theorem.

Theorem 4.3. *If $m > 1$, a fixob of $\langle \mathbf{m}, \mathbf{n} \rangle$ is isomorphic to the identity endofunctor if (and only if) it induces an automorphism of $\mathfrak{S}_{\mathbf{m}}$.*

B.1. Proof of Theorem 4.3 for $\langle \mathbf{m}, \mathbf{m} \cup \{\mathbf{m}\}\rangle$ with $\mathbb{F}\mathfrak{S}_{\mathbf{m}\cup\{\mathbf{m}\}} = \langle \tau \rangle$ for an involution τ. Let $\mathbb{F}$ induce an automorphism of $\mathfrak{S}_{\mathbf{m}}$. In the case $\langle \mathbf{m}, \mathbf{m} \cup \{\mathbf{m}\}\rangle$, we can always uniquely extend each $\varphi \in \mathfrak{S}_{\mathbf{m}}$ to $\mathbf{m} \cup \{\mathbf{m}\}$, and we must have

$$\begin{tikzcd}
\mathbf{m} \arrow[r, hook, "i"] \arrow[d, "\varphi"] & \mathbf{m}\cup\{\mathbf{m}\} \arrow[d, "\overline{\varphi}"] & & \mathbf{m} \arrow[r, tail, "\mathbb{F}i"] \arrow[d, "\mathbb{F}\varphi"] & \mathbf{m}\cup\{\mathbf{m}\} \arrow[d, "\tau^{\operatorname{sgn}\varphi}"] \\
\mathbf{m} \arrow[r, hook, "i"] & \mathbf{m}\cup\{\mathbf{m}\} & \longmapsto & \mathbf{m} \arrow[r, tail, "\mathbb{F}i"] & \mathbf{m}\cup\{\mathbf{m}\}
\end{tikzcd} .$$

We conclude that $\mathbb{F}\varphi$ is some fixed involution when φ runs through odd permutations in $\mathfrak{S}_{\mathbf{m}}$; since $\mathbb{F}$ induces an automorphism of $\mathfrak{S}_{\mathbf{m}}$, $\mathfrak{S}_{\mathbf{m}}$ contains just one odd involution, and the only possibility is $\langle \mathbf{2}, \mathbf{3}\rangle$. Then $\mathbb{F}\mathfrak{S}_{\mathbf{3}} = \langle \tau \rangle$ with $\tau \neq \mathrm{Id}_{\mathbf{3}}$, for with $\tau = \mathrm{Id}_{\mathbf{3}}$ we would have

$$\begin{tikzcd}
\mathbf{2} \arrow[r, hook, "i"] \arrow[d, "(0{,}1)"] & \mathbf{3} \arrow[d, "(0{,}1)"] & & \mathbf{2} \arrow[r, tail, "\mathbb{F}i"] \arrow[d, "(0{,}1)"] & \mathbf{3} \arrow[d, equal] \\
\mathbf{2} \arrow[r, hook, "i"] & \mathbf{3} & \longmapsto & \mathbf{2} \arrow[r, tail, "\mathbb{F}i"] & \mathbf{3}
\end{tikzcd}$$

which is impossible. Let $\tau = (a, b)$ with c its fixed point. By Theorem 3.8, we then have $\mathbb{F}f = \ulcorner c \urcorner$ for all non isomorphism $f : \mathbf{3} \to \mathbf{3}$. But now, for $k = 0, 1$

$$\begin{tikzcd}
 & \mathbf{2} \arrow[dr, hook, "i"] & & & & \mathbf{2} \arrow[dr, hook, "\mathbb{F}i"] & \\
\mathbf{2} \arrow[ur, "\ulcorner k \urcorner"] \arrow[rr, "\ulcorner k \urcorner"] \arrow[dr, hook, "i"'] & & \mathbf{3} & \longmapsto & \mathbf{2} \arrow[ur, "\ulcorner k \urcorner"] \arrow[rr, "\mathbb{F}\ulcorner k \urcorner"] \arrow[dr, hook, "\mathbb{F}i"'] & & \mathbf{3} \\
 & \mathbf{3} \arrow[ur, "\ulcorner k \urcorner"'] & & & & \mathbf{3} \arrow[ur, "\ulcorner c \urcorner"'] &
\end{tikzcd} .$$

The commutativity of the lower triangles implies $\mathbb{F}\ulcorner 0 \urcorner = \mathbb{F}\ulcorner 1 \urcorner = \ulcorner c \urcorner$, which, with the commutativity of the upper triangles, implies $\mathbb{F}i(0) = \mathbb{F}i(1) = c$, contradicting the injectivity of $\mathbb{F}i$.

So, if $m > 1$ there exist no NTF of $\langle \mathbf{m}, \mathbf{m} \cup \{\mathbf{m}\}\rangle$ inducing an automorphism of $\mathfrak{S}_{\mathbf{m}}$.

B.2. Proof of Theorem 4.3 for $\langle \mathbf{m}, \mathbf{m} \cup \{\mathbf{m}\}\rangle = \langle \mathbf{3}, \mathbf{4}\rangle$ with $\mathbb{F}\mathfrak{S}_{\mathbf{4}} \simeq \mathfrak{S}_{\mathbf{3}}$.. Let $\mathbb{F}$ induce an automorphism of $\mathfrak{S}_{\mathbf{3}}$. Composing if necessary with an endofunctor of $\langle \mathbf{3}, \mathbf{4}\rangle$ isomorphic to the identity, we may suppose that $\mathbb{F}$ induces

$$\begin{pmatrix} (0,1) & (0,2) & (0,3) \\ (0,1) & (0,2) & (1,2) \end{pmatrix}$$

on $\mathfrak{S}_{\mathbf{4}}$ (and that $\mathbb{F}$ fixes $\mathrm{End}_{\mathbf{3}}$). Then by Theorem 3.8, $\mathbb{F}\ulcorner k \urcorner = \ulcorner 3 \urcorner$ for any constant map $\ulcorner k \urcorner : \mathbf{4} \to \mathbf{4}$, and so, for $k = 0, 1$, we have

$$\begin{tikzcd}
 & \mathbf{3} \arrow[dr, hook, "i"] & & & & \mathbf{3} \arrow[dr, hook, "\mathbb{F}i"] & \\
\mathbf{3} \arrow[ur, "\ulcorner k \urcorner"] \arrow[rr, "\ulcorner k \urcorner"] \arrow[dr, hook, "i"'] & & \mathbf{4} & \longmapsto & \mathbf{3} \arrow[ur, "\ulcorner k \urcorner"] \arrow[rr, "\mathbb{F}\ulcorner k \urcorner"] \arrow[dr, hook, "\mathbb{F}i"'] & & \mathbf{4} \\
 & \mathbf{4} \arrow[ur, "\ulcorner k \urcorner"'] & & & & \mathbf{4} \arrow[ur, "\ulcorner 3 \urcorner"'] &
\end{tikzcd} . \qquad (\triangle)$$

