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PENTAGON PARTITIONS OF POLYGONS AND A SPECIAL CLASS OF PLANAR MAPS

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RÉSUMÉ. Nous présentons ici une bijection géométrique entre les partitions d'un polygone en pentagones et une famille de cartes planaires pointées, appelées les cartes d'ordre un (chaque arête possède au moins une extrémité incidente à la face extérieure), dont l'arête pointée est un isthme.

ABSTRACT. A geometrical one-to-one correspondence is given between partitions of a rooted polygon into pentagons and a family of rooted planar maps, called planar maps of order one (each edge having at least one extremity incident to the exterior face), with a bridge root edge.

1. Introduction. In 1870, E. Schröder raised the following question: How many different parenthesizing possibilities is there for a sum of n terms? He split this question in four combinatorial problems (case I to case IV) (see [6]). Case I and II involve a non-commutative sum of n terms. Case I is about whole parenthesizing, in which each pair of brackets encloses a sum of exactly two terms or expressions included in brackets. Case II is about part parenthesizing, in which each pair of brackets encloses a sum of two or more terms or expressions included in brackets. Third and fourth cases deal respectively with first and second problems, but with a commutative sum of n terms.

In 1940, I. M. H. Etherington illustrated these problems with convex polygons partitions (see [4]). He gave an equation characterizing the generating function which enumerates the partitions of a convex $(n+1)$-gon, the final sub-polygons being (a_i+1)-gons, $i \geq 1$, (in which a_i, $i \geq 1$, are given positive integers). The resolution of this equation in a particular case ($i = 1$, $a_1 = 4$) gives the number of partitions into pentagons for a given polygon. In this paper a shorter demonstration to obtain the enumeration is presented. The resulting enumeration of partitioned $(3n+2)$-gons into pentagons is $(4n)!/(n!(3n+1)!)$.

In 1984, D. Arquès obtained the same value in the enumeration of rooted planar maps of order one (*i.e.* for planar maps in which each edge has at least one extremity incident to the exterior face) with n edges including a bridge root edge (see [1]). A new and direct geometrical proof to enumerate this set of maps is presented. As the set of polygons partitioned into pentagons and the set of planar maps of order one

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with a bridge root edge, are objects that can be drawn on the sphere, one could expect a geometrical one-to-one correspondence between these two families, the number of edges being taken into account. We exhibit such a one-to-one correspondence.

Bijections involving polygons partitions have already been studied, as for example the one exhibited by R. P. Stanley between convex $(n+2)$-gon partitions and some standard Young tableaux (see [7]). A similar problem to the one presented here, a geometrical one-to-one correspondence between a family of hypermaps and partitions of polygons (see [2]), was shown by D. Arquès and A. Giorgetti in 1997. But the method could not be adapted to our problem. Starting from a hypermap belonging to a given family, they transformed the hypermap into a partitioned polygon, by inserting edges and vertices, opening some hyperfaces and gluing some edges. Thus a new method had to be developed and is presented in this paper.

In Section 2 of this paper, an enumeration of polygons partitioned into pentagons, according to edge's number, is presented. Then in Section 3, planar maps of order one with n edges including a bridge root edge are enumerated. For this enumeration, a geometrical proof is shown. This geometrical aspect is also used in Section 4 to exhibit a one-to-one correspondence between the two families enumerated in Sections 2 and 3.

2. Enumeration of pentagon partitions of polygons. A polygon with $3n+2$ edges can be partitioned into pentagons. The number of partitions into pentagons of a given rooted polygon can be obtained easily using a simple geometrical argument. There are several other proofs of this enumeration. We have already explained the proof showed by I. M. H. Etherington [4]. R. P. Stanley [8] exhibited a bijection between plane S-trees (any non-endpoint vertex has degree in a set S of positive integers) with l vertices and m endpoints and partitions of a $m+1$-gon into $l-m$ regions, each a k-gon with $k-1 \in S$. He also gave an enumeration of these S-trees. In the case $S = \{4\}$, $m = 3n+1$ and $l = 4n+1$, we obtain the number of partitions of a $3n+2$-gon into pentagons.

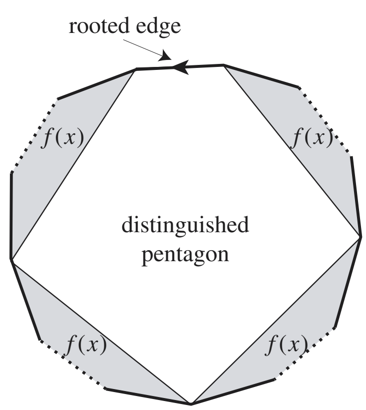


Figure 1. Convex polygon partitioned into pentagons

Definition 1. A rooted polygon is a polygon with a selected edge (see Figure 1), called the root edge of the polygon.

A partition of the polygon corresponds to cuts along diagonals of the polygon, reiterated until only triangles are left or stopped earlier. When r cuts have been made, the original polygon is divided into $r+1$ sub-polygons. Geometrically, a partition may be described as a set of r diagonals which do not intersect (except perhaps at the vertices). The polygon itself ($r = 0$) is included among its partitions. Two partitions of the same rooted polygon are non-identical if relatively to the root edge the two sets of cutting diagonals are different.

A partition of the polygon into pentagons is a partition in which the original polygon is divided into pentagons only. We point out that a convex rooted polygon with $3n+2$ edges (we will call it a $(3n+2)$-gon), will be separated into n pentagons.

Notation 1. Let Π_n be the set of partitions of a rooted $3n+2$-gon into n pentagons and $\Pi = \cup_{n\geq 0}\Pi_n$. Π_0 is the set reduced to the unique partition of the flat polygon (polygon with only two sides).

Let $f(x)$ be the generating function for the number of convex polygons partitions into pentagons, the degree of x representing the number of edges of the polygon minus one. The coefficient p_n of x^{3n+1} in $f(x)$ is then the number of partitions of the rooted $3n+2$-gon into n pentagons.

Theorem 1. *The number of partitions of a* $(3n+2)$*-gon into pentagons is*

$$p_n = \frac{1}{3n+1}\binom{4n}{n}.$$

Proof. Let P be an element of Π_n ($n > 0$). If the pentagon including the root edge in P is selected, there are left four polygons partitioned into pentagons, glued to the four non-root sides of the selected pentagon (see Figure 1). The four partitions obtained, describe the following set: $\sum_{i_1+i_2+i_3+i_4=n-1} \Pi_{i_1} \times \Pi_{i_2} \times \Pi_{i_3} \times \Pi_{i_4}$ in which $\Pi_{i_1} \times \Pi_{i_2} \times \Pi_{i_3} \times \Pi_{i_4}$ represents the set of partitions of Π_n obtained in gluing four partitions chosen respectively in $\Pi_{i_1}, \Pi_{i_2}, \Pi_{i_3}, \Pi_{i_4}$ to the four sides (other than the root side) of the selected pentagon.

As Π represents the set of polygons partitions into pentagons, this implies that there is a one-to-one correspondence between $\Pi \setminus \Pi_0$ and Π^4.

We deduce the following expression for $f(x)$: $f(x) = x + f(x)^4$. This equation can be solved with the Lagrange's inversion and we obtain the following result: coefficient p_n of x^{3n+1} in $f(x) = \frac{1}{3n+1}\binom{4n}{n}$. $\square$

3. Enumeration of planar maps of order one with a bridge root edge. A set of definitions related to planar maps of order one are given in Section 3 (see [1, 3] for further details). In the next section, some vertices in the planar maps of order one are selected and are used in Section 3.3 to obtain a decomposition of the set of maps of order one with a non-bridge root edge. Therefrom, we easily deduce an enumeration of this set in Section 3.4. This is used in Section 3.5 to find an enumeration of the set of planar maps of order one with a bridge root edge.

From now on, a counterclockwise orientation on the sphere is considered.

3.1. Definitions.

Definition 2. A planar map is a partition of the sphere of $\mathbb{R}^3$ in three finite sets of cells:

- the set of vertices, which are dots on the sphere;

- the set of edges, which are simple open Jordan arcs, pairwise disjoint. Their extremities (coinciding or not) are vertices;
- the set of faces, which are simply connected domains, the borders being the union of vertices and edges.

Two cells are incident if one is in the border of the other.

We call half-edge of a planar map, an oriented edge of this map. There is a natural association of the half-edge with its initial and final vertices, its underlying edge and the opposite half-edge.

