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SOME MECHANICAL AND MATHEMATICAL ASPECTS ON TRANSONIC FLOWS

M. POGU AND G. TOURNEMINE

RÉSUMÉ. Dans cet article on étudie des écoulements stationnaires en régime transsonique. On propose une modélisation de ceux-ci puis une approche fonctionnelle du problème. Afin de contraindre les ondes de choc à apparaître, on introduit une inégalité de saut mettant en jeu la composante tangentielle du flux de masse. Une nouvelle méthode de résolution basée sur la minimisation d'une fonctionnelle agissant sur un espace compact est proposée.

ABSTRACT. In this paper steady transonic flows are considered. A modelization and a functional approach are proposed. In order to constrain the shock waves to appear, a jump inequality, involving the tangential component of the mass flux is introduced. A new solution procedure is given by minimizing a functional acting on a compact set.

Introduction. This article is concerned with steady transonic plane flows of a non-dissipative compressible ideal gaz around a profile set in unbounded atmosphere. For background notions, see for instance [1, 2, 3, 4]. Generally, transonic flows are the seat of weak shock waves. More precisely, if ε denotes the shock strength, it is shown [3] that the increasing of the jump with entropy is of order ε^3, and that the same holds for the vorticity. It is then advisable to deal with isentropic and irrotational flows. Under these conditions, the equations governing the velocity field reduce to a quasi-linear first order partial differential system of elliptic hyperbolic type. From the functional standpoint, the question of existence of a solution to such systems is still with us. The main difficulty derives from the lack of monotonicity, see for instance for this notion [4, 5, 6]. In the previous literature on the subject [7] the existence of a solution to the system is proved under significant conjectures. To the best of our knowledge, passing over these conjectures, only partial results are proved within functional frames. Usually the associated methods lead essentially to approach the equation of continuity while the irrotational character is preserved: the vorticity equates zero, see, for instance, [8, 9, 10, 11, 12, 13] and Subsection 1.2.

In this article, we propose a model with a functional approach following a different way. The equation of continuity holds exactly while the vorticity is minimized. Further, in order to constrain the shock waves to appear, a jump inequality, involving the

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tangential component of the mass flux is introduced as soon as the governing equations are set. Let us describe the main advantages of the proposed method. The equation of continuity holds exactly. The minimization of the vorticity may imply it can be small, this is in accordance with flows having weak shock waves and are therefore weakly rotational downstream these waves. Let us point out that the minimization of the divergence of the mass flux might imply solely this divergence equates zero, without warranting the conservation of mass holds necessarily.

The method furnishes namely the sonic lines and shock waves. Let us mention that most of the arguments concerning the functional approach become applicable to general situations associated with quasi-linear first order partial differential systems of mixed type, that is: elliptic, hyperbolic.

1. Statements. Steady transonic plane flows of a non-dissipative compressible ideal gaz around a given profile $\mathcal{P}$ are considered. The flows are assumed to be symmetrical so that we can reduce the study to the upper-half plane. In the frame of reference $(0; x_1, x_2)$, see Figure 1, the boundary $\Gamma_{\mathcal{P}}$ of $\mathcal{P}$ cuts the x_1-axis at point $N\left(-c/2, 0\right)$ and at point $T\left(c/2, 0\right)$, where c is the chord of $\mathcal{P}$. The governing equations are studied in the domain Ω which is bounded by introducing the artificial boundary $\widetilde{\Gamma}$. To simplify we take $\widetilde{\Gamma}$ as the half circle centered at 0 and of fixed radius R, large enough. Thus the boundary $\partial\Omega$ of Ω consists of $\widetilde{\Gamma}$, and of the curve Γ prolonging $\Gamma_{\mathcal{P}}$ to the x_1-axis. The unit outward normal to $\partial\Omega$ is denoted by $\vec{n}_e = (n_{e_1}, n_{e_2})$. The components n_{e_1} and n_{e_2} are taken here with respect to $(0; x_1, x_2)$. The body forces are supposed negligible as usual.

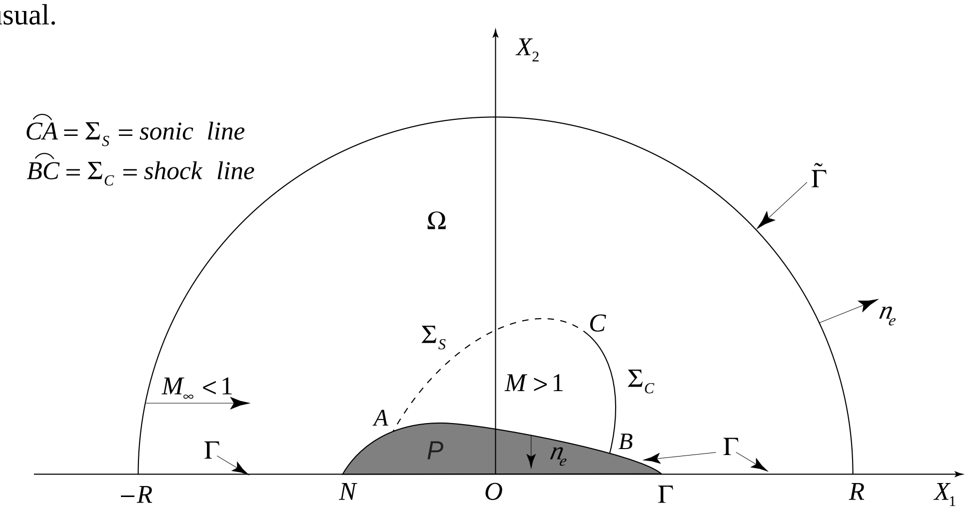

Figure 1.

Before describing the model constructed in this article (Section 2), let us recall some notions on the isentropic flows including weak shock waves (Subsection 1.1), then on the system to solve if we only want approach the equation of continuity (Subsection 1.2).