From the commutativity of the lower triangles we get $\mathbb{F}\ulcorner 0 \urcorner = \mathbb{F}\ulcorner 1 \urcorner = \ulcorner 3 \urcorner$, and from the commutativity of the upper triangles, $\mathbb{F}i(0) = \mathbb{F}i(1) = 3$, contradicting the injectivity of $\mathbb{F}i$. Therefore, there are no NTF of $\langle \not 3, \not 4 \rangle$ inducing an automorphism of $\mathfrak{S}_{\not 3}$.

Collecting the above information, Theorem 4.3 is true in the case $\langle \not m, \not m \cup \{\not m\} \rangle$.

B.3. Proof of Theorem 4.3 for the cases $\langle \not m, \not n \rangle$ with $n \geq m + 2$. Let $\mathbb{F}$ induce an automorphism of $\mathfrak{S}_{\not m}$.

The first case is when $\mathbb{F}\mathfrak{S}_{\not n} = \langle \tau \rangle$ for some involution τ. Since $n \geq m + 2$, any $\varphi \in \mathfrak{S}_{\not m}$ may be extended to an even automorphism of $\mathfrak{S}_{\not n}$, and we have

$$\begin{tikzcd} \not m \arrow[r, hook, "i"] \arrow[d, "(0{,}1)"'] & \not n \arrow[d, "\overline{\varphi}\ \text{(even)}"] \\ \not m \arrow[r, hook, "i"] & \not n \end{tikzcd} \quad \longmapsto \quad \begin{tikzcd} \not m \arrow[r, tail, "\mathbb{F}i"] \arrow[d, "\mathbb{F}(0{,}1)"'] & \not n \arrow[d, equal] \\ \not m \arrow[r, tail, "\mathbb{F}i"] & \not n \end{tikzcd}$$

which is in contradiction with $\mathbb{F}$ inducing an automorphism of $\mathfrak{S}_{\not m}$. Therefore, there are no NTF of $\langle \not m, \not n \rangle$ inducing an automorphism of $\mathfrak{S}_{\not m}$, with $\mathbb{F}\mathfrak{S}_{\not n} \simeq \mathfrak{S}_{\not 2}$ and $n \geq m + 2$.

The second case is $\langle \not 2, \not 4 \rangle$, with $\mathbb{F}\mathfrak{S}_{\not 4} \simeq \mathfrak{S}_{\not 3}$ and, without loss of generality, $\mathbb{F}$ fixing $\mathfrak{S}_{\not 2}$. But this is also impossible, and the argument is the same as for $\langle \not 3, \not 4 \rangle$ above, replacing $\not 3$ by $\not 2$ in $(\triangle)$ and the lines following it. Therefore, there are no non trivial fixobs of $\langle \not 2, \not 4 \rangle$ inducing an automorphism of $\mathfrak{S}_{\not 2}$, with $\mathbb{F}\mathfrak{S}_{\not 4} \simeq \mathfrak{S}_{\not 3}$.

This completes the proof of Theorem 4.3. $\square$

4.1.3 Characterizing the trivial fixobs of $\langle \not m, \not n, \not p, \dots \rangle$. An immediate consequence of the above results on the fixobs of $\langle \not m, \not n \rangle$ is the following:

Proposition 4.4. *Given an at-least-two-element set $\{\not m, \not n, \not p \dots\}$, with $m < n < p < \dots$, a fixob of $\langle \not m, \not n, \not p, \dots \rangle$ is isomorphic to the identity endofunctor if, and only if, for some $\not i \in \{\not m, \not n, \not p, \dots\}$, with $1 < i$, it induces an automorphism of $\mathfrak{S}_{\not i}$.*

4.2. Characterizing NTF of $\langle \not m, \not n \rangle$ and $\langle \not m, \not n, \not p, \dots \rangle$.

4.2.1 The classification theorem. In view of Theorems 4.2 and 4.3, we suppose that $m > 1$ with $\mathbb{F}$ not inducing an automorphism of $\mathfrak{S}_{\not m}$.

Theorem 3.8 stated that an NTF F of $\mathrm{End}_{\not n}$ is determined by the data of a non invertible endomorphism Φ of $\mathfrak{S}_{\not n}$ together with the data of a $\Phi\mathfrak{S}_{\not n}$-idempotent $h : \not n \to \not n$ through the rule

$$Fg = \begin{cases} \Phi g & \text{if } g \text{ is an isomorphism} \\ h & \text{if } g \text{ is not an isomorphism.} \end{cases}$$

Let h have m fibers; then, the data of this $\Phi\mathfrak{S}_{\not n}$-idempotent h with m fibers canonically amounts to the data of an epimorphism $f : \not n \to \not m$ given by $f(t) = \sigma k$ if $t \in F_k$ (f stands for $\underline{f}$iber) for some fixed σ in $\mathfrak{S}_{\not m}$, together with an injection $\not m \xrightarrow{v} \not n$, describing the value of h (v stands for $\underline{v}$alue) on $F_k : h(t) = v(\sigma k)$ for all $t \in F_k$; that is, it amounts to