A rooted planar map is a planar map with a distinguished half-edge $\tilde{b}$, called root half-edge.

We define α (respectively σ) as the permutation on the set of half-edges which associates each half-edge to its opposite half-edge (respectively to the first one met when turning counterclockwise around its initial vertex).

We name $\bar{\sigma}$ the permutation $\sigma \circ \alpha$ on the set of half-edges. b being a half-edge, $\bar{\sigma}^*(b)$ is the oriented side of a face of the map (the face incident to the right side of b).

A bridge is an edge incident on both sides to the same face.

If $\tilde{b}$ is not a bridge, we will call exterior face (respectively interior face) of a rooted planar map, the face $\bar{\sigma}^*(\tilde{b})$ (respectively $\bar{\sigma}^*(\alpha(\tilde{b}))$) located on the right side (respectively on the left side) of the root half-edge $\tilde{b}$.

Let $b_1 = \alpha(\tilde{b})$. Then $\bar{\sigma}^*(b_1) = (b_1, \ldots, b_r)$ is the circuit (oriented border) of half-edges incident on their right to the interior face. We call respectively $(s_1, \ldots, s_r)$ the initial vertices of $(b_1, \ldots, b_r)$, not necessarily all distinct (in Figure 2, $s_3 = s_5$, $s_7 = s_9$, $s_{11} = s_{12}$ and $s_{13} = s_1$).

Definition 3. [A special class of planar maps, maps of order one]

A rooted planar map is said to be of order one, if each edge has at least one extremity incident to the exterior face.

Let b_i be a half-edge belonging to the interior face. There are two cases:

I. b_i has both extremities incident to the exterior face (see $b_1, b_2, b_9, b_{10}, b_{11}, b_{12}, b_{13}$ in Figure 2).

II. b_i has one extremity non-incident to the exterior face (the other being incident to the exterior face by definition of order one maps) (see $b_3, b_4, b_5, b_6, b_7, b_8$ in Figure 2).

A map of order one is of type I if, and only if, every edges of the interior face of this map belong to case I.

Notation 2. $\{p\}$ is the planar map reduced to a single vertex.

Let $\mathcal{I}_n$ (respectively $\mathcal{N}_n$) be the set of planar maps of order one with n edges, the root edge being a bridge (respectively the root edge not being a bridge) and $\mathcal{I} = \cup_{n \geq 1} \mathcal{I}_n$ (respectively $\mathcal{N} = \cup_{n \geq 1} \mathcal{N}_n$).

The set Λ of rooted planar maps of order one is: $\Lambda = \{p\} \cup \mathcal{I} \cup \mathcal{N}$.

By convention the name of the half-edge is shown on the figures near its initial extremity.

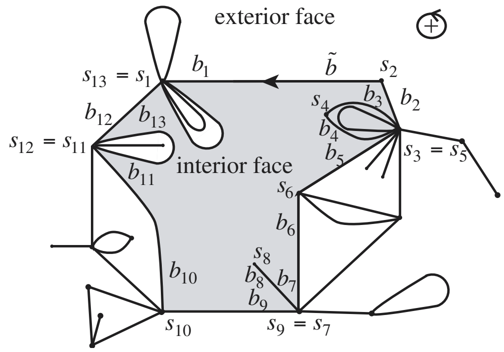


Figure 2. A planar map belonging to $\mathcal{N}$

3.2. Characteristic vertices in maps of order one. We now present some transformations on a map M of Λ that will allow us to introduce, in Definition 5, some characteristics of the vertices. These definitions are used in the next section (3.3).

Definition 4. Let M be any map of Λ and $\tilde{b}$ its root half-edge. The new rooted maps are created as follow:

- $M^{(1)}$ (see Figure 3), by adding a root half-edge r to M (which is the root half-edge of $M^{(1)}$), such as:
  1. its final extremity is glued to the initial vertex of $\tilde{b}$, with $\bar{\sigma}(r) = \tilde{b}$;
  2. its initial extremity is glued to the final vertex of the last half-edge e_1 belonging to $\bar{\sigma}^{-1*}(\tilde{b}) = (\bar{\sigma}^{-1}(\tilde{b}), \bar{\sigma}^{-2}(\tilde{b}), \ldots, \bar{\sigma}(\tilde{b}), \tilde{b})$, with $\bar{\sigma}(e_1) = r$ and such as $M^{(1)}$ is of type I.
- $M^{(i)}$ $(i = 2, 3)$ (see Figure 3), by adding a root half-edge r to M (which is the root half-edge of $M^{(i)}$, $i = 2, 3$), such as:
  1. its initial extremity is glued to the initial vertex of $\tilde{b}$, with $\sigma(r) = \tilde{b}$;
  2. its final extremity is glued to the final vertex of the last half-edge e_i $(i = 2, 3)$ belonging to $\bar{\sigma}^*(\tilde{b}) = (\tilde{b}, \ldots, \bar{\sigma}^{-1}(\tilde{b}))$, with $\bar{\sigma}(e_i) = \alpha(r)$ $(i = 2, 3)$ and such as $M^{(2)}$ is of type I and $M^{(3)}$ is of order one.

Lemma 1. *Let M be any map of Λ and $\tilde{b}$ its root half-edge. A map $M^{(i)}$, $i = 1, 2, 3$, issued from M as described in Definition 4 can always be build.*

Proof. We can always construct the following maps, by adding a root half-edge r to M. r is added in two different ways, giving two different maps:

1. the final extremity is glued to the initial vertex of $\tilde{b}$, with $\bar{\sigma}(r) = \tilde{b}$ (see the first drawing in the first row of Figure 3). Its initial extremity is glued to the final vertex of the half-edge $he = \bar{\sigma}^{-1}(\tilde{b})$ belonging to $\bar{\sigma}^{-1*}(\tilde{b})$, with $\bar{\sigma}(he) = r$ and the map obtained is then of type I.
2. The initial extremity is glued to the initial vertex of $\tilde{b}$, with $\sigma(r) = \tilde{b}$ (see the first drawing in the second and third rows of Figure 3). Its final extremity is glued to the final vertex of the half-edge $he = \tilde{b}$ belonging to $\bar{\sigma}^*(\tilde{b})$, with $\bar{\sigma}(he) = \alpha(r)$ and the map obtained is then of type I and consequently of order one.