1.1. Isentropic flows including weak shock waves. The velocity field acts from Ω into $\mathbb{R}^2$. The limit speed is taken to be unity. The density ρ and the magnitude $|\vec{v}|$ of the velocity are connected by the isentropic relation:

$$\rho\left(|\vec{v}|^2\right) = \rho_0 \left(1 - |\vec{v}|^2\right)^{1/(\gamma-1)}, \quad |\vec{v}| \in (0,1). \tag{1}$$

In the latter, the density ρ_0 at the stagnation point and the ratio $\gamma > 1$ of specific heats are given constants. The critical speed reads as:

$$v_c = \left(\frac{\gamma - 1}{\gamma + 1}\right)^{1/2}. \tag{2}$$

Function $\vec{v}$ satisfies the equations and conditions below:

$$\begin{cases} \operatorname{div}\left(\rho\left(|\vec{v}|^2\right)\vec{v}\right) = 0 & \text{in } \Omega \quad (3.1) \\ \operatorname{rot}\vec{v} = 0 & \text{in } \Omega \quad (3.2) \\ 0 \le |\vec{v}| \le 1 & \text{in } \Omega \quad (3.3) \\ \rho\left(|\vec{v}|^2\right)\vec{v}\cdot\vec{n}_e = g & \text{on } \partial\Omega \quad (3.4) \end{cases}$$

Relation (3.1) is the equation of continuity. Let us denote by v_1, v_2 the components of $\vec{v}$ with respect to $(0; x_1, x_2)$ and by ∂_1 and ∂_2 the first partial derivatives with respect to x_1 and x_2. In (3.2) we set $\operatorname{rot}\vec{v} = \partial_1 v_2 - \partial_2 v_1$. We introduce Equation (3.2) since the flow is supposed isentropic here. Condition (3.3) means the magnitude of the velocity does not exceed the limit speed. In (3.4), the given function g satisfies:

$$\begin{cases} g = 0 & \text{on } \Gamma \quad (4.1) \\ g = t_\infty n_{e_1} & \text{on } \widetilde{\Gamma} \quad (4.2) \end{cases}$$

In (4.2) we have set: $t_\infty = \rho\left(u_\infty^2\right)u_\infty$, with u_∞ taken fixed within the interval $]\,0, v_c\,[$. Equality (4.2) simulates that the flow is uniform at infinity with velocity $(u_\infty, 0)$. Accordingly artificial boundary $\widetilde{\Gamma}$ is set at finite distance far enough from $\mathcal{P}$, and not intersecting the supersonic pockets. Moreover, g obeys the necessary condition

$$\int_{\partial\Omega} g\,d\Gamma = 0. \tag{5}$$

The latter is imposed by (3.1) and (3.4).

1.2. System to solve - the equation of continuity being approached. A possible functional approach of (3) consists in minimizing expressions equivalent to the lefthand side in (3.1) while Equation (3.2) remains preserved. Then this approach leads to find $\vec{v}$ so that

$$\begin{cases} \operatorname{div}\left(\rho\left(|\vec{v}|^2\right)\vec{v}\right) = \underline{d} & \text{in } \Omega \quad (6.1) \\ \operatorname{rot}\vec{v} = 0 & \text{in } \Omega \quad (6.2) \\ 0 \le |\vec{v}| \le 1 & \text{in } \Omega \quad (6.3) \\ \rho\left(|\vec{v}|^2\right)\vec{v}\cdot\vec{n}_e = g & \text{on } \partial\Omega \quad (6.4) \end{cases}$$

holds.

In (6.1), function $\underline{d}$ is a parameter to be minimized using a suitable norm.

Let us mention some results concerning the study of (6) within the frame of Sobolev spaces.

a) Among the possible solutions to (6) (if they exist) one considers those which satisfy an entropy condition. It is shown these solutions obey a variational inequality. The latter is solved by minimizing the associated functional on a convex compact subset equipped with an entropy condition. See for instance, [8, 9, 13].

b) The possible solutions to (6) (if they exist) obey an equality, expressed in terms of the projection of the mass flux onto a linear subspace. By means of a suitable norm, one proves the gap between the projection and zero is accurate to any $\varepsilon > 0$. See, for instance, [4, 10, 11, 12].

2. Construction of the model. Since the studied flow is transonic then a curve Σ divides Ω into a subsonic zone: Ω_Σ^- and a supersonic zone: Ω_Σ^+ (see Figure 2). The supersonic zones corresponds to the supersonic pockets. An oriented unit tangent to Σ is denoted by τ_Σ. Let us define the functions $\vec{v}_\Sigma^\pm$ deduced from $\vec{v}$ by setting

$$\begin{aligned} \Omega_\Sigma^\pm &\longrightarrow \mathbb{R}^2 \\ x &\longmapsto \vec{v}(x)\,. \end{aligned} \tag{7}$$

Curve Σ is partitioned into the sonic line Σ_s and the shock line Σ_c. The construction of these curves will be precised in Subsection 2.3.

The jump of a quantity a through Σ_c is denoted by $[[a]] = a^- - a^+$. Here a^- (a^+ respectively) corresponds with the limit of a, from the left (from the right) of shock line Σ_c.

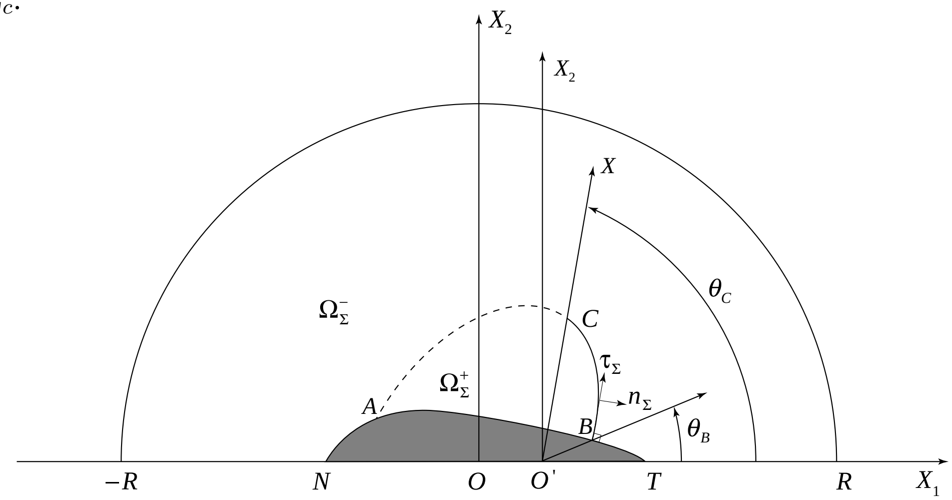

Figure 2.

2.1. System to solve - the vorticity being approached. Let us introduce the mass flux

$$\vec{q}(\vec{v}) = \rho\left(|\vec{v}|^2\right)\vec{v}. \tag{8}$$

Function $\vec{v}$ is sought to satisfy:

$$
\begin{cases}
\operatorname{div}(\vec{q}(\vec{v})) = 0 & \text{in } \Omega & (9.1)\\
\operatorname{rot}\vec{v} = \underline{r} & \text{in } \Omega & (9.2)\\
0 \le \left|\vec{v}_\Sigma^-\right| \le v_c & \text{in } \Omega_\Sigma^- & (9.3)\\
v_c \le |\vec{v}_\Sigma^+| \le 1 & \text{in } \Omega_\Sigma^+ & (9.4)\\
[[\vec{q}(\vec{v})]] \cdot \vec{t}_\Sigma \ge \sigma & \text{on } \Sigma_c & (9.5)\\
\vec{q}(\vec{v}) \cdot \vec{n}_e = g & \text{on } \partial\Omega & (9.6)
\end{cases}
$$

In (9.2) function $\underline{r}$ is a parameter to be minimized using a suitable norm.