> *the data of an injection v and a surjection f with $f \circ v = \mathrm{Id}_{\not m}$ (setting $h = v \circ f$):*

$$
\begin{tikzcd}
\not m \arrow[r, tail, "v"] \arrow[rr, bend left, "\mathrm{Id}_{\not m}"] & \not n \arrow[r, two heads, "f"] \arrow[rr, bend right, "h"'] & \not m \arrow[r, tail, "v"] & \not n
\end{tikzcd}
\tag{4.1}
$$

We will call f the fiber component of h, and v its value component. Saying that h is $\Phi\mathfrak{S}_{\not n}$-idempotent just amounts then (besides idempotency) to the fact that the following diagram is commutative for all $\sigma \in \Phi\mathfrak{S}_{\not n}$

$$
\begin{tikzcd}
 & \not n \arrow[dd, "\sigma"] \arrow[dr, two heads, "f"] & \\
\not m \arrow[ur, tail, "v"] \arrow[dr, tail, "v"'] & & \not m \\
 & \not n \arrow[ur, two heads, "f"'] &
\end{tikzcd}
$$

Defining an endofunctor F of $\mathrm{Ens}_{<\not n}$ with a pair $(\Phi, h = v \circ f)$, the notation being as above, yields a way of defining an NTF of $\langle \not m, \not n \rangle$ as follows:

$$
\mathbb{F} : g \longmapsto \begin{cases} \mathrm{Id}_{\not m} & \text{if } g : \not m \to \not m \\ v & \text{if } g : \not m \to \not n \\ f & \text{if } g : \not n \to \not m \\ Fg & \text{if } g : n \to \not n \end{cases}
\tag{4.2}
$$

It is easy to check that this indeed defines a functor.

Let us give two examples of such a fixob $\mathbb{F}$.

$$
\begin{bmatrix} m = \not 3 \\ n = \not 4 \end{bmatrix} : \ \mathbb{F}g = \begin{cases} \mathrm{Id}_{\not 3} & \text{if } g : \not 3 \to \not 3 \\ \begin{pmatrix} 0 & 1 & 2 \\ 0 & 1 & 2 \end{pmatrix} & \text{if } g : \not 3 \to \not 4 \\ \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 0 \end{pmatrix} & \text{if } g : \not 4 \to \not 3 \\ \mathrm{Id}_{\not 4} & \text{if } g : \not 4 \to \not 4 \end{cases}
$$

$$
\begin{bmatrix} m = \not 3 \\ n = \not 5 \end{bmatrix} : \ \mathbb{F}g = \begin{cases} \mathrm{Id}_{\not 3} & \text{if } g : \not 3 \to \not 3 \\ \begin{pmatrix} 0 & 1 & 2 \\ 2 & 3 & 4 \end{pmatrix} & \text{if } g : \not 3 \to \not 5 \\ \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 2 & 2 & 0 & 1 & 2 \end{pmatrix} & \text{if } g : \not 5 \to \not 3 \\ \left\{ \begin{array}{ll} (0,1)^{\operatorname{sgn} g} & \text{for } g \text{ an isomorphism} \\ \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 2 & 2 & 2 & 3 & 3 \end{pmatrix} & \text{otherwise} \end{array} \right\} & \text{if } g : \not 5 \to \not 5 \end{cases}
$$

These examples show that NTF exist for some $\langle \not m, \not n \rangle$. In fact:

Theorem 4.5. [1] *NTF of* $\langle \not{m}, \not{n} \rangle$ *are all of the form* (4.2) *for some unique pair* $(\Phi, h = v \circ f)$ *made of an endomorphism* Φ *of* $\mathfrak{S}_{\not{n}}$ *and a decomposition* $h = v \circ f$ *of an* $\operatorname{Im}\Phi$-*idempotent as in* (4.1). *In this unique pair, the value component* v *of* h *is the value of* $\mathbb{F}$ *on arrows* $\not{m} \to \not{n}$, *and the fiber component of* h *is the value of* $\mathbb{F}$ *on arrows* $\not{n} \to \not{m}$.

[2] *Each NTF* F *of* $\langle \not{n} \rangle$ *extend to* $m!$ *NTF of* $\langle \not{m}, \not{n} \rangle$ *where* m *is the number of fibers of the value* h *of* F *on* $\operatorname{End}_{\not{n}} - \mathfrak{S}_{\not{n}}$ (see Theorem 3.8). *These* $m!$ *possible extensions are determined by the* $m!$ *possible decompositions of* h *described in* (4.1).

4.2.2 Proof of Theorem 4.5. Let $\mathbb{F}$ be an NTF of $\langle \not{m}, \not{n} \rangle$.

4.2.2.A First we suppose that $\mathbb{F}\mathfrak{S}_{\not{n}} = \langle \tau \rangle$ for an involution $\tau \in \mathfrak{S}_{\not{n}}$. In this case, the proof follows from observations A.1 to A.6.

••– *A.1.*– For each injection $\not{m} \overset{k}{\rightarrowtail} \not{n}$, $\mathbb{F}k$ imbeds each fiber of $\alpha_{\not{m}}$ into a fiber of $\alpha_{\not{n}}$, sending fixed points of $\alpha_{\not{m}}$ to fixed points of $\alpha_{\not{n}}$, which in particular implies that different fibers go into different fibers, and thus, that $\alpha_{\not{n}}$ has *at least* m fibers. This results immediately from

$$\begin{array}{ccc} \not{m} & \overset{k}{\rightarrowtail} & \not{n} \\ {\scriptstyle \ulcorner 0 \urcorner}\downarrow & & \downarrow{\scriptstyle \ulcorner k0 \urcorner} \\ \not{m} & \underset{k}{\rightarrowtail} & \not{n} \end{array} \quad \longmapsto \quad \begin{array}{ccc} \not{m} & \overset{\mathbb{F}k}{\rightarrowtail} & \not{n} \\ {\scriptstyle \alpha_{\not{m}}}\downarrow & & \downarrow{\scriptstyle \alpha_{\not{n}}} \\ \not{m} & \underset{\mathbb{F}k}{\rightarrowtail} & \not{n} \end{array} .$$