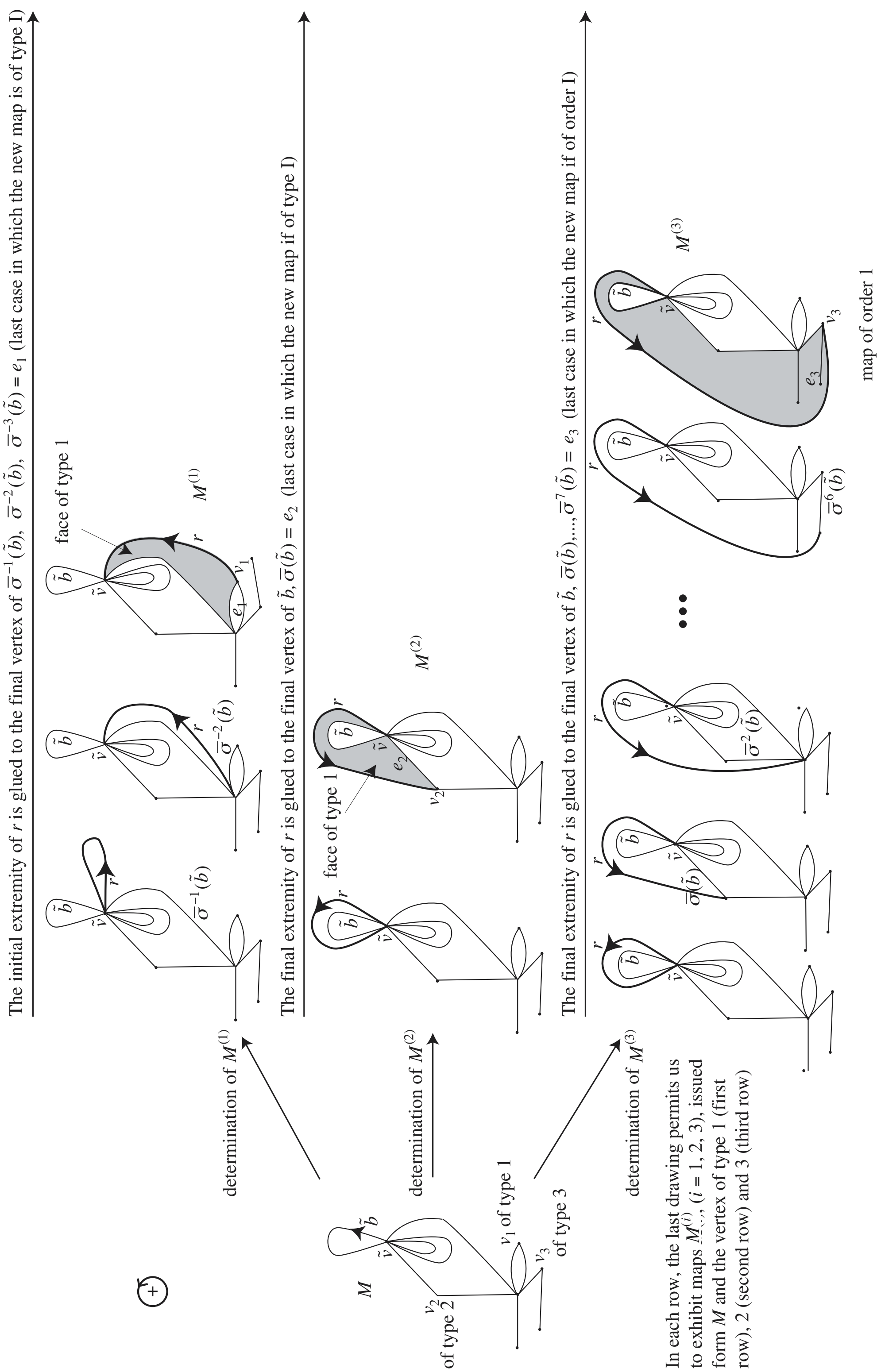


Figure 3. Exhibition of map $M^{(1)}$ (respectively $M^{(i)}$, $i = 2, 3$) issued from M and the associated vertex of type i, $i = 1, 2, 3$,

Following 1, one obtains Map $M^{(1)}$, he being now the last half-edge belonging to $\bar{\sigma}^{-1*}(\tilde{b})$, such as $M^{(1)}$ is of type one. Following 2, one obtains Map $M^{(2)}$, he being now the last half-edge belonging to $\bar{\sigma}^{*}(\tilde{b})$, such as $M^{(2)}$ is of type I. Following 2, one obtains Map $M^{(3)}$, he being now the last half-edge belonging to $\bar{\sigma}^{*}(\tilde{b})$, such as $M^{(3)}$ is of order one. $\square$

Definition 5. Let M be any map of Λ and $M^{(i)}$, $i = 1, 2, 3$, the maps deduced from M by Definition 4. The final vertex v_i, $i = 1, 2, 3$, of the half-edge e_i, $i = 1, 2, 3$, described in Definition 4, is said to be of type i, $i = 1, 2, 3$, in M (see Figure 3).

3.3. A topological decomposition of planar maps of $\mathcal{N}$. A topological decomposition of a map N of $\mathcal{N}$ into two maps of $\mathcal{I} \times \mathcal{I} \cup \{p\}$ is given. At first N is split into two maps (see cut algorithm). One of these maps belongs to $\mathcal{I}$ (see Theorem 2) and the other one can be transformed bijectively into a map belonging to $\mathcal{I} \cup \{p\}$ (see Theorem 3).

3.3.1. Exterior sub-maps.

Lemma 2. [König, Hamiltonian circuit] *Let N be a planar map in which the root half-edge is not a bridge. From the sequence $\bar{\sigma}^{*}(\tilde{b})$, a unique sub-sequence of half-edges constituting a circuit, including $\tilde{b}$, that does not cross two times the same edge or the same vertex can be extracted.*

Proof. The proof being straightforward is omitted. $\square$

The sub-sequence described in the previous lemma (see in Figure 4 the circuit drawn in bold) is a rooted polygon dividing the plane into two open connected domains, one exterior, including the point at infinity, and one interior. Let $(g_1, \ldots, g_r)$ be r maximal planar maps incident respectively to $(s_1, \ldots, s_r)$ (see Definition 2) and formed exclusively of half-edges belonging to the exterior domain (see $g_1 = g_{13}$, $g_3 = g_5$, $g_7 = g_9$ and g_{10} drawn dotted in Figure 4). g_i, $1 \leq i \leq r$, can be reduced to $\{p\}$ (in Figure 4, we have $g_2 = g_4 = g_6 = g_8 = g_{11} = g_{12} = \{p\}$).

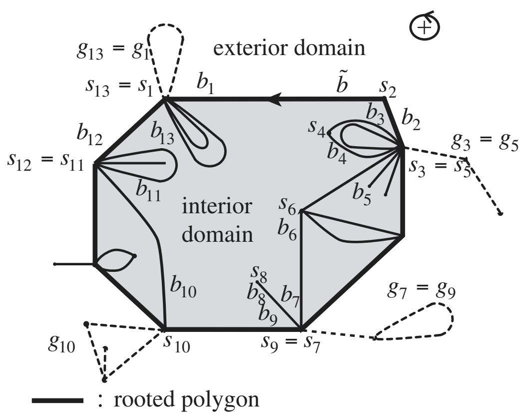


Figure 4. A planar map belonging to $\mathcal{N}$

3.3.2. Decomposition of the map N into two maps. The following cut algorithm describes how to cut the polygon of N into two parts.

Cut algorithm
Let $\bar{\sigma}^{-1*}(b_1) = (b_r, b_{r-1}, \ldots, b_1)$ (in Figure 9, $r = 13$);
Let b be the half-edge b_r;
while b distinct from b_1 and b belongs to case I
$b = \bar{\sigma}^{-1}(b)$;

(a) If b is the half-edge b_1 /* all half-edges $b_r, b_{r-1}, \ldots, b_1$ belong to case I */ then {a cut after b_2 in the circuit $\bar{\sigma}^{-1*}(b_1)$ is put, *i.e.* between b_2 and b_1, such as g_2 stays glued to the root vertex;} (see Figure 5)

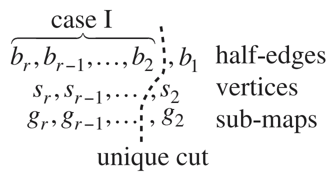


Figure 5. Case (a) of the cut algorithm

(b) else {a first cut between $b_{j+1} = \bar{\sigma}(b)$ and $b_j = b$ ($j + 1 = 1$ when $j = r$) is put, g_{j+1} going along with b_j (see Figure 9, $j = 8$);
Let $\bar{\sigma}^*(b_1) = (b_2, b_3, \ldots, b_r, b_1)$;
Let b be the half-edge b_2;
while b belongs to case I
$b = \bar{\sigma}(b)$;
A second cut is put between $b_{i-1} = \bar{\sigma}^{-1}(b)$ and $b_i = b$, such as g_i goes along with b_i (see Figure 9, $i = 3$);} (see Figure 6)

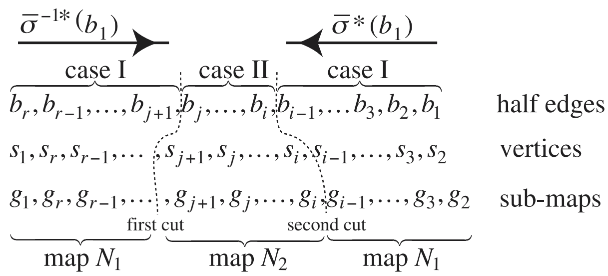


Figure 6. Case (b) of the cut algorithm

End of algorithm.

Decomposition of N. As N belongs to $\mathcal{N}$ and the two cuts (coinciding or not) are incident to the interior and exterior face of N, we obtain two maps. Let us call the map including b_1, N_1, rooted in $\tilde{b}$ and the map including b_i and b_j (that can be reduced to $\{p\}$ if we are in case (a) of the cut algorithm), N_2, rooted in b_i.

Notation 3. The removal of $\tilde{b}$ in N_1 disconnects the map into two sub-maps: the sub-map N_{11} glued to the final vertex of $\tilde{b}$ in N_1, rooted in $\bar{\sigma}(\tilde{b})$, and the sub-map N_{12} glued to the root vertex in N_1, rooted in $\sigma(\tilde{b})$.

Let k be such as s_k is the vertex of type 3 in N_2. N_{2bis} is the sub-map N_2 deprived of g_k.