Requirement (9.2) is in accordance with the assumption already mentioned in the introduction, and concerning the magnitude of the entropy, that way, of the vorticity via Crocco's theorem. Conditions (9.3) and (9.4) are introduced to involve the domain Ω_Σ^- (Ω_Σ^+) where the fluid is lower (exceeds) the speed of sound. Condition (9.5) constrains a shock wave to appear on Σ_c by means of a function σ chosen such as:

$$
\begin{cases}
\sigma > 0 & \text{on } \Sigma_c & (10.1)\\
\sigma' < 0 & \text{on } \Sigma_c & (10.2)
\end{cases}
$$

Conditions (10.1) and (10.2) ensure that the shock strength exists actually, and is decreasing along Σ_c (we will move from B to C, with $\sigma = 0$ at C only). More details concerning the motivation of (9.5), (10) will be given in the next subsection.

The problem now is to find function $\vec{v}$ satisfying (9), curve Σ dividing Ω and function σ obeying (10).

Let us point out the fundamental difficulty. When considering subsonic flows, conditions (9) reduce to (9.1), (9.2), (9.6) with: $0 \le |\vec{v}| \le v_c$, see for instance [4]. This essential difference makes precisely the difficulty to solve (9) taken as a whole: monotonicity properties are no more available, see [4] for further explanations.

2.2. The shock condition introduced in the model. Before developing the solution method of (9), let us give some lines on the motivation of condition (9.5). One has

$$[[\vec{q}]] = [[\vec{q} \cdot \vec{n}_\Sigma]]\vec{n}_\Sigma + [[\vec{q} \cdot \vec{\tau}_\Sigma]]\vec{\tau}_\Sigma, \qquad \text{on } \Sigma_c \tag{11.1}$$

here $\vec{n}_\Sigma$ is a unit normal to Σ_c and oriented from Ω_Σ^+ to Ω_Σ^-.

From (9.1) one obtains:

$$[[\vec{q}]] \cdot \vec{n}_\Sigma = 0 \qquad \text{on } \Sigma_c. \tag{11.2}$$

From (11.1) and (11.2) it follows

$$[[\vec{q}]] = [[\vec{q} \cdot \vec{\tau}_\Sigma]]\vec{\tau}_\Sigma \qquad \text{on } \Sigma_c. \tag{11.3}$$

Introducing the projection on $\vec{\tau}_\Sigma$ of the jump condition associated with Euler's equation, one deduces from (8) and (11.3)

$$[[\vec{q}]] = [[\rho\left(|\vec{v}|^2\right)]]\,(\vec{v} \cdot \vec{\tau}_\Sigma)\,\vec{\tau}_\Sigma \qquad \text{on } \Sigma_c. \tag{11.4}$$

The second principle of the thermodynamics implies density ρ increases through Σ_c that is

$$\rho\left(|\vec{v}_\Sigma^-|^2\right) > \rho\left(|\vec{v}_\Sigma^+|^2\right). \tag{11.5}$$

The observation of the shock polar, namely, leads to consider the inequality:

$$\vec{v}\cdot\vec{\tau}_\Sigma > 0 \text{ on } \Sigma_c. \tag{11.6}$$

Gathering (11.4), (11.5) and (11.6) it is inferred:

$$[[\vec{q}]]\cdot\vec{\tau}_\Sigma > 0\,,\ \text{on } \Sigma_c. \tag{11.7}$$

Inequality (11.7) has been taken into account by (9.5) precisely.

From the above considerations it follows that our aim is not to deal with the Rankine-Hugoniot relations taken as a whole, but to propose a model using a significant part of these relations.

2.3. Curves dividing the domain. Owing to informations from mechanics (see for instance [3] for these informations) the shape of Σ is simulated to the best, see Figure 2. We will choose the equation of Σ such that:

(α) T does not belong to the closure of Ω_Σ^+ (see (12.5) below).
(β) Σ does not reduce to a point of $\Gamma_{\mathcal{P}}$ (see (12.6) below).
(γ) Σ lies under curve $\widetilde{\Sigma}$ with $\Sigma\cap\widetilde{\Gamma}=\emptyset$ (see (12.7) below).
(δ) Σ is concave (see (12.8) below).

In order to simplify, given strictly positive numbers (small enough) will generally be denoted by α.

Let $0'$ be the point of coordinates $(\ell, 0)$, we assume:

$$-\frac{c}{2}+\alpha \le \ell \le \frac{c}{2}-\alpha. \tag{12.1}$$

We denote by (r,θ) the polar coordinates with respect to $(0; x_1, x_2)$, and by I the interval $[\,\alpha, \pi-\alpha]$. The equations of curves $\Gamma_{\mathcal{P}}$, Σ, $\widetilde{\Gamma}$ read:

$$r = F_\ell(\theta),\ r = F(\theta),\ r = \widetilde{F}_\ell(\theta),\ \theta\in I. \tag{12.2}$$

It is supposed:

$$\alpha \le F(\theta),\ \theta\in I, \tag{12.3}$$

and

$$F_\ell \in C^2(I),\ F\in C^2(I). \tag{12.4}$$

Actually (12.4) will result from hypothesis (19) in the sequel.

Now, we can expressed conditions (α)-(δ) by the following:

$$F(\theta) \le \left(\frac{c}{2}-\alpha\right)-\ell \tag{12.5}$$

$$F_\ell\left(\frac{\pi}{2}\right)+\alpha \le F\left(\frac{\pi}{2}\right) \tag{12.6}$$

$$F(\theta) \leq \widetilde{F}_\ell(\theta) - \alpha,\ \theta \in I \tag{12.7}$$

$$\alpha \leq \frac{1}{F(\theta)} + \left(\frac{1}{F(\theta)}\right)'' ,\ \theta \in I. \tag{12.8}$$

From (12.8), it follows Σ_c cuts Γ at a unique point B. The polar angle θ_B belongs to $]\,0, \frac{\pi}{2}\,[$, adaptating the α's in (12.1), (12.3) if necessary.

The shock line is the part of Σ described by $\theta \in (0, \theta_c)$ where:

$$\theta_B + \alpha \leq \theta_c \leq \frac{\pi}{2} - \alpha. \tag{12.9}$$

Let us set $\theta_c = (O'x_1, O'X)$, (12.8) implies the axis $0'X$ cuts Σ at a unique point C.