••– *A.2.*– Given $\not{m} \underset{l}{\overset{k}{\rightrightarrows}} \not{n}$, then we have $\mathbb{F}k = \mathbb{F}l$ or $\mathbb{F}l = \tau \circ \mathbb{F}k$; hence, since $\alpha_{\not{n}}$ is $\langle \tau \rangle$-idempotent, all $\mathbb{F}k$ ($k : \not{m} \to \not{n}$ an injection) imbed a given fiber of $\alpha_{\not{m}}$ into the same fiber of $\alpha_{\not{n}}$. This results from

$$\begin{array}{ccc} \not{m} & \overset{k}{\rightarrowtail} & \not{n} \\ \| & & \downarrow{\scriptstyle \varphi} \\ \not{m} & \underset{l}{\rightarrowtail} & \not{n} \end{array} \quad \longmapsto \quad \begin{array}{ccc} \not{m} & \overset{\mathbb{F}k}{\rightarrowtail} & \not{n} \\ \| & & \downarrow{\scriptstyle \tau^{\operatorname{sgn}\varphi}} \\ \not{m} & \underset{\mathbb{F}l}{\rightarrowtail} & \not{n} \end{array} ,$$

where φ is an appropriate element of $\mathfrak{S}_{\not{n}}$.

••– *A.3.*– $\mathbb{F}$ is constant on non-injective maps $f : \not{m} \to \not{n}$; more precisely, $\mathbb{F}f$ sends any fiber of $\alpha_{\not{m}}$ with fixed point t to $\mathbb{F}i(t)$ (where $i : \not{m} \hookrightarrow \not{n}$; any other injection could do as well by A.2). Indeed, any non injective $f : \not{m} \to \not{n}$ factors through a non-isomorphism $u : \not{m} \to \not{m}$ and an injection $k : \not{m} \rightarrowtail \not{n}$, and $\mathbb{F}$ transforms

$$\begin{tikzcd} & \not{m} \arrow[dr, tail, "k"] & \\ \not{m} \arrow[ur, "u"] \arrow[rr, "f"] & & \not{n} \end{tikzcd}$$

into

$$\begin{tikzcd} & \not{m} \arrow[dr, tail, "\mathbb{F}k"] & \\ \not{m} \arrow[ur, "\alpha_{\not{m}}"] \arrow[rr, "\mathbb{F}f"] & & \not{n} \end{tikzcd} .$$

The result follows from A.2.

••– *A.4.*– For each $f : \not n \to \not m$, $\mathbb{F}f$ maps each fiber of $\alpha_{\not n}$ to one point, in such a way that the fiber with fixed point of the form $\mathbb{F}i(t)$ is mapped to t. This is so because f, being not an isomorphism, factors through a non isomorphism $u : \not n \to \not n$, and $\mathbb{F}$ transforms

$$\begin{tikzcd} & \not n \arrow[dr, "g"] & \\ \not n \arrow[ur, "u"] \arrow[rr, "f"] & & \not m \end{tikzcd}$$

into

$$\begin{tikzcd} & \not n \arrow[dr, "\mathbb{F}g"] & \\ \not n \arrow[ur, "\alpha_{\not n}"] \arrow[rr, "\mathbb{F}f"] & & \not m \end{tikzcd},$$

which implies that $\mathbb{F}f$ quotients fibers of $\alpha_{\not n}$ to singletons. Moreover, again because f is not injective, there exists a non isomorphism $u : \not m \to \not m$ and an injection $k : \not m \rightarrowtail \not n$ such that $u = f \circ k$, and then $\mathbb{F}$ transforms

$$\begin{tikzcd} \not m \arrow[d, tail, "k"'] \arrow[dr, "u"] & \\ \not n \arrow[r, "f"] & \not m \end{tikzcd}$$

into

$$\begin{tikzcd} \not m \arrow[d, tail, "\mathbb{F}k"'] \arrow[dr, "\alpha_{\not m}"] & \\ \not n \arrow[r, "\mathbb{F}f"] & m \end{tikzcd};$$

therefore, if t is a fixed point of $\alpha_{\not m}$, $\mathbb{F}f$ sends $\mathbb{F}k(t) = \mathbb{F}i(t)$ (see A.2) to t.

••– *A.5.*– The fibers of $\alpha_{\not m}$ are singletons (*i.e.* $\alpha_{\not m} = \mathrm{Id}_{\not m}$). Therefore, $\mathbb{F}$ is null on $\mathrm{End}_{\not n}$ and (by A.2 and A.3) is constant on $\hom(\not m, \not n)$. Indeed, given a left inverse e to an injection $k : \not m \rightarrowtail \not n$,

$$\begin{tikzcd} \not m \arrow[r, tail, "k"] \arrow[rr, bend left, "\mathrm{Id}_{\not m}"] & \not n \arrow[r, two heads, "e"] & \not m \end{tikzcd}$$

is transformed into

$$\begin{tikzcd} \not m \arrow[r, tail, "\mathbb{F}k"] \arrow[rr, bend left, "\mathrm{Id}_{\not m}"] & \not n \arrow[r, two heads, "\mathbb{F}e"] & \not m \end{tikzcd}.$$

Since $\mathbb{F}k$ imbeds fibers of $\alpha_{\not m}$ into fibers of $\alpha_{\not n}$, and since $\mathbb{F}e$ quotients fibers of $\alpha_{\not n}$ to singletons (their "base point"), fibers of $\alpha_{\not m}$ are singletons.

••– *A.6.*– $\alpha_{\not n}$ has m fibers, and therefore, from A.4, $\mathbb{F}$ is constant on $\hom(\not n, \not m)$. Indeed, we have

$$\begin{tikzcd} \not n \arrow[r, "g"] \arrow[rr, bend left, "f"] & \not m \arrow[r, "h"] & \not n \end{tikzcd} \quad \longmapsto \quad \begin{tikzcd} \not n \arrow[r, "\mathbb{F}g"] \arrow[rr, bend left, "\alpha_{\not n}"] & \not m \arrow[r, tail, "\mathbb{F}h"] & \not n \end{tikzcd}$$

(because $h \circ g$ is not an isomorphism), which implies that $\alpha_{\mathbf{n}}$ has *at most* m fibers. But, from A.1, it has *at least* m fibers. The result follows.