Proposition 1. properties of maps N_1 and N_2 (see Figure 9)

a) $\tilde{b}$ *belongs in* N_1 *to a bridge edge, so* N_1 *belongs to* $\mathcal{I}$.
b) N_2 *belongs to* Λ.
c) s_{j+1} *is the vertex of type 1 in* N_{11}.
d) s_i *is the vertex of type 2 in* N_{12}.
e) s_{j+1} *is the vertex of type 3 in* N_{2bis} *(see Figure 7).*

Proof. a) and b) are straightforward.

c) is a consequence of the cut algorithm as $\tilde{b}$, with its root vertex glued to the vertex s_{j+1}, represents the half-edge r if M is here the map N_{11} (see Definition 4).

d) is a consequence of the cut algorithm as $\tilde{b}$, with its final vertex glued to the vertex s_i, represents the half-edge r if M is here the map N_{12} (see Definition 4).

e) Let us construct from the map N_{2bis}, the map $N^{(3)}_{2bis}$ (see Definition 4). $N^{(3)}_{2bis}$ is obtained by gluing the final extremity of the root half-edge r to the final extremity of b_j, s_{j+1}. In fact this map is of order one, as N is, and s_{j+1} is of type 3 in N_{2bis}. Besides note that as b_j belongs to case II in N, its initial extremity s_j is not incident to the exterior face of N. Therefore if a root half-edge r is added to N_{2bis} with its initial extremity glued to the initial extremity of b_i, with $\sigma(r) = b_i$, and its final extremity glued to the final extremity of $\bar{\sigma}(b_j)$, with $\bar{\sigma}(\bar{\sigma}(b_j)) = \alpha(r)$, the map obtain is no more of order one (see Figure 7). In that case, both the extremities of b_j do not belong to the exterior face. □

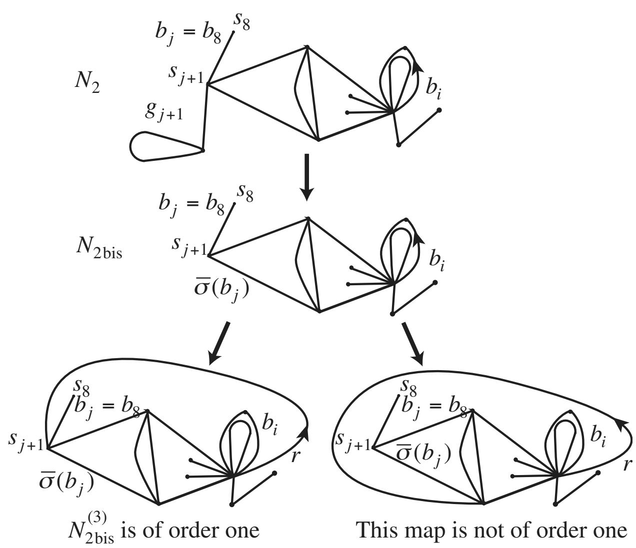


Figure 7. Search of the vertex of type 3 in N_{2bis}

Theorem 2. *The family of maps* N_1 *obtained by the previous decomposition is* $\mathcal{I}$.

Proof. N_1 being obviously a map of $\mathcal{I}$, it remains to prove that N_1 can be any map of $\mathcal{I}$, *i.e.* it remains to prove that for any map of $\mathcal{I}$ the two vertices v_1 and v_2 can be found, which when glued to another map N_2 allow to recover map N. N_{11} and N_{12} belong to Λ as N_1 belongs to $\mathcal{I}$. Then from Lemma 1, it is known that there exists a unique vertex of type 1 in N_{11} and a unique vertex of type 2 in N_{12}. We saw in the cut algorithm that these vertices are the vertices v_1 and v_2 (see Figure 9). We now know where to attach map N_1 to reconstruct map N. $\square$

Theorem 3. *The set of maps N_2 is in one-to-one correspondence with $\mathcal{I} \cup \{p\}$.*

Proof. Let us consider that N_2 is not reduced to a single vertex (case (a) of the cut algorithm). Note that the place of the second cutting is showed by the root half-edge of N_2. As it is, the place of the first cutting can not be recovered. For that N_2 must be transformed.

a- Transformation of a map N_2 not reduced to $\{p\}$ into a map belonging to $\mathcal{I}$ (see Figure 9)

Step 1 To transform N_2 into a map belonging to $\mathcal{I}$, the final extremity s_{i+1} of the root half-edge b_i is glued to s_{j+1} such as b_j does not belong to the exterior domain but map g_{j+1} does (see step 1 in Figure 9). So as s_{j+1} is of type 3 in N_2 deprived of g_{j+1} (see Proposition 1.e), it is still of type 3 in this new map deprived of g_{j+1} for the same reasons. Map N_{21} is obtained.

Step 2 Then b_i with map g_{j+1} are untied from s_{j+1} (see step 2 in Figure 9). A map N_{22} is obtained. N_{22} belongs to $\mathcal{I}$. Let us call N_{221} the sub-map glued to the final vertex of the root edge, this root edge excluded. It can be any map of Λ, as it corresponds to map g_{j+1} incident to s_{j+1} in N_2. Let call N_{222} the map N_{22} deprived of N_{221}. It belongs to Λ. s_{j+1} being still of type 3 in N_{222}, N_{21} can be recovered from N_{22}.

It will be shown now that any map of $\mathcal{I}$ can be transform into a map N_2 and thus prove that the set of maps N_2 is in bijection with $\mathcal{I}$.

b- Reciprocal transformation

Let N_{22} be any map of $\mathcal{I}$, with its root half-edge called b_i. Thanks to a we know how to recover N_{21} from N_{22}. The vertex v of type 3 in N_{222} must be found (see Figure 9). Lemma 1 leads to the existence of v. When v is founded, the final extremity of the root half-edge b_i is glued to v such as the sub-map g_{j+1} incident to the final vertex of b_i, belongs to the exterior domain. N_{21} has been recovered (see step 3 in Figure 9) and v is still of type 3 in N_{21} deprived of g_{j+1}. To recover N_2 from N_{21}, the sub-set S of all edges that must be unglued from v and glued to a new vertex s_{i+1} must be found (S corresponds to all the edges that have an extremity equal to s_{i+1} in N_2). In the ordered set $\sigma^{-1*}(\alpha(b_i))$ in N_{21}, starting from $\alpha(b_i)$, there are two cases as v is of type 3 in N_{21} deprived of g_{j+1} (see Figure 8):

- There is a loop l such as $\sigma(l) = \alpha(l)$, then l corresponds to b_j. Then S is the set of all half-edges met in $\sigma^{-1*}(\alpha(b_i))$ until meeting l for the first time (meaning b_j).
- There is a half-edge with its final vertex non-incident to the exterior face. This last half-edge corresponds to $\alpha(b_j)$. Then S is the set of all half-edges met in $\sigma^{-1*}(\alpha(b_i))$ until meeting $\alpha(b_j)$, $\alpha(b_j)$ excluded.

Therefore to recover N_2 from N_{21}, v is untied with all half-edges belonging to S (see step 4 in Figure 9). $\square$

3.4. Enumeration of $\mathcal{N}$ and Λ. From the precedent topological decomposition an expression of the generating function of $\mathcal{N}$ is obtained in Theorem 4.

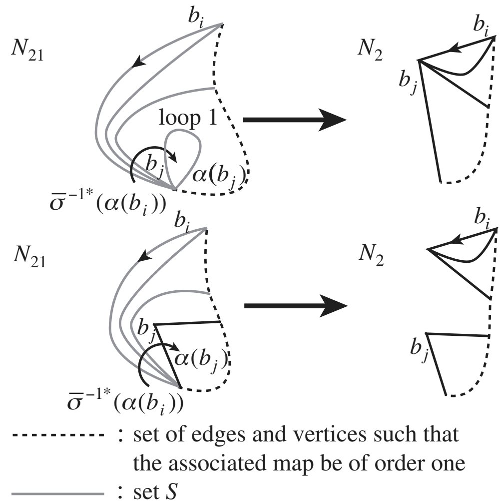


Figure 8. Different cases of maps N_{21} and recovering of N_2

Notation 4. Let G_1 be the generating function for planar maps of order one. Let F_1 (respectively I_1) be the generating function for planar maps of order one with a non-bridge (respectively bridge) root edge, *i.e.* the generating function of $\mathcal{N}$ (respectively $\mathcal{I}$). The degree of z represents the number of edges. Henceforth in equations, we will consider implicitly that generating functions are function of z (writing I_1 instead of $I_1(z)$, and so on...).