The slip condition (see (9.6) and (4.1)), combined with the properties of the shock polar, shows that this curve has a foot at B, and is oriented perpendicularly to $\Gamma_{\mathcal{P}}$ at B. This is taken into account by choosing tangent $\vec{\tau}_\Sigma$ in a parallel direction to normal $\vec{n}_e$, that is:

$$F(\theta_B)F_\ell(\theta_B) + F'(\theta_B)F'_\ell(\theta_B) = 0. \tag{12.10}$$

Sometimes, it will be useful to write Σ in terms of cartesian coordinates with respect to $(0; x_1, x_2)$. From (12.2) the equation of Σ reads

$$\Sigma(x) = 0 \tag{12.11}$$

where $\Sigma(x) = F(\theta) - r$, with

$$\theta(x) = \begin{cases} r - \text{Arctg}\left(\dfrac{-x_2}{x_1 - \ell}\right), & x_1 < \ell \\ \dfrac{\pi}{2}, & x_1 = \ell \\ \text{Arctg}\left(\dfrac{x_2}{x_1 - \ell}\right), & x_1 > \ell \end{cases} \tag{12.12}$$

$$r(x) = \sqrt{(x_1 - \ell)^2 + x_2^2}. \tag{12.13)}$$

From (12.1), expressions (12.12), (12.13)) are considered only when (x_1, x_2) does not equate $(\ell, 0)$.

It is supposed that curves Σ are within a fixed domain Ω_0, included in Ω, so that assumptions and conditions of this subsection hold true up to the closure of Ω (see Figure 3).

2.4. Decomposition of the vorticity field constraining the shocks to occur. In this subsection we introduce the decomposition

$$\vec{q} = \vec{u} + \vec{w}_{\Sigma\sigma}$$

with

$$\text{div}\,\vec{q} = 0.$$

Functions $\vec{u}$ are selected smooth, namely they involve no shock through Σ, while $w_{\Sigma\sigma}$ is constructed to simulate the shocks.

Now let us show we can construct $\vec{u}$ and $\vec{w}_{\Sigma\sigma}$, actually.

a) We denote by g^1 the function obtained from g by substituting t_∞ for 1 into (4) that is

$$g_1 = 0 \quad \text{on } \Gamma$$
$$g^1 = n_{e_1} \quad \text{on } \widetilde{\Gamma}.$$

Introducing a function $\eta \in C^\infty\left(\overline{\Omega}\right)$ such that:

$$\eta = 1 \text{ in a neighbourhood of } \widetilde{\Gamma}$$
$$\eta = 0 \text{ in a neighbourhood of } \Gamma_{\mathcal{P}}$$

setting:

$$u_1^1 = \partial_2(\eta x_2),\ u_2^1 = -\partial_1(\eta x_2)$$

one obtains a function $\vec{u}^1$ satisfying:

$$\begin{cases} \vec{u}^1 \in C^2\left(\overline{\Omega}\right) & \\ \operatorname{div}\vec{u}^1 = 0 & \text{in } \Omega \\ \vec{u}^1 \cdot \vec{n}_e = g^1 & \text{on } \partial\Omega\,. \end{cases} \tag{13.1}$$

The function:

$$\vec{u} = t_\infty \vec{u}^1 \tag{13.2}$$

is so that:

$$\begin{cases} \vec{u} \in C^2\left(\overline{\Omega}\right) & (13.3) \\ \operatorname{div}\vec{u} = 0 \quad \text{in } \Omega & (13.4) \\ \vec{u} \cdot \vec{n}_e = g \quad \text{on } \partial\Omega\,. & (13.5) \end{cases}$$

holds.

b) We set $I_0 = [\varphi_{B_0}, \varphi_{A_0}]$ (see Figure 3 and the end of Subsection 2.3).

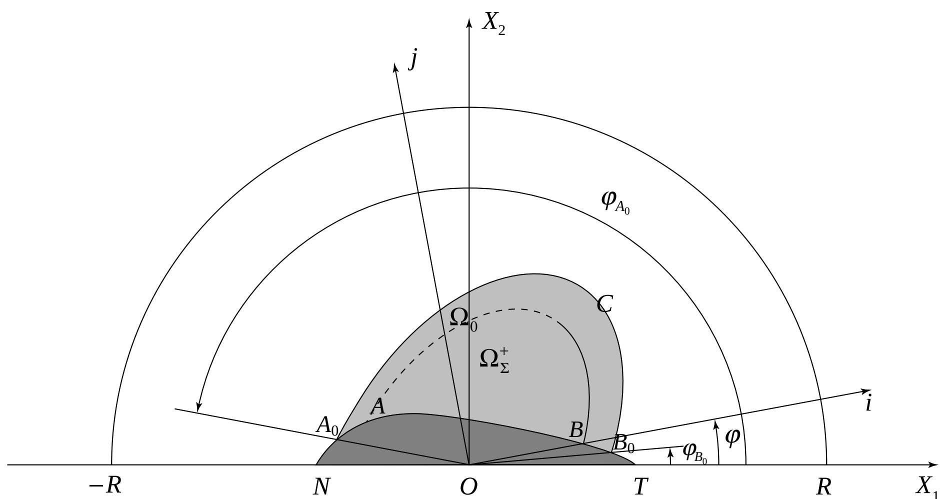

Figure 3.

Function $\vec{w}_{\sigma\Sigma}$ is constructed involving σ (see (9.5), (10)) and Σ by the following procedure:

i) Introduce a function $\sigma : \varphi \mapsto \sigma(\varphi)$.

We suppose

$$\sigma \in C^1(I_0). \tag{14.1}$$

Let us denote by φ_c the polar angle of C with respect to $(0; x_1, x_2)$. In order to perform the passage: $(0; x_1, x_2) \to (0'; x_1, x_2)$, the decomposition: $\overline{OC} = \overline{OO'} + \overline{O'C}$ can be used.