It comes from A.1 to A.6 that, when $\mathbb{F}\mathfrak{S}_{\mathbf{n}} = \langle\tau\rangle$, we have the data of an $\mathbb{F}\mathfrak{S}_{\mathbf{n}}$-idempotent map $\alpha_{\mathbf{n}}$, with m fibers indexed by $\mathbf{m}$, and that $\mathbb{F}$ is described by (4-1). This completes the proof of Theorem 4.5 in the case $\mathbb{F}\mathfrak{S}_{\mathbf{n}} = \langle\tau\rangle$.

4.2.2.B. Next, we suppose that $\mathbf{n} = \mathbf{4}$ with $\mathbb{F}\mathfrak{S}_{\mathbf{4}} \simeq \mathfrak{S}_{\mathbf{3}}$. This concerns case $\langle\mathbf{1},\mathbf{4}\rangle$ and cases $\langle\mathbf{2},\mathbf{4}\rangle$ & $\langle\mathbf{3},\mathbf{4}\rangle$ that will be considered together.

B.1.– Theorem 4.2 describes the nontrivial fixobs of $\langle\mathbf{1},\mathbf{4}\rangle$, and they correspond, in the context of Theorem 4.5, to the idempotent component of the NTF on $\mathbf{n}$ being a constant map. So Theorem 4.5 is proved for $\langle\mathbf{1},\mathbf{4}\rangle$.

On the other hand, there are no diagram of the following form, commutative for all σ in $\mathbb{F}\mathfrak{S}_{\mathbf{4}} \simeq \mathfrak{S}_{\mathbf{3}}$:

$$\begin{tikzcd} & \mathbf{4} \arrow[dr, "r"] \arrow[dd, "\sigma"] & \\ \mathbf{m} \arrow[ur, hook, "f"] \arrow[dr, hook, "f"'] & & \mathbf{m} \\ & \mathbf{4} \arrow[ur, "r"'] & \end{tikzcd} \qquad (\mathbf{m} = \mathbf{2}, \mathbf{3})$$

for the injection of such a decomposition must take its value in the fixed points of $\mathbb{F}\mathfrak{S}_{\mathbf{4}}$, and there is just one fixed point. Therefore, to complete the proof of Theorem 4.5, we must show that

$$\langle\mathbf{2},\mathbf{4}\rangle \textit{ and } \langle\mathbf{3},\mathbf{4}\rangle \textit{ have no NTF } \mathbb{F} \textit{ such that } \mathbb{F}\mathfrak{S}_{\mathbf{4}} \simeq \mathfrak{S}_{\mathbf{3}} \tag{4.3}$$

(*i.e.* we must show that Theorem 4.5 is vacuously true in these cases).

In fact, it is sufficient to prove that a non-trivial fixob of $\langle\mathbf{m},\mathbf{4}\rangle$, ($\mathbf{m} = \mathbf{2}, \mathbf{3}$), would kill $\mathrm{End}_{\mathbf{m}}$ for then we would have

$$\begin{tikzcd} & \mathbf{m} \arrow[dr, hook, "i"] & \\ \mathbf{m} \arrow[ur, "\ulcorner 0 \urcorner"] \arrow[dr, hook, "i"'] & & \mathbf{4} \\ & \mathbf{4} \arrow[ur, "\ulcorner 0 \urcorner"'] & \end{tikzcd} \longmapsto \begin{tikzcd} & \mathbf{m} \arrow[dr, "\mathbb{F}i"] & \\ \mathbf{m} \arrow[ur, equal] \arrow[dr, "\mathbb{F}i"'] & & \mathbf{4}, \\ & \mathbf{4} \arrow[ur, "\ulcorner c \urcorner"'] & \end{tikzcd}$$

with c being the fixed point of $\mathbb{F}\mathfrak{S}_{\mathbf{4}}$, which contradicts the injectivity of $\mathbb{F}i$.

B.2.– The following lemma completes the proof of Theorem 4.5.

Lemma 4.6. *If there exists a non trivial fixob* $\mathbb{F}$ *of* $\langle\mathbf{m},\mathbf{4}\rangle$, $\mathbf{m} = \mathbf{2}, \mathbf{3}$, *with* $\mathbb{F}\mathfrak{S}_{\mathbf{4}} \simeq \mathfrak{S}_{\mathbf{3}}$, *then* $\mathbb{F}$ *kills* $\mathrm{End}_{\mathbf{m}}$.

Let $\mathbb{F}$ be a non trivial fixob of $\langle\mathbf{m},\mathbf{4}\rangle$, $\mathbf{m} = \mathbf{2}, \mathbf{3}$, with $\mathbb{F}\mathfrak{S}_{\mathbf{4}} \simeq \mathbf{3}$.

1. *Case* $\mathbf{m} = \mathbf{2}$. Let us suppose the contrary; then $\mathbb{F}$ would not kill $\mathrm{End}_{\mathbf{2}}$ and, by Theorem 4.3, would not induce an automorphism of $\mathfrak{S}_{\mathbf{2}}$. This implies that $\mathbb{F}\mathfrak{S}_{\mathbf{2}} = \{\mathrm{Id}_{\mathbf{2}}\}$ so that $\mathbb{F}(\mathrm{End}_{\mathbf{2}} - \mathfrak{S}_{\mathbf{2}})$ would be $\{\ulcorner v \urcorner\}$ for some v. Let $\mathbb{F}(\mathrm{End}_{\mathbf{4}} - \mathfrak{S}_{\mathbf{4}}) = \{\ulcorner u \urcorner\}$. Then (commutative) diagrams of the form

$$\begin{tikzcd} \mathbf{2} \arrow[r, hook, "i"] \arrow[d, "p"'] & \mathbf{4} \arrow[r, "g"] \arrow[d, "q"] & \mathbf{2} \arrow[dl, hook, "i"] \\ \mathbf{2} \arrow[r, hook, "i"] & \mathbf{4} & \end{tikzcd}$$