Theorem 4. a) *The generating function F_1 of $\mathcal{N}$ is: $F_1 = I_1 \times (1 + I_1)$.*
b) *The generating function G_1 of Λ is: $G_1 = (1 + I_1)^2$.*

Proof. Two lemmas are needed (see Lemmas 3 and 4 in a)) to prove Theorem 4 (see b)).

a) From the decomposition described in the precedent section, a generating function of $\mathcal{N}$ is obtained.

Lemma 3. *Maps N_1 are enumerated by the generating function I_1.*

Proof. The proof is straightforward since by Theorem 2, the set of maps N_1 is in one to one correspondence with $\mathcal{I}$. □

Lemma 4. *Maps N_2 are enumerated by the generating function $1 + I_1$.*

Proof. The proof is straightforward since by Theorem 3, the set of maps N_2 is in one to one correspondence with $\mathcal{I} \cup \{p\}$. □

As N is the union of N_1 and N_2, we obtain $F_1 = I_1 \times (1 + I_1)$.

b) The set of maps of order one Λ is the union of $\mathcal{I}$, $\mathcal{N}$ and $\{p\}$. So, $G_1 = I_1 + F_1 + 1 = I_1 + I_1 \times (1 + I_1) + 1 = (1 + I_1)^2$. □

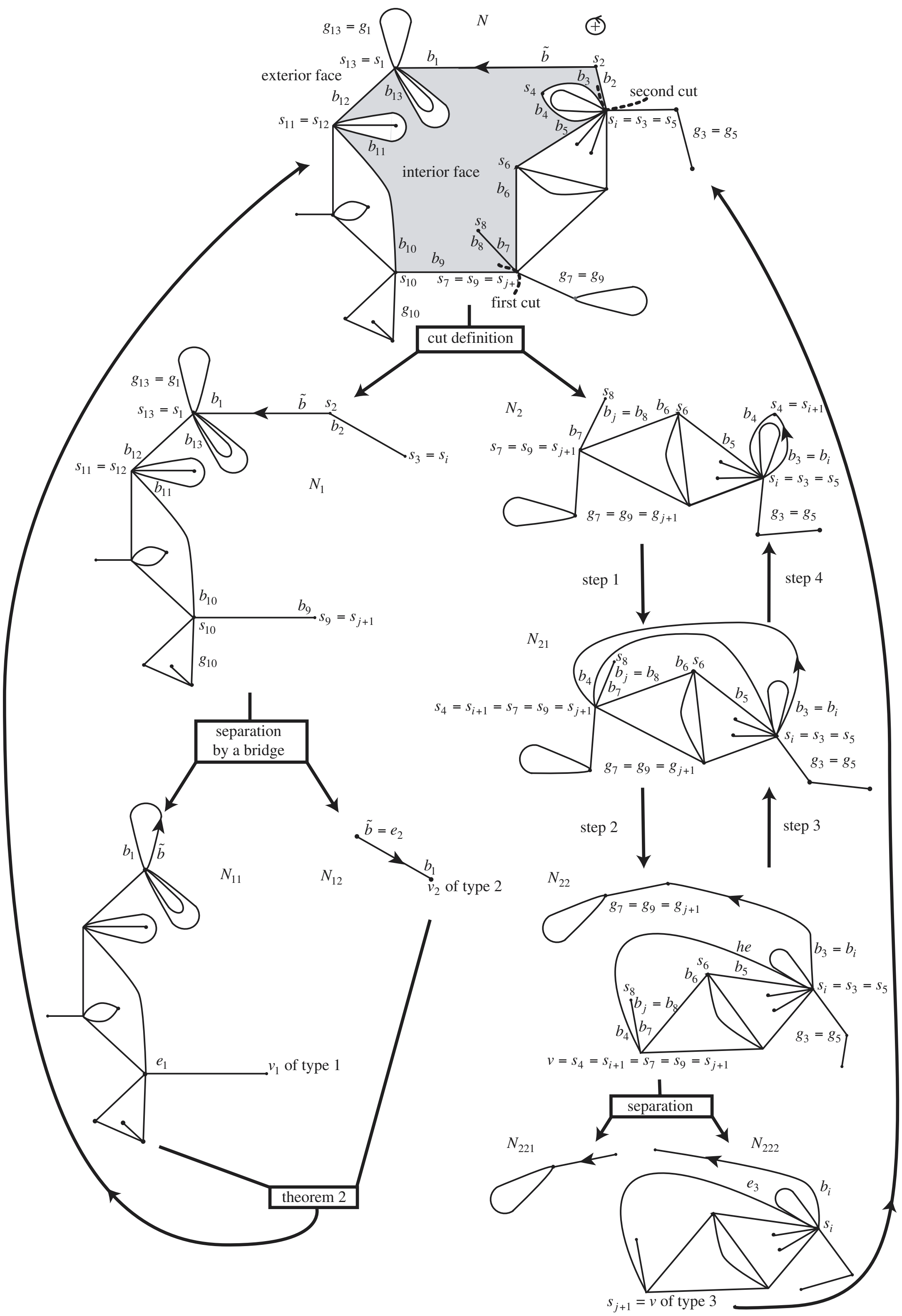


Figure 10. Decomposition of map of Figure 2 into sub-maps of order one

3.5. Enumeration of $\mathcal{I}$. We deduce from 3.4 an enumeration of $\mathcal{I}$.

Theorem 5. *The generating function of $\mathcal{I}$ is: $I_1 = z(1 + I_1)^4$.*

And thus the number of planar maps of order one with n edges including a bridge root edge is given by: $1/(3n + 1)\binom{4n}{n}$.

Proof. As $G_1 = (1 + I_1)^2$ and $I_1 = zG_1^2$, it follows that $I_1 = z(1 + I_1)^4$. Applying the Lagrange's inversion to this later equation, the following result is obtained: coefficient of z_n in $I_1 = 1/(3n + 1)\binom{4n}{n}$. □

4. A geometrical one-to-one correspondence between Π_n and $\mathcal{I}_n$. First of all let us recall that a $(3n + 2)$-gon is partitioned into n pentagons and that the set of partitions of a $(3n + 2)$-gon and the set of maps of order one with a bridge root edge have the same number of elements. So there is a natural idea which is to associate one edge of a map belonging to $\mathcal{I}_n$ with a pentagon. The first step is to transform the root bridge into a first pentagon with one root side. We show this first step in Section 4.1.

But then we have to know what to do with the two remaining planar maps of order one. If each map of order one can be split into two maps belonging to $\mathcal{I} \cup \{p\}$ (which seems possible as $G_1 = (1 + I_1)^2$ from Theorem 4), four maps belongings to $\mathcal{I} \cup \{p\}$ are obtained. We reiterate the transformation of the bridge root edge of each of these maps into a new pentagon and the four rooted edges of this four pentagons are glued to the four non-root sides of the first pentagon. Then this decomposition and gluing can be reiterated until each edge has been transformed into a pentagon. Therefore each bridge becomes a pentagon and each pentagon represents a bridge. In Section 4.2, it is seen how to transform bijectively a map belonging to Λ into two maps belonging to $\mathcal{I} \cup \{p\}$. Then in Section 4.3, it is shown how to transform a map of $\mathcal{I}_n$ into a unique partition of Π_n. It will then be easily seen that this transformation is a one-to-one correspondence.

Notation 5. Let us name p_1 the root edge of the pentagon, and p_1, p_2, p_3, p_4, p_5 the five sides of the pentagon as showed in Figure 10. Let I be a map of $\mathcal{I}_n$ with a root half-edge $\tilde{b}$, m_1 the map of order one glued to the root vertex v_1 of I ($\tilde{b} \notin m_1$), and m_2 the map of order one attached to the final extremity v_2 of $\tilde{b}$ ($\alpha(\tilde{b}) \notin m_2$).

4.1. First stage of the bijective transformation of a map of $\mathcal{I}_n$ into a map of Π_n. The root edge $\tilde{b}$ becomes a pentagon. Then the vertex of m_1 that was glued to v_1, is glued at the intersection of p_4 and p_5, and the vertex of m_2 that was glued to v_2, is glued at the intersection of p_2 and p_3 (see Figure 10).