Further, we assume

$$\begin{cases} \sigma(\varphi) = 0\,, & \varphi \in (\varphi_c, \varphi_{A_0}) \qquad (14.2) \\ \sigma(\varphi) \geq \sigma_0(\varphi)\,, & \varphi \in (\varphi_{B_0}, \varphi_c) \qquad (14.3) \\ \sigma'(\varphi) \leq -\alpha\,, & \varphi \in (\varphi_{B_0}, \varphi_c) \qquad (14.4) \end{cases}$$

here the given function σ_0 belongs to $C^1(I_0)$ and satisfies:

$$\begin{cases} \sigma_0(\varphi) = 0\,, & \varphi \in (\varphi_c, \varphi_{A_0}) \\ \sigma_0(\varphi) \geq 0\,, & \varphi \in (\varphi_{B_0}, \varphi_c) \\ \sigma_0(\varphi_c) = 0, & \\ \sigma_0'(\varphi_c) = 0\,, & \end{cases}$$

ii) Consider the equation of $\Gamma_{\mathcal{P}}$ and Σ in polar coordinates with respect to $(0; x_1, x_2)$, that is

$$r = F_{\mathcal{P}}(\varphi),\ r = F_\Sigma(\varphi). \tag{15.1}$$

It is assumed

$$F_{\mathcal{P}} \in C^2(I_0),\ F_\Sigma \in C^2(I_0). \tag{15.2}$$

By means of these expressions, we define on $\overline{\Omega}_0$ (see the end of Subsection 2.3) the function

$$\psi = -(r - F_{\mathcal{P}})\sigma \left[1 - \left(\frac{r - F_{\mathcal{P}}}{F_\Sigma - F_{\mathcal{P}}}\right)\right]. \tag{15.3}$$

Let us note that in (15.3), function ψ is not the stream function of the flow.

iii) Calculate in Ω_Σ^+:

$$\partial_r \psi = -\sigma \left[1 - \left(\frac{r - F_{\mathcal{P}}}{F_\Sigma - F_{\mathcal{P}}}\right)\right] + (r - F_{\mathcal{P}})\sigma \frac{1}{(F_\Sigma - F_{\mathcal{P}})} \tag{16.1}$$

$$\begin{aligned} \partial_\varphi \psi = \sigma F_{\mathcal{P}}' \left[1 - \left(\frac{r - F_{\mathcal{P}}}{F_\Sigma - F_{\mathcal{P}}}\right)\right] - (r - F_{\mathcal{P}})\sigma' \left[1 - \left(\frac{r - F_{\mathcal{P}}}{F_\Sigma - F_{\mathcal{P}}}\right)\right] \\ - (r - F_{\mathcal{P}})\sigma \left[\frac{F_{\mathcal{P}}'}{(F_\Sigma - F_{\mathcal{P}})} + \frac{r - F_{\mathcal{P}}}{(F_\Sigma - F_{\mathcal{P}})^2}(F'''_\Sigma - F'''_{\mathcal{P}})\right]. \end{aligned} \tag{16.2}$$

On Σ, one has

$$\begin{cases} \partial_r \psi &= -\sigma \\ \partial_\varphi &= \sigma F_\Sigma'\,. \end{cases} \tag{16.3}$$

In relations (16.4) and (17) below, the coordinates are taken with respect to $(0; i, j)$ (see Figure 3).

$$\vec{\tau}_\Sigma = \frac{1}{\left[1 + \left(\frac{F'_\Sigma}{F_\Sigma}\right)^2\right]^{\frac{1}{2}}} \begin{pmatrix} \frac{F'_\Sigma}{F_\Sigma} \\ 1 \end{pmatrix} \tag{16.4}$$

iv) Set

$$\vec{w}_{\Sigma\sigma} = \begin{cases} 0 & \text{in } \Omega^-_\Sigma \\ \left(\frac{1}{r}\partial_\varphi\psi, -\partial_r\psi\right) & \text{in } \Omega^+_\Sigma\,. \end{cases} \tag{17}$$

From (17), (15.1), (16.3), (16.4) the relations

$$\begin{cases} \operatorname{div}\vec{w}_{\Sigma\sigma} = 0 & \text{in } \Omega & (18.1) \\ \vec{w}_{\Sigma\sigma}\cdot\vec{n}_e = 0 & \text{on } \partial\Omega & (18.2) \\ \vec{w}^+_{\Sigma\sigma}\cdot\vec{\tau}_\Sigma = -\left[1 + \left(\frac{F'_\Sigma}{F_\Sigma}\right)^2\right]^{\frac{1}{2}}\sigma & \text{on } \Sigma & (18.3) \end{cases}$$

hold.

Conditions (14) and (18.3) warrant (10) and (9.5), while (13.5) and (18.2) imply (9.6).

3. Outlines of the solution method.

a) The construction of Subsection 2.4 enables us to consider a vector field: $\vec{q} = \vec{u} + \vec{w}_{\Sigma\sigma}$ satisfying:

$$\begin{cases} \operatorname{div}\vec{q} = 0 & \text{in } \Omega\,, \\ [[\vec{q}\,]]\cdot\vec{\tau}_\Sigma \geq \sigma & \text{on } \Sigma_c\,, \\ \vec{q}\cdot\vec{n}_e = g & \text{on } \partial\Omega\,. \end{cases}$$

b) Equations of continuity (9.1) remains preserved. Actually we determine by solving Equations (8) for a given $\vec{q}$. The roots $K^-(\vec{q})$ and $K^+(\vec{q})$ are selected wether we are in Ω^-_Σ (Condition (9.3)) or in Ω^+_Σ (Condition (9.4)).

c) Introducing the t-uples:

$$z = (\ell, F_\Sigma, \varphi_c, u, \sigma)$$

condition (9.2) is approached to the best by minimizing the magnitude of rot $\vec{v}(z)$ in a suitable norm. Compactness arguments are applied, involving embedding theorems within Sobolev spaces.

4. Some basic notations. Concerning the introductory matter and properties of Sobolev spaces, used in the following, see for instance [14, 15].

Let us give some notations. The vectorial terms are no more quoted by arrows in order to simplify. Let p be an integer, the space $L^2(\mathcal{O})$ of square summable functions acting from an open set $\mathcal{O} \subset \mathbb{R}^2$ to $\mathbb{R}^p$ is equipped with the inner product:

$$(u, v)_{0,\mathcal{O}} = \int_{\mathcal{O}} u \cdot v dx.$$

The associated norm is denoted by

$$|u|_{0,\mathcal{O}} = \left(\int_{\mathcal{O}} |u|^2 dx \right)^{1/2}.$$

The absolute value of a real number or the Euclidean norm of a vector will be denoted by $|\ \cdot\ |$ independently.

Let m be an integer, the Sobolev space $H^m(\mathcal{O})$ is supplied with the usual inner product:

$$(u, v)_{m,\mathcal{O}} = \sum_{|\mu| \leq m} (\partial^\mu u, \partial^\mu v)_{0,\mathcal{O}},$$

here ∂^μ denotes the partial derivatives ∂_1, ∂_2 of order μ with respect to x_1, x_2. The corresponding norm reads:

$$|u|_{m,\mathcal{O}} = \left(\sum_{|\mu| \leq m} |\partial^\mu u|^2_{0,\mathcal{O}} \right)^{1/2}.$$

In the sequel, given strictly positive numbers, chosen large enough, will be denoted by N for some expressions, independently.