are transformed into (commutative)

$$\begin{tikzcd}
\mathbf{2} \arrow[r, tail, "\mathbb{F}i"] \arrow[d, "\mathbb{F}p"'] \arrow[dr, phantom, "\ulcorner u \urcorner"] & \mathbf{4} \arrow[r, "g"] \arrow[d] & \mathbf{2} \arrow[dl, "\mathbb{F}i"] \\
\mathbf{2} \arrow[r, tail, "\mathbb{F}i"] & \mathbf{4} &
\end{tikzcd}$$

since q is not an isomorphism. But for g the section of i with $g(2) = g(3) = 0$, we have $p = \mathrm{Id}_{\mathbf{2}}$ and we get

$$\begin{tikzcd}
\mathbf{2} \arrow[r, tail, "\mathbb{F}i"] \arrow[d, equal] \arrow[dr, phantom, "\ulcorner u \urcorner"] & \mathbf{4} \arrow[d] \\
\mathbf{2} \arrow[r, tail, "\mathbb{F}i"] & \mathbf{4}
\end{tikzcd}.$$

This is a ontradiction.

2. *Case* $\mathbf{m} = \mathbf{3}$. Let us suppose the contrary; then $\mathbb{F}$ would not kill $\mathrm{End}_{\mathbf{3}}$ and, by Theorem 4.3, would not induce an automorphism of $\mathfrak{S}_{\mathbf{3}}$. This implies that $\mathbb{F}\mathfrak{S}_{\mathbf{3}} = \langle \tau \rangle$ for some involution τ. Composing $\mathbb{F}$ with an endofunctor isomorphic to the identity if necessary, we may suppose in all generality that $\mathbb{F}\mathfrak{S}_{\mathbf{3}} = \langle (0,1) \rangle$ or $\{\mathrm{Id}_{\mathbf{3}}\}$, and that $\mathbb{F}$ induces $\begin{pmatrix} (0,1) & (0,2) & (0,3) \\ (0,1) & (0,2) & (1,2) \end{pmatrix}$ on $\mathfrak{S}_{\mathbf{4}}$. In the case $\mathbb{F}\mathfrak{S}_{\mathbf{3}} \neq \{\mathrm{Id}_{\mathbf{3}}\}$,

$$\begin{tikzcd}
\mathbf{3} \arrow[r, hook, "i"] \arrow[d, "{(0,k)}"'] & \mathbf{4} \arrow[d, "{(0,k)}"] \\
\mathbf{3} \arrow[r, hook, "i"] & \mathbf{4}
\end{tikzcd}$$

would be transformed into

$$\begin{tikzcd}
\mathbf{3} \arrow[r, tail, "\mathbb{F}i"] \arrow[d, "{(0,1)}"'] & \mathbf{4} \arrow[d, "{(0,k)}"] \\
\mathbf{3} \arrow[r, tail, "\mathbb{F}i"] & \mathbf{4}
\end{tikzcd}$$

$(k = 1, 2)$, which contradicts the injectivity of $\mathbb{F}i$. In the case $\mathbb{F}\mathfrak{S}_{\mathbf{3}} = \{\mathrm{Id}_{\mathbf{3}}\}$, essentially the same argument as for $\mathbf{m} = \mathbf{2}$ yields a contradiction. This completes the proof of the and of Theorem 4.5. $\square$

4.2.3 The case of $\langle \mathbf{m}, \mathbf{n}, \mathbf{p} \rangle$. Theorem 4.5 states among other things that non trivial fixobs of $\langle \mathbf{m}, \mathbf{n} \rangle$ kill $\mathrm{End}_{\mathbf{m}}$; this is the key fact in proving Theorem 4.7 below, which is the main consequence of Theorem 4.5.

Theorem 4.7. *Given* $0 < m < n < p$, *all fixobs of* $\langle \mathbf{m}, \mathbf{n}, \mathbf{p} \rangle$ *are isomorphic to the identity.*

Proof. Let $\mathbb{F}$ be a fixob of $\langle \mathbf{m}, \mathbf{n}, \mathbf{p} \rangle$. It induces fixobs $\mathbb{F}_1$ of $\langle \mathbf{m}, \mathbf{n} \rangle$ and $\mathbb{F}_2$ of $\langle \mathbf{n}, \mathbf{p} \rangle$.

Let us suppose that $\mathbb{F}_2$ is isomorphic to the identity, say $\mathbb{F}_2 = \widehat{\varphi}_{\mathbf{n},\mathbf{p}}$ with $\varphi_{\mathbf{n},\mathbf{p}} = \{\varphi_{\mathbf{n}}, \varphi_{\mathbf{p}}\}$. It follows from Theorem 4.1 that $\mathbb{F}_1$ is isomorphic to the identity, say $\mathbb{F}_1 = \widehat{\varphi}_{\mathbf{m}\mathbf{n}}$ with $\varphi_{\mathbf{m},\mathbf{n}} = \{\varphi_{\mathbf{m}}, \varphi_{\mathbf{n}}\}$. Factoring morphisms $\mathbf{m} \to \mathbf{p}$ through $\mathbf{n}$, we immediately conclude that for all $f : \mathbf{m} \to \mathbf{p}$, $\mathbb{F}f = \varphi_{\mathbf{p}} \circ f \circ \varphi_{\mathbf{m}}^{-1}$ and similarly for morphisms $\mathbf{p} \to \mathbf{m}$; therefore $\mathbb{F} = \widehat{\varphi}_{\mathbf{m},\mathbf{n},\mathbf{p}}$ with $\varphi_{\mathbf{m},\mathbf{n},\mathbf{p}} = \{\varphi_{\mathbf{m}}, \varphi_{\mathbf{n}}, \varphi_{\mathbf{p}}\}$.

Using Theorem 4.3, a similar reasoning proves that if $\mathbb{F}_1$ is isomorphic to the identity, then $\mathbb{F}$ is also isomorphic to the identity.