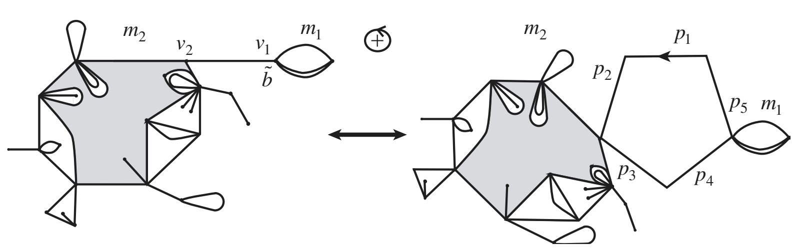


Figure 10. First stage of the geometrical one-to-one transformation between a map belonging to $\mathcal{I}_n$ and a partition belonging to Π_n

4.2. A geometrical one-to-one correspondence between Λ and $(\mathcal{I}_n \cup \{p\})^2$. The set Λ_n of planar maps of order one with n edges, $n > 0$, is split into three sets, $\{p\}$, $\mathcal{I}_n$ and $\mathcal{N}_n$. This one to one correspondence is detailed in three parts:

1. $\{p\}$ is placed into one-to-one correspondence with $\{p\}^2$.
2. $\mathcal{I}_n$ is placed into one-to-one correspondence with $\mathcal{I}_n \times \{p\}$ (see Figure 11).

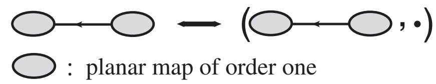


Figure 11. Decomposition of a map of $\mathcal{I}$ into a pair of maps, the first belonging to $\mathcal{I}$ and the second is $\{p\}$

3. $\mathcal{N}_n$ is put into one-to-one correspondence with $(\mathcal{I}_n \cup \{p\}) \times \mathcal{I}_n$. To show geometrically this correspondence, we start with the decomposition of a map of $\mathcal{N}$ showed in Section 3.3. This decomposition gives two maps N_1 and N_2, in which the root edge of N belongs to N_1. N_1 belongs to $\mathcal{I}$, and N_2 can be bijectively transformed into a map N_{22} of $\mathcal{I} \cup \{p\}$ (see Section 3.3.2). Therefore a map belonging to $\mathcal{N}$ has been bijectively associated to two maps, one belonging to $\mathcal{I} \cup \{p\}$ and one to $\mathcal{I}$ (see Figure 12).

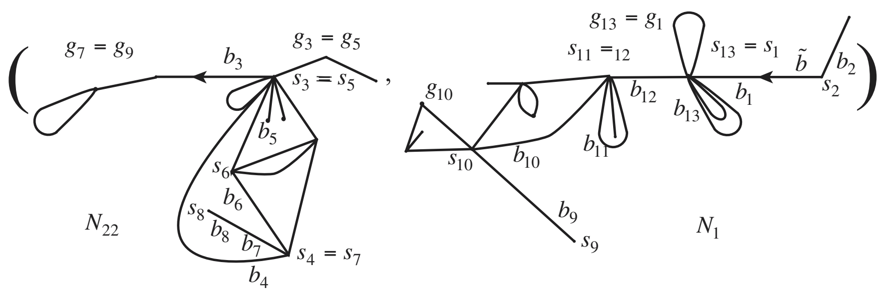


Figure 12. Decomposition of map of Figure 2, into two maps belonging to $(\mathcal{I} \cup \{p\}) \times \mathcal{I}$

Example. A particular case

Let us recall that if each half-edge of $\bar{\sigma}^*(b_1)$ belongs to case I (see Definition 3) only, just one cut is put between b_1 and b_2, such as g_2 goes with b_1. Thus one map belonging to $\mathcal{I}$ and one map N_{22} reduced to $\{p\}$ are obtained (see Figure 13).

Figure 13. Decomposition of the map glued to the root vertex of the map showed in Figure 10, into a pair of maps, $\{p\}$ and one belonging to $\mathcal{I}$

4.3. Bijective transformation of a map of $\mathcal{I}_n$ into a map of Π_n. $\sigma(\tilde{b})$ (respectively $\bar{\sigma}(\tilde{b})$) becomes the root half-edge of map m_1 (respectively m_2). Section 4.2 shows that m_i, $i = 1, 2$, are bijectively associated to pairs of maps (m_{i_1}, m_{i_2}), each map belonging

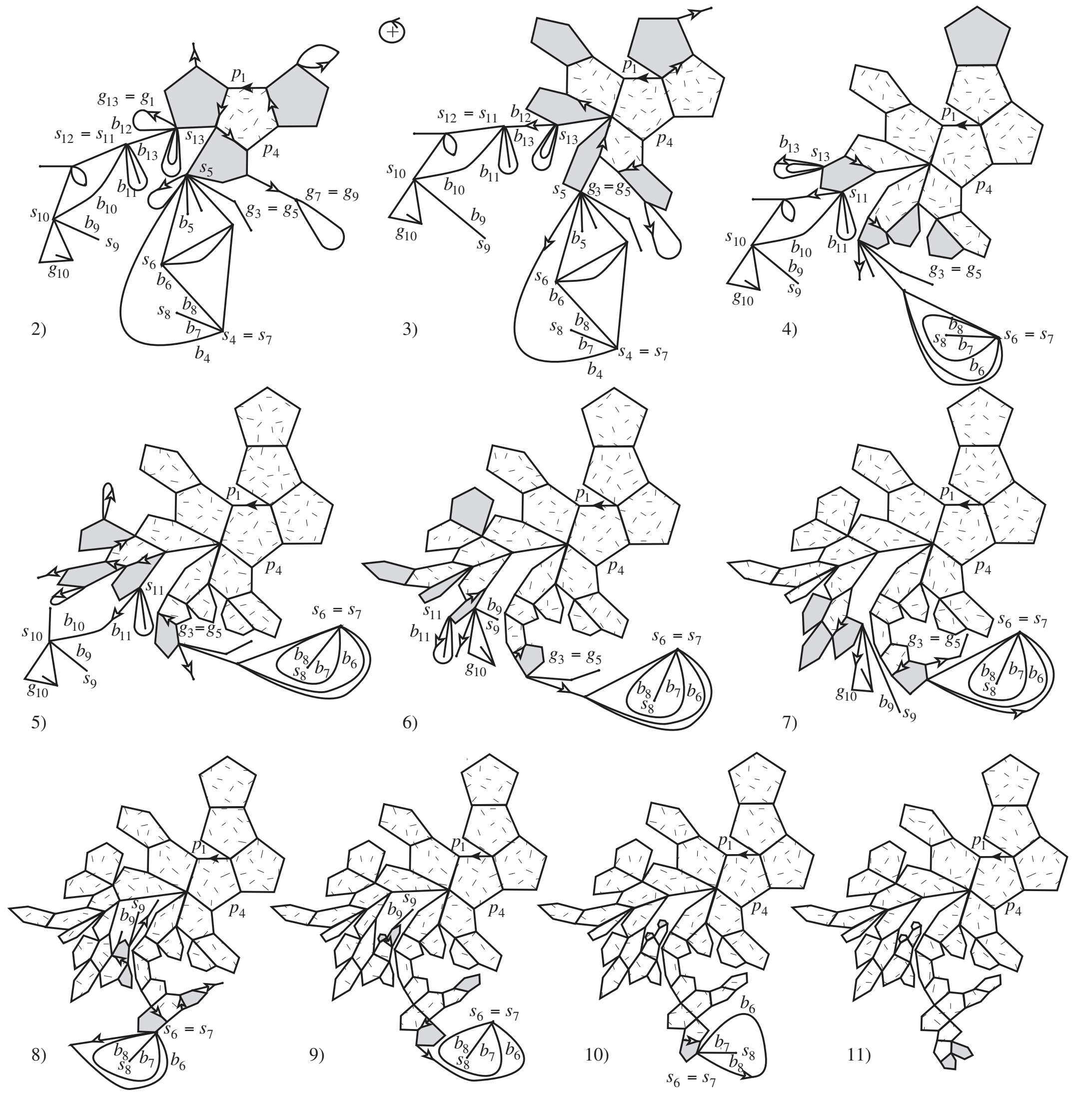


Figure 14. Second to last stage of the geometrical one-to-one transformation between map on the left of Figure 10 and a partition belonging to Π_n

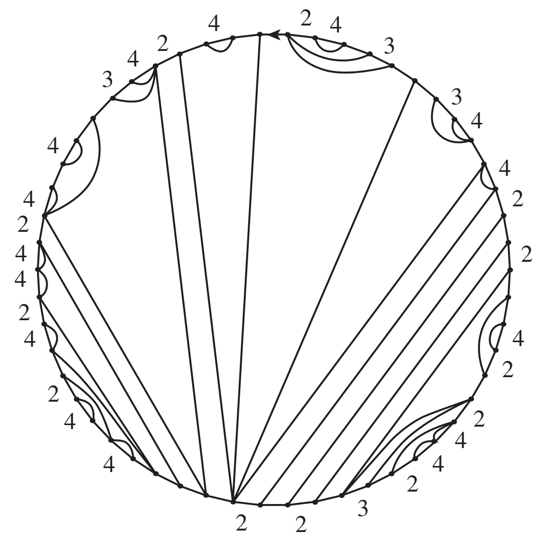
to $\mathcal{I} \cup \{p\}$. So after step showed in Section 4.1 we glue:

- the root half-edge of m_{1_1} to p_5,
- the root half-edge of m_{1_2} to p_4,
- the root half-edge of m_{2_1} to p_2,
- the root half-edge of m_{2_2} to p_3.