5. Minimization of the magnitude of the vorticity. We observe that expression $\operatorname{rot} v$ (introduced already in Subsection 3.c) is non-linear and not necessarily convex with respect to t-uple z. Owing to the minimization of the magnitude of $\operatorname{rot} v$, it is usually advisable to consider compactness arguments. The latter will imply, on the one hand, some tuple components are continuous or derivable, on the other hand, some subsets are suitably embedded inducing compactness properties. Some preliminary results concerning the bounds are required.

5.1. Preliminary results. It is assumed that F_ℓ, F, σ, $F_{\mathcal{P}}$, F_Σ (introduced in (12.2), (14.1), (15.1) respectively) satisfy:

$$F_\ell \in H^3(I), \tag{19.1}$$

$$F \in H^3(I): \ |F|_{3,I} \leq N, \tag{19.2}$$

$$\sigma \in H^2(I_0): \ |\sigma|_{2,I_0} \leq N, \tag{19.3}$$

$$|F_{\mathcal{P}}|_{3,I_0} \leq N, \tag{19.4}$$

$$|F_\Sigma|_{3,I_0} \leq N. \tag{19.5}$$

By means of embedding theorems, we deduce that: (19.1)-(19.5) implies (12.4), (14.1), (15.2) respectively.

From the expressions of ρ, v_c, $q(v)$ given in (1), (2) and (8) respectively, it follows that:

$$|q(v)| \leq t_c,$$

where we have set:

$$t_c = \rho(v_c^2)v_c. \tag{20.1}$$

Coming back to the definition of t_∞ ((4.2) below) one has in the same way:

$$t_\infty \leq t_c. \tag{20.2}$$

Proposition 1. *Parameters N and t_∞ can be chosen so that function u (introduced in (13.2)) satisfies:*

$$|u| \leq t_c \quad \textit{in } \Omega_\Sigma^- \,, \tag{20.3}$$

$$|u + w_{\Sigma\sigma}| \leq t_c \quad \textit{in } \Omega_\Sigma^+ \,,$$

independently of Σ.

Proof. From (13.2) one has:

$$|u| \leq t_\infty N_0 \quad \text{in } \Omega \tag{1i}$$

with $N_0 = \sup\{|u^1(x)|\,;\ x \in \overline{\Omega}\}$.

In order to estimate $|w_{\Sigma\sigma}|$ it sufficies from relation (17) to bound $|\nabla_\psi|$ in Ω_Σ^+, independently of Σ. Thus, we have:

$$|w_{\Sigma\sigma}| \leq |\nabla_\psi|. \tag{2i}$$

Using an embedding theorem (mentioned by E.T.) (for the latter, see for instance [14] or [15] and the associated references), Relation (19.4) (respectively (19.5)) implies:

$$|F_{\mathcal{P}}| \leq N\,,\ \text{(respectively } |F_\Sigma| \leq N). \tag{3i}$$

Since one has:

$$r \geq F_{\mathcal{P}} \quad \text{in } \Omega_\Sigma^+\,,$$

from (3i), it follows:

$$r - F_{\mathcal{P}} \leq N. \tag{4i}$$

Since on Ω_Σ^+, the inequalities:

$$r \geq F_{\mathcal{P}},\ r \leq F_\Sigma$$

hold, we obtain:

$$\left(\frac{r - F_{\mathcal{P}}}{F_\Sigma - F_{\mathcal{P}}}\right) \leq 1\,. \tag{5i}$$

Combining estimates (4i), (5i) and (16.1), (16.2), leads to:

$$|\nabla\psi| \leq C[(1 + |F'_{\mathcal{P}}| + |F'_\Sigma|)\sigma + N|\sigma'|]\,.$$

Applying E.T., Relations (19.3), (19.4), (19.5) imply:

$$|\nabla\psi| \leq C(N + N^2). \tag{6i}$$

Gathering estimates (1i), (2i), (6i), one has:

$$\begin{cases} |u| \leq t_\infty N_1\,, \\ \text{and} \\ |u + w_{\sigma\Sigma}| \leq C(N + N^2)\,. \end{cases}$$

This enables us to take t_∞ and N small enough so that (20.3) holds. □

From (13.2) one has:

$$|u|_{2,\Omega} \leq N_1 t_\infty\,, \tag{20.4}$$

with $N_1 = |u^1|_{2,\Omega}$.

It will be useful in the sequel to assume angle φ_c (see Subsection 2.4.b) is chosen such as:

$$\varphi_c \in [\varphi_1, \varphi_2], \tag{20.5}$$

where φ_1 and φ_2 are given constants within $]\,0, \pi\,[$.

For technical reasons some of the estimates and conditions, stated until now, are recorded (see: the constructions in Subsections 2.3 and 2.4, and the groups of relations (19), (20)). We introduce the t-uples $z = (\ell, F_\Sigma, \varphi_c, u, \sigma)$ with components satisfying:

$$\begin{cases} \ell \in \left[-\dfrac{c}{2} + \alpha, \dfrac{c}{2} - \alpha\right] & (21.1) \\ |F_\Sigma|_{3,I_0} \leq N & (21.2) \\ \varphi_c \in [\varphi_1, \varphi_2] \text{ where } \varphi_1,\ \varphi_2 \text{ are given constants} & (21.3) \\ |u|_{2,\Omega} \leq N_1 t_\infty & (21.4) \\ |\sigma|_{3,I_0} \leq N & (21.5) \end{cases}$$

so that the corresponding conditions:

$$\begin{cases} F(\theta)\left(\dfrac{c}{2} - \ell\right) - \alpha \\ \alpha \leq F(\theta), \quad \theta \in I \\ F_\ell\left(\dfrac{\pi}{2}\right) + \alpha \leq F\left(\dfrac{\pi}{2}\right) \\ F(\theta) \leq \widetilde{F}_\ell(\theta) - \alpha, \quad \theta \in I \\ \alpha \leq \dfrac{1}{F(\theta)} + \left(\dfrac{1}{F(\theta)}\right)'', \quad \theta \in I \\ \theta_B + \alpha \leq \theta_c \leq \dfrac{\pi}{2} - \alpha \\ F(\theta_B)F_\ell(\theta_B) + F'(\theta_B)F'_\ell(\theta_B) = 0 \end{cases} \tag{22.1}$$

$$\begin{cases} \operatorname{div} u = 0 & \text{in } \Omega \\ u \cdot n_e = g & \text{on } \partial\Omega \end{cases} \tag{22.2}$$

$$\begin{cases} \sigma(\varphi) = 0, & \varphi \in (\varphi_c, \varphi_{A_0}) \\ \sigma(\varphi) \geq \sigma_0(\varphi), & \varphi \in (\varphi_{B_0}, \varphi_c) \\ \sigma'(\varphi) \leq -\alpha, & \varphi \in (\varphi_{B_0}, \varphi_c) \end{cases} \tag{22.3}$$

$$\begin{cases} \operatorname{div} w_{\Sigma\sigma} = 0 & \text{in } \Omega \\ w_{\Sigma\sigma} \cdot n_e = 0 & \text{on } \partial\Omega \\ \overset{+}{w}_{\Sigma\sigma} \cdot \tau_\Sigma = -\left[1 + \left(\dfrac{F_{\Sigma'}}{F_\Sigma}\right)^2\right]^{\frac{1}{2}} \sigma & \text{on } \Sigma \end{cases} \tag{22.4}$$

hold.