Let us suppose now that there exists a fixob $\mathbb{F}$ not isomorphic to the identity. Then, from the discussion above, neither $\mathbb{F}_2$ nor $\mathbb{F}_1$ can be isomorphic to the identity. By Theorem 4.5, $\alpha_{\mathbf{m}}$ and $\alpha_{\mathbf{n}}$ (see the beginning of section 4 for the notation) are respectively $\mathrm{Id}_{\mathbf{m}}$ and $\mathrm{Id}_{\mathbf{n}}$, *i.e.* $\alpha_{\mathbf{m}}$ and $\alpha_{\mathbf{n}}$ have fibers which are singletons. But fibers of $\alpha_{\mathbf{n}}$ are indexed by $\mathbf{m}$, and thus $\mathbf{m} = \mathbf{n}$: contradiction. $\square$

An immediate consequence is that any fixob of a full subcategory of Ens generated by at least three non empty finite sets of different cardinals can be but a trivial fixob; in particular, $\mathrm{Ens}_{<\kappa}$, with $\kappa \geq 4$, has but trivial fixobs.

4.2.4 An effective enumeration. It results from the above study that, for each $1 \leq m \leq n$, there exist NTF of $\langle \mathbf{m}, \mathbf{n} \rangle$. More precisely, the being as in Theorem 4.5:

Theorem 4.8. [1] *The following algorithm yields all NTF of $\langle \mathbf{m}, \mathbf{n} \rangle$:*

(A) *if $\mathbf{n} \neq \mathbf{4}$:*

$$\left[\begin{array}{l} \textit{(1) Choose a surjection } \mathbf{n} \overset{f}{\twoheadrightarrow} \mathbf{m}\ ; \\ \textit{set } F_k = f^{-1}(k) \textit{ for each } k. \\ \textit{(2) Choose a right inverse } \mathbf{n} \overset{v}{\rightarrowtail} \mathbf{m} \textit{ to } f \\ \textit{set } v_k = v(k) \textit{ for each possible } k. \end{array}\right] \qquad \textit{Set } h = v \circ f;$$

$$\left[\begin{array}{l} \textit{(3) Partition each } F_i \textit{ into pairs and singleton,} \\ \textit{each ``base point'' } v_k \textit{ forming a singleton. Let} \\ \{a_1, b_1\}, \{a_2, b_2\}, \ldots, \{a_k, b_k\} \textit{ be the pairs} \\ \textit{in this partition;} \\ \textit{set } \tau = (a_1, b_1)(a_2, b_2)\ldots(a_k, b_k) \textit{ (product} \\ \textit{of transpositions). (cf. } \text{Figure 1 below)} \end{array}\right] \qquad \textit{Set } \Phi(\sigma) = \tau^{\operatorname{sgn}\sigma}.$$

When $(\Phi, h = v \circ f)$ runs through the set of all possible values, (4.2) *runs without repetition through all NTF $\mathbb{F}$ of $\langle \mathbf{m}, \mathbf{n} \rangle$ with $\mathbb{F}\mathfrak{S}_{\mathbf{n}} \simeq \mathfrak{S}_{\mathbf{2}}$.*

(B) *if $\mathbf{n} = \mathbf{4}$:*

$$\left[\begin{array}{l} \textit{B.1} \quad \textit{Do as in } \text{A} \textit{ to obtain all NTF with } \mathbb{F}\mathfrak{S}_{\mathbf{4}} \not\simeq \mathfrak{S}_{\mathbf{2}}. \\ \textit{B.2} \quad \textit{Do as follows to obtain all NTF with } \mathbb{F}\mathfrak{S}_{\mathbf{4}} \simeq \mathbf{3}\textit{, which exist} \\ \textit{only in the case } \langle \mathbf{1}, \mathbf{4} \rangle \text{ (see (4.3))}\textit{:} \\ \qquad \left\{\begin{array}{l} \text{(1) } \textit{Choose a ``base point'' of } \mathbf{4}\textit{, say } d. \\ \textit{Let } \{a, b, c\} \textit{ be the remaining element.} \\ \textit{Set} \left\{\begin{array}{l} f : \mathbf{4} \to \mathbf{1} \\ v = \ulcorner d \urcorner : \mathbf{1} \to \mathbf{4} \\ h = v \circ f \end{array}\right. \\ \text{(2)} \textit{Choose an endomorphism } \Phi \textit{ of } \mathfrak{S}_{\mathbf{4}} \textit{ killing } \mathrm{Aut}_{\{a,b,c\}} \end{array}\right. \end{array}\right.$$

When $(\Phi, h = v \circ f)$ runs through the set of all possible values, (4.2) *runs without repetition through all NTF $\mathbb{F}$ of $\langle \mathbf{m}, \mathbf{4} \rangle$. There are* 24 *of them when $\mathbb{F}\mathfrak{S}_{\mathbf{4}} \simeq \mathfrak{S}_{\mathbf{3}}$, which happens only with $m = 1$.*

[2] *The union over all m, with $0 < m \leq n$, of the restrictions to $\langle \not{n} \rangle$ of all the above NTF of $\langle \not{m}, \not{n} \rangle$ is the set of NTF of $\langle \not{n} \rangle$:*

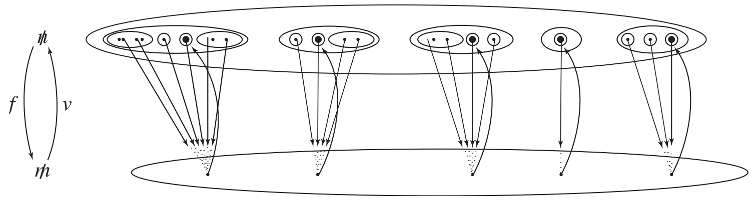


Figure 1.

5. Conclusion. Theorems 3.8, 4.5 and 4.7 yield a full description of fixobs of any full subcategory $\langle \not{m} \rangle$, $\langle \not{m}, \not{n} \rangle$, $\langle \not{m}, \not{n}, \not{p}, \dots \rangle$ (with $m < n < p < \dots$) of the category of finite sets and maps and an algorithm for enumerating them.