Then each of these root half-edges are transformed into pentagons and the two maps glued to their root vertex and final extremity are rooted and glued as in Section 4.1 to this new pentagon (see Figure 14). This process is reiterated until a polygon partitioned into pentagons is obtained (see Figures 14 and 15). Each step of this transformation is a one-to-one correspondence as it is always possible to go back. In fact, the orientation of the sphere gives an easy way to go back. Each time that a pentagon is created, it is implicitly rooted, and so can be transformed back to a root half-edge.

5. Conclusion. A one-to-one correspondence between rooted polygons partitions and the set of rooted planar maps of order one with a bridge root edge has been shown with the use of a geometrical proof. The difficulty lay in the fact of finding for any rooted planar map of order one a decomposition into two maps, each one belonging to the set of rooted planar maps of order one with a bridge root edge, the map reduced to a unique vertex included, $\mathcal{I} \cup \{p\}$. It is interesting to note that the work on rooted polygons partitions is linked to the theory of rooted planar maps, which has many results found in an extensive literature (see for example [1], [3], [5], [10], [11]).

Let remark that the bijection between 4-trees with $4n+1$ vertices and $3n+1$ endpoints and partitions of a polygon into pentagons given by R. P. Stanley [8], leads to a new bijection between the set of planar maps of order one with a bridge root edge and 4-trees with $4n+1$ vertices and $3n+1$ endpoints.



An edge with number *n* written beside represents actually *n* edges

Figure 15. Last stage of Figure 14 giving the convex polygon issued from map situated on the left of Figure 10

Résumé substantiel en français.

Énumération des partitions d'un polygone en pentagones. Un polygone convexe pointé ayant $3n+2$ arêtes ($(3n+2)$-gone), se découpe en n pentagones.

Théorème 1. *Le nombre de partitions d'un* $(3n+2)$*-gone en pentagones est*

$$p_n = \frac{1}{3n+1}\binom{4n}{n}.$$

Soit Π_n l'ensemble des partitions d'un $(3n+2)$-gone pointé en n pentagones.

1. Énumération de carte planaires d'ordre 1 dont l'arête pointée est un isthme. On choisit une orientation anti-trigonométrique sur la sphère.

1.1. Définitions.

Définition 1. Soit α (respectivement σ) la permutation sur l'ensemble des brins qui associe à chaque brin, son brin opposé (respectivement le premier brin rencontré en tournant autour du sommet pointé dans le sens positif). Alors $\bar{\sigma}$ est la permutation $\sigma \circ \alpha$.

Si $\tilde{b}$ n'est pas un isthme (arête incidente sur ses deux côtés à une même face), nous appellerons face extérieure (respectivement face intérieure) d'une carte planaire pointée, la face $\bar{\sigma}^*(\tilde{b})$ (respectivement $\bar{\sigma}^*(\alpha(\tilde{b}))$) située sur le côté droit (respectivement sur le côté gauche) du brin pointé $\tilde{b}$.

Soit $b_1 = \alpha(\tilde{b})$. Alors $\bar{\sigma}^*(b_1) = (b_1, \ldots, b_r)$ est le circuit orienté des brins incidents sur leur droite à la face intérieure. Nous appellerons respectivement $(s_1, \ldots, s_r)$ les sommets initiaux de $(b_1, \ldots, b_r)$, non nécessairement tous distincts (à la figure 2, $s_3 = s_5$, $s_7 = s_9$, $s_{11} = s_{12}$ et $s_{13} = s_1$).

Définition 2. (*Les cartes d'ordre un*) Une carte planaire pointée est dite d'ordre un si chaque arête a au moins une extrémité incidente à la face extérieure.

Soit b_i le brin appartenant à la face intérieure. Il y a deux cas :

I. b_i a ses deux extrémités incidentes à la face extérieure (voir b_1, b_2, b_9, b_{10}, b_{11}, b_{12}, b_{13} à la figure 2).

II. b_i a une extrémité non-incidente à la face extérieure (l'autre étant incidente à la face extérieure par définition des cartes d'ordre un) (voir $b_3, b_4, b_5, b_6, b_7, b_8$ à la figure 2).

Une carte d'ordre un est de type I si et seulement si chaque arête de la face intérieure appartient au cas I.

Notation 6. $\{p\}$ est la carte planaire réduite à un unique sommet.

Soit $\mathcal{I}_n$ (respectivement $\mathcal{N}_n$) l'ensemble des cartes planaires d'ordre un avec n arêtes, l'arête pointée étant un isthme (respectivement l'arête pointée n'étant pas un isthme) et $\mathcal{I} = \cup_{n\geq 1}\mathcal{I}_n$ (respectivement $\mathcal{N} = \cup_{n\geq 1}\mathcal{N}_n$).

L'ensemble Λ des cartes planaires pointées d'ordre un est : $\Lambda = \{p\} \cup \mathcal{I} \cup \mathcal{N}$.

Par convention, le nom du brin est montré sur la figure près de son extrémité initiale.

1.2. Une décomposition topologique des cartes planaires de $\mathcal{N}$.

1.2.1. Sous-carte extérieure.

Lemme 1. (König, circuit hamiltonien)

Soit N une carte planaire dont le brin pointé n'est pas un isthme. De la séquence $\bar{\sigma}^*(\tilde{b})$, une unique sous-séquence de brins constituant un circuit, incluant $\tilde{b}$, qui ne traverse pas deux fois la même arête ou le même sommet, peut être extraite.

La sous-séquence décrite dans le précédent lemme (voir à la figure 4 le circuit dessiné en gras) est un polygone divisant le plan en deux domaines ouverts connexes, un extérieur, incluant le point à l'infini, et un intérieur. Soit $(g_1, \ldots, g_r)$ r cartes planaires maximales incidentes respectivement aux sommets $(s_1, \ldots, s_r)$ (voir définition 1), composées exclusivement de brins appartenant au domaine extérieur (voir $g_1 = g_{13}$, $g_3 = g_5$, $g_7 = g_9$ et g_{10} dessinés en pointillé à la figure 4). g_i, $1 \leq i \leq r$, peut être réduit à $\{p\}$ (à la figure 4, on a $g_2 = g_4 = g_6 = g_8 = g_{11} = g_{12} = \{p\}$).

1.2.2. Décomposition de la carte N en deux cartes. Nous présentons une décomposition topologique d'une carte N de $\mathcal{N}$ en deux cartes de $\mathcal{I} \times \mathcal{I} \cup \{p\}$. L'algorithme de coupure suivant décrit comment couper le polygone de N en deux parties.

Algorithme de coupure
Soit $\bar{\sigma}^{-1*}(b_1) = (b_r, b_{r-1}, \ldots, b_1)$ (à la figure 9, $r = 13$)
Soit b le brin b_r;
Tant que b est distinct de b_1 ou que b appartient au cas I
$b = \bar{\sigma}^{-1}(b)$;

(a) Si b est le brin b_1 /* tous les brins $b_r, b_{r-1}, \ldots, b_1$ appartiennent au cas I */
Alors {une coupure est faite après b_2 dans le circuit $\bar{\sigma}^{-1*}(b_1)$, *i.e.* entre b_2 et b_1, tel que g_2 reste collé au sommet pointé;} (voir figure 5)

(b) sinon {un première coupure est faite entre $b_{j+1} = \bar{\sigma}(b)$ et $b_j = b$ ($j + 1 = 1$ lorsque $j = r$), g_{j+1} restant avec b_j (voir figure 9, $j = 8$);
Soit $\bar{\sigma}^*(b_1) = (b_2, b_3, \ldots, b_r, b_1)$;
Soit b le brin b_2;
Tant que b appartient au cas I
$b = \bar{\sigma}(b)$;
Une deuxième coupure est faite entre $b_{i-1} = \bar{\sigma}^{-1}(b)$ et $b_i = b$, tel que g_i reste avec b_i (voir figure 9, $i = 3$);} (voir figure 6)

Fin de l'algorithme.