We denote by G the set of the aforementioned t-uples. Let us point out that estimates (20.3), (20.4) imply G is non-empty.

5.2. Compactness arguments. It is advisable (see estimates (21)) to introduce the following:

- the space $E = \mathbb{R} \times H^{2-\varepsilon}(I_0) \times \mathbb{R} \times H^{2-\varepsilon}(\Omega) \times H^{3-\varepsilon}(I_0)$ equipped with the norm:

$$|z|_E = |\ell| + |F_\Sigma|_{H^{3-\varepsilon}(I_0)} + |\varphi_c| + |u|_{H^{3-\varepsilon}(\Omega)} + |\sigma|_{H^{3-\varepsilon}(I_0)}$$

  here $\varepsilon > 0$ is taken small enough.
- the subset E_s of t-uples $z = (\ell, F_\Sigma, \varphi_c, u, \sigma)$ satisfying (21).

Proposition 2. *E_s is a compact subset of E, that is: from any sequence (z_n) of elements belonging to E_s, one can extract a subsequence (denoted by z_n alike) converging in E_s, under norm $|\cdot|_E$.*

Proof. Let us consider a sequence $z_n = (\ell_n, F_{\Sigma_n}, \varphi_{c_n}, u_n, \sigma_n)$ in E_s. From (21.1), (21.3), by applying the Bolzano-Weierstrass theorem it is deduced:

$$\ell_n \xrightarrow{\mathbb{R}} \ell$$

$$\varphi_{c_n} \xrightarrow{\mathbb{R}} \varphi_c$$

with (ℓ, φ_c) satisfying (21.1), (21.3).

From (21.2), (21.4), (21.5) and applying a weak compactness result (for the latter see for instance, [14] or [15]), one obtains:

$$\begin{cases} F_{\Sigma_n} \xrightarrow{H^3(I_0)} F & \text{weakly} \\ u_n \xrightarrow{H^2(\Omega)} u & \text{weakly} \\ \sigma_n \xrightarrow{H^3(I_0)} \sigma & \text{weakly} \end{cases} \tag{23}$$

with (F, u, σ) satisfying (21.2), (21.4), (21.5). Let $\varepsilon > 0$, applying an embedding theorem (see [15] for instance), (23) imply

$$\begin{cases} F_{\Sigma_n} \xrightarrow{H^{3-\varepsilon}(I_0)} F \\ u_n \xrightarrow{H^{2-\varepsilon}(I_0)} u \\ \sigma_n \xrightarrow{H^{3-\varepsilon}(I_0)} \sigma \, . \end{cases} \qquad \square \tag{24}$$

Let us recall G is the set of the t-uples belonging to E_s so that (22) holds.

Proposition 3. *G is a compact subset of E.*

Proof. Since the t-uples of G satisfy (21), (22), it sufficies to show this set is a closed subset of E_s under $|\cdot|_E$. Let us consider a subsequence

$$z_n = (\ell_n, F_{\Sigma_n}, \varphi_{c_n}, u_n, \sigma_n)$$

in G.

One has $z_n \in E_s$ obviously. From Proposition 2, it follows

$$z_n \xrightarrow{|\cdot|_E} z,$$

that is convergence (24) holds. Further, one obtains

$$F_{\Sigma_n} \xrightarrow{H^2(I_0)} F \tag{25.1}$$

$$\sigma_n \xrightarrow{H^2(I_0)} \sigma \tag{25.2}$$

From the latter, applying (E.T.), we have

$$F_{\Sigma_n} \xrightarrow{C^0(I_0)} F \tag{25.3}$$

$$\sigma_n \xrightarrow{C^0(I_0)} \sigma \tag{25.4}$$

here $C^0(I_0)$ is the space of continuous functions acting from I_0 into $\mathbb{R}$. From (25.3), fixing θ, term $r_n = F_{\Sigma_n}(\theta)$ converges towards $r = F(\theta)$, as n increases to infinity. The last equation defines a curve Σ so that: $F = F_\Sigma$, with:

$$F_{\Sigma_n}(\theta) \longrightarrow F_\Sigma(\theta)\,, \quad \forall \theta \in I_0\,. \tag{25.5}$$

In the same way, (25.4) implies

$$\sigma_n(\theta) \longrightarrow \sigma(\theta)\,, \quad \forall \theta \in I_0\,. \tag{25.6}$$

From convergences (25), it follows F_Σ and σ satisfy (22.1)–(22.3). Coming back to (17), (16.1), (16.2), the two first equalities in (22.4) are deduced.

Let us show the last condition in (22.4) is preserved at the limit.

Taking $\xi \in \mathcal{D}(\Omega)$ with support $(\xi) \cap \Sigma_n \neq \emptyset$, from (18.3) one has:

$$\int_{\Sigma_n} (w_{\Sigma_n \sigma_n} \cdot \tau_{\Sigma_n}) \xi d\Gamma = - \int_{\varphi_{B_n}}^{\varphi_{A_n}} \left[1 + \left(\frac{F'_{\Sigma_n}}{F_{\Sigma_n}} \right)^2 (\varphi) \right]^{\frac{1}{2}} \sigma(\varphi) \xi d\varphi.$$

That way, convergences (25) imply the last equality remains preserved at the limit. □

5.3. Determination of the velocity from the mass flux. For a given q, the solutions to (8) are obtained from (20.3) by setting:

$$\begin{cases} v_\Sigma^\pm = h \pm \left(|q^\pm|^2\right) q^\pm \\ \quad \text{with} \\ \qquad |v_\Sigma^-| \in (0, v_c) \\ \qquad |v_\Sigma^+| \in (v_c.1) \\ \qquad q^- = u & \text{in } \Omega_\Sigma^- \\ \qquad q^+ = u + w_{\Sigma\sigma} & \text{in } \Omega_\Sigma^+ . \end{cases} \tag{26}$$

Let us introduce the continuous functions:

$$K^\pm(\lambda) = h^\pm(|\lambda|^2)\lambda \,, \quad \lambda \in \mathbb{R}^2.$$

In total the expression of the velocity field reads:

$$v(z) = K^-(u)\chi_\Sigma^- + K^+(u + w_{\Sigma\sigma})\chi_\Sigma^+ \tag{27}$$

where $\chi_\Sigma^\pm$ is the indicator of $\Omega_\Sigma^\pm$.

5.4. The functional to minimize. Let us consider the functional

$$\begin{aligned} G &\xrightarrow{J} \mathbb{R}^+ \\ z &\longmapsto J(z) = |\operatorname{rot} v(z)|_{-1} \end{aligned} \tag{28}$$

here the dual space $H^{-1}(\Omega)$ of H_0^1 is supplied with the norm $|\cdot|_{-1}$ (for these notions concerning the dual spaces, see for instance [14] or [15]). We want to establish the existence result hereunder.

Theorem. *There exists $\underline{z} \in G$ satisfying:*

$$J(\underline{z}) = \inf\{J(z) \,, \quad z \in G\}. \tag{29}$$

Proof. From (27), (28), functional J results in composing the mappings:

$$\begin{aligned} G &\xrightarrow{T_1} L^2(\Omega) \\ z &\longmapsto v(z) \end{aligned}$$

$$L^2(\Omega) \xrightarrow{T_2} H^{-1}(\Omega)$$
$$v \longmapsto \mathrm{rot}\, v$$
$$H^{-1}(\Omega) \xrightarrow{T_3} \mathbb{R}^+$$
$$w \longmapsto |\mathrm{rot}\, w|_{-1}.$$

Since a norm is continuous, so is function T_3. From the definition of space $H^{-1}(\Omega)$ (see the previous reference concerning dual spaces), T_2 is continuous from $(L^2(\Omega), |\cdot|_0)$ into $(H^{-1}(\Omega), |\cdot|_{-1})$ (recall $|\cdot|_0$ has already been introduced in Section 4).

Let us show that T_1 is well defined. From the expression of $K^{\pm}$ in (27) one has alike:

$$v(z) = v_\Sigma^- \chi_\Sigma^- + v_\Sigma^+ \chi_\Sigma^+.$$

From (26) it follows:

$$|v(z)| \leq 2. \tag{1i}$$

Convergence (25.3) implies:

$$\chi_{\Sigma_n}^{\pm} \longrightarrow \chi_\Sigma^{\pm}, \quad \forall x \in \Omega. \tag{2i}$$

Let us introduce the function, acting from Ω_Σ^+ into Ω_0, defined by setting:

$$\Pi_{w_{\Sigma\sigma}} = (\partial_2\psi, -\partial_1\psi),$$

here function ψ has been considered in (15.3). Substituting (Σ, σ) for (Σ_n, σ_n), $\Pi_{w_{\Sigma_n\sigma_n}}$ is defined in the same way.

From (15.1), (15.2) and (25.1), (25.2) one has:

$$\Pi_{w_{\Sigma_n\sigma_n}}(x) \longrightarrow \Pi_{w_{\Sigma\sigma}}(x) \qquad (\forall x \in \Omega_0). \tag{3i}$$

Let us use the identity

$$K^+(u_n + w_{\Sigma_n\sigma_n})\chi_{\Sigma_n}^+ = K^+(u_n + \Pi_{w_{\Sigma_n\sigma_n}})\chi_{\Sigma_n}^+ \quad \text{in } \Omega_{\Sigma_n}^+$$

since K^+ is continuous, convergences (25.1), (25.2), (2i), (3i) imply

$$K^+(u_n + w_{\Sigma_n\sigma_n})\chi_{\Sigma_n}^+ \longrightarrow K^+(u + w_{\Sigma\sigma})\chi_\Sigma^+. \tag{4i}$$

In the same way, one obtains

$$K^-(u_n)\chi_{\Sigma_n}^- \longrightarrow K^-(u)\chi_\Sigma^-. \tag{5i}$$

Gathering (4i) and (5i) leads to:

$$v(z_n) \longrightarrow v(z) \qquad \forall x \in \Omega. \tag{6i}$$

From (1i) and (6i), the Lebesgue's dominated convergence theorem shows that function T_1 is continuous. $\square$

6. Concluding remarks. The flow is symmetrical (see Section 1). This does not mean a loss of generality, owing to the main difficulty involved by the mixed type (elliptic hyperbolic) of the governing equations. However let us mention that non-symmetrical flows could be dealt with, using the procedure of [10, 11], in order to take into account the Kutta-Joukowski condition.

The constructed model constrains the shocks to occur. The resulting equations are rather difficult to solve so far that no performing functional spaces lead to a complete solution (to the best of our knowledge). Under these unfavorable conditions, we have reduced the solution to that of the minimization of a suitable functional. However a basic difficulty remains: the functional and the corresponding domain are not convex generally. This has been removed by applying compactness arguments in the frame of Sobolev spaces, while the main features associated with the shocks remain preserved, (see the choice of subset G, (22) before).

Finally it is shown (see the theorem in Subsection 5.4) the functional reaches its minimum (or its minima).

Concerning the minimization of non-convex functionals acting on Hilbert spaces, let us mention the results of [16].

To the best of our knowledge, the modelization leading to (9) (Section 2) and the corresponding solution procedure (Section 3) provide a new way to study transonic flows from the functional point of view.

Résumé substantiel en français. Cet article concerne les écoulements transsoniques plans, stationnaires d'un gaz idéal non dissipatif qui s'établissent autour d'un profil placé en atmosphère infinie. L'écoulement à l'infini amont est donné uniforme et les forces volumiques sont supposées négligeables. En général, les écoulements transsoniques sont le siège d'ondes de choc faibles. Plus précisément, si ε désigne la force du choc, le saut d'entropie est de l'ordre de ε^3 ; il en va de même pour le tourbillon (voir [3]). Il est donc avisé de traiter des écoulements transsoniques irrotationnels et isentropiques. Dans ces conditions, les équations gouvernant le champ des vitesses se réduisent à un système d'équations aux dérivées partielles quasi linéaire du premier ordre de type elliptique-hyperbolique. Du point de vue de l'analyse fonctionnelle, la question de l'existence d'une solution pour un tel système se pose toujours. Dans la littérature sur le sujet l'existence d'une solution est montrée sous des conjectures significatives (voir [7] par exemple). À notre connaissance, si l'on ne tient pas compte de ces dernières, seuls des résultats partiels sont établis dans les cadres fonctionnels introduits. Habituellement les méthodes associées conduisent pour l'essentiel à approcher l'équation de la conservation de la masse alors que le caractère irrotationnel est conservé: le tourbillon s'annule, voir par exemple les références [8] à [13] et le sous-paragraphe 1.2.

Dans cet article, nous proposons un modèle