Combinatorial results, based on Theorems 4.8 are developed in a forthcoming paper, such as the following result; in this theorem, C_n^{i*k} is the number of i pairwise disjoint subsets of $\not{n}$ each of cardinal k, that is

$$\mathrm{C}_n^{i*k} = \frac{\mathrm{C}_n^k \mathrm{C}_{n-k}^k \dots \mathrm{C}_{n-(i-1)k}^k}{i!}$$

and $a \div b$ is the euclidean quotient of a by b.

Theorem 5.1. [1] *For $\not{n}$ not equal to $\not{4}$, the number of NTF of $\langle \not{n} \rangle$ is*

$$\sum_{s=0}^{(n+1)\div 2} \sum_{h=0}^{n-2s} \mathrm{C}_n^{s*2} \mathrm{C}_{n-2s}^h h^{n-s-h}$$

[2] *For $\not{n}$ equal to $\not{4}$, this number must be increased by* 24 *due to the* 24 *NTF of $\langle \not{1}, \not{4} \rangle$ with $\mathbb{F}\mathfrak{S}_{\not{4}} \simeq \mathfrak{S}_{\not{3}}$; this gives*

$$24 + \sum_{h=0}^{4} \mathrm{C}_4^h h^{4-h} + 6 \sum_{h=0}^{2} \mathrm{C}_2^h h^{3-h} = 89$$

Résumé substantiel en français. L'objet de cet article est l'existence d'endofoncteurs des sous-catégories pleines de Ens fixant les objets (les « fixobs ») mais non isomorphes à l'endofoncteur identité. Ces endofoncteurs sont appelés « NTF » ("Non Trivial Fixobs").

On présente des résultats partiels sur ce problème, qui, dans sa forme générale ou dans le cas de la sous-catégorie pleine des ensembles infinis reste à résoudre: nous résolvons complètement le problème dans le cas des sous-catégories pleines engendrées par des ensembles finis. Plus précisément, les NTF des sous-catégories pleines engendrées par un ensemble fini ou par deux ensembles finis de cardinaux différents sont totalement décrits, et il est établi que les sous-catégories pleines engendrées par au moins trois ensembles finis non vides de cardinaux différents ne possèdent pas de NTF. Comme il est expliqué à la section 1.3, le problème général de l'existence de NTF se pose naturellement dans le contexte de la recherche de quantificateurs non « standards ».

Les résultats essentiels sont donnés ci-dessous. Notons $\langle X, Y, \dots \rangle$ la sous-catégorie pleine engendrée par X, Y, *etc.* —ici des ensembles finis). Étant donné un sous-groupe H de $\S X$, une application $u : X \to X$ est dite H-idempotente si elle commute avec tous les éléments de H.

- Cas des sous-catégories pleines $\langle X \rangle$:
  1. Tout fixob de $\langle X \rangle$, induisant un automorpisme de $\mathfrak{S}_X$ est isomorphe à l'endofoncteur identité.
  2. Les NTF de $\langle X \rangle$ sont classifiés par le paires (Φ, h) où Φ est un endomorphisme non inversible de $\mathfrak{S}_X$ and H est un $\Phi\mathfrak{S}_X$-idempotent; plus précisément, les NTF de $\langle X \rangle$ sont exactement les fixobs de la forme

$$Ff = \begin{cases} \Phi f & \text{si } f \in \mathfrak{S}_X \\ h & \text{sinon} \end{cases}$$

- Cas des sous-catégories pleines $\langle X, Y \rangle$, $X \not\simeq Y$:
  3. Un fixob de $\langle X, Y \rangle$, est isomorphe à l'endofoncteur identité si et seulement si il induit un automorphisme de $\mathfrak{S}_Y$; si $|X| > 1$, un fixob de $\langle X, Y \rangle$, est isomorphe à l'endofoncteur identité si et seulement si il induit un automorphisme de $\mathfrak{S}_Y$.
  4. Supposons $0 < |X| < |Y|$, disons $|X| = m$ et $|Y| = n$. Un NTF F de $\langle Y \rangle$ correspondant à une paire (Φ, h) (voir (2.) ci dessus) se prolonge en un NTF de $\langle X, Y \rangle$ si et seulement si l'idempotent h a m fibres, et ces NTF de $\langle X, Y \rangle$ sont de la forme

$$\mathbb{F}g = \begin{cases} \mathrm{Id}_X & \text{si } g : X \to X \\ v & \text{si } g : X \to Y \\ f & \text{si } g : Y \to X \\ FG & \text{si } g : Y \to Y \end{cases}$$

  où $f : Y \to X$ est surjectif et $v : X \to Y$ est injectif avec $\mathrm{Id}_X = f \circ v$ et $h = v \circ f$. Les NTF de $\langle X, Y \rangle$ sont classifiés par les paires $(\Phi, h = v \circ f)$ où f et v sont comme ci-dessus, c'est-à-dire par les paires formées d'un NTF de $\langle X \rangle$ dont l'idempotent a m fibres et d'une factorisation de cet idempotent via X en une surjection suivie de son inverse à droite.
  5. La description (4.) est effective, et permet de compter les NTF de $\langle X, Y \rangle$; leur nombre est

$$\sum_{s=0}^{(n+1)\div 2} \sum_{h=0}^{n-2s} \mathrm{C}_n^{s*2} \mathrm{C}_{n-2s}^h h^{n-s-h}$$

  Si $n = |Y| = 4$, il faut ajouter 24 à ce nombre.
- Cas des sous-catégories pleines $\langle X, Y, Z, \dots \rangle$:
  6. Si $0 < |X| < |Y| < |Y| \dots$, alors $\langle X, Y, Z, \dots \rangle$ n'admet aucun fixob; en d'autres mots, trois (ou plus) ensembles finis non vides de cardinaux différents engendrent une sous-catégorie pleine sans NTF.

Les démonstrations des résultats ont un caractère très ensembliste; dans un double travail en cours, on s'intéresse aux résultats qui restent vrais au sens de la logique intuitionniste, et aux questions de combinatoire et d'arithmétique qui s'y rattachent naturellement.

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