Décomposition de N. Comme N appartient à $\mathcal{N}$ et que les deux coupures (coïncidant ou non) sont incidentes aux faces intérieure et extérieure de N, on obtient deux cartes. Appelons la carte contenant b_1, N_1, pointée en $\tilde{b}$ et la carte contenant b_i et b_j (qui peut être réduite à $\{p\}$ si on se trouve dans le cas (a) de l'algorithme de coupure), N_2, pointée en b_i.

Notation 7. La suppression de $\tilde{b}$ dans N_1 déconnecte la carte en deux sous-cartes : la sous-carte N_{11}, collée au sommet final de $\tilde{b}$ dans N_1, pointée en $\bar{\sigma}(\tilde{b})$, et la sous-carte N_{12}, collée au sommet pointé dans N_1, pointée en $\sigma(\tilde{b})$.

Théorème 2. *La famille de cartes N_1 obtenue par la décomposition précédente est $\mathcal{I}$.*

Théorème 3. *L'ensemble des cartes N_2 est en bijection avec $\mathcal{I} \cup \{p\}$.*

Preuve. Considérons que N_2 n'est pas réduite à un simple sommet (cas (a) de l'algorithme de coupure). Pour transformer la carte N_2 en une carte de $\mathcal{I}$, on réalise les deux étapes suivantes :

É. 1 : l'extrémité finale s_{i+1} du brin pointé b_i est collée à s_{j+1} telle que b_j n'appartienne pas au domaine extérieur, tandis que g_{j+1} en fait parti (voir l'étape 1 de la figure 9). On obtient une carte N_{21}.

É. 2 : Alors b_i avec la carte g_{j+1} est détaché de s_{j+1} (voir l'étape 2 de la figure 9). On obtient une carte N_{22} qui appartient à $\mathcal{I}$.

Cette transformation est bijective. $\square$

1.3. Énumération de $\mathcal{N}$, Λ et $\mathcal{I}$.

Théorème 4. a) *La fonction génératrice F_1 de $\mathcal{N}$ est :* $F_1(z) = I_1(z) \times (1 + I_1(z))$.
b) *La fonction génératrice G_1 de Λ est :* $G_1(z) = (1 + I_1(z))^2$.

Théorème 5. *La fonction génératrice de $\mathcal{I}$ est :* $I_1(z) = z(1 + I_1(z))^4$.

Ainsi le nombre de cartes planaires d'ordre un avec n arêtes incluant une arête pointée isthme est : $1/(3n+1)\binom{4n}{n}$.

2. Une bijection géométrique entre Π_n et $\mathcal{I}_n$.

Notation 8. Soit p_1 l'arête pointée du pentagone, et p_1, p_2, p_3, p_4, p_5 les cinq côtés du pentagones comme décrit à la figure 10. Soit I une carte de $\mathcal{I}_n$ avec son brin pointé $\tilde{b}$, m_1 la carte d'ordre un collée au sommet pointé v_1 de I ($\tilde{b} \notin m_1$), et m_2 la carte d'ordre un rattachée à l'extrémité finale v_2 de $\tilde{b}$ ($\alpha(\tilde{b}) \notin m_2$).

2.1. Première étape de la transformation bijective d'une carte de $\mathcal{I}_n$ en une carte de Π_n. L'arête pointée $\tilde{b}$ devient un pentagone. Alors le sommet de m_1 qui était attaché à v_1, est collé à l'intersection de p_4 et p_5, et le sommet de m_2 qui était attaché à v_2, est collé à l'intersection de p_2 et p_3 (voir figure 10).

2.2. Une bijection géométrique entre Λ et $(\mathcal{I}_n \cup \{p\})^2$. L'ensemble Λ_n des cartes planaires d'ordre un avec n arêtes, $n > 0$, est séparé en trois ensembles, $\{p\}, \mathcal{I}_n$ et $\mathcal{N}_n$. Cette bijection se détaille en trois parties :

1. $\{p\}$ est mis en bijection avec $\{p\}^2$.
2. $\mathcal{I}_n$ est mis en bijection avec $\mathcal{I}_n \times \{p\}$ (voir figure 11).
3. $\mathcal{N}_n$ est mis en bijection avec $(\mathcal{I}_n \cup \{p\}) \times \mathcal{I}_n$. Pour montrer géométriquement cette bijection, nous commençons avec la décomposition d'une carte de $\mathcal{N}$, montré au paragraphe 1.2. Cette décomposition donne deux cartes N_1 et N_2, où l'arête pointée de N appartient à N_1. N_1 appartient à $\mathcal{I}$, et N_2 peut être transformée bijectivement en une carte N_{22} de $\mathcal{I} \cup \{p\}$ (voir section 2.2). Ainsi une carte appartenant à $\mathcal{N}$ a été associée bijectivement à deux cartes, une appartenant à $\mathcal{I} \cup \{p\}$ et l'autre à $\mathcal{I}$ (voir figure 12).

2.3. Transformation bijective d'une carte de $\mathcal{I}_n$ en une carte de Π_n. $\sigma(\tilde{b})$ (respectivement $\bar{\sigma}(\tilde{b})$) devient le brin pointé de la carte m_1 (respectivement m_2). La section 2.2 montre que les cartes m_i, $i = 1, 2$, sont associés bijectivement à des paires de cartes (m_{i_1}, m_{i_2}), chaque carte appartenant à $\mathcal{I} \cup \{p\}$. D'où après l'étape montrée en 2.1, on colle :

- le brin pointé de m_{1_1} à p_5;
- le brin pointé de m_{1_2} à p_4;
- le brin pointé de m_{2_1} à p_2;
- le brin pointé de m_{2_2} à p_3.

Alors chacun de ces brins pointés est transformé en pentagone et les deux cartes, collées à leur sommets pointés et extrémité finale, sont pointées et collées comme montré au 2.1, au nouveau pentagone (voir figure 14). Ce procédé est répété jusqu'à ce qu'un polygone découpé en pentagones soit obtenu (voir figures 14 et 15). Chaque étape de cette transformation est une bijection car on peut toujours revenir en arrière. En fait, l'orientation de la sphère permet un retour en arrière simple; chaque fois qu'un pentagone est créé, il est implicitement pointé, et peut donc être retransformé en un brin pointé.

REFERENCES

1. D. Arquès, *Une relation fonctionnelle nouvelle sur les cartes planaires pointées*, J. Combin. Theory Ser. B **39** (1985), 27–42.
2. D. Arquès and A. Giorgetti, *Une bijection géométrique entre une famille d'hypercartes et une famille de polygones énumérées par la série de Schröder*, Formal power series and algebraic combinatorics (Vienna, 1997), pp. 14–25; Discrete Math. **217** (2000), 17–32.
3. R. Cori, *Un code pour les graphes planaires et ses applications*, Astérisque **27** (1975).
4. I. M. H. Etherington, *Some problems of non-associative combinations (I)*, Edinburgh Math. Notes (1941); no. 32, 1–6.
5. Y. Liu, *Enumerative theory of maps*, Kluwer Acad. Publishers, 2000.
6. E. Schröder, *Vier combinatorische probleme*, Z. Math. Phys. **15** (1870), 371–376.
7. R. P. Stanley, *Polygon dissections and standard Young tableaux*, J. Combin. Theory Ser. A **76** (1996), 175–177.
8. R. P. Stanley, *Enumerative combinatorics*, Cambridge Studies in Advanced Mathematics, vol. 62, Cambridge University Press, Cambridge, 1999.
9. L. Takács, *Enumeration of rooted trees and forests*, Math. Sci. **18** (1993), 1–10.
10. W. T. Tutte, *A census of planar maps*, Canad. J. Math. **15** (1963), 249–271.
11. W. T. Tutte, *On the enumeration of planar maps*, Bull. Amer. Math. Soc. **74** (1968), 64–74.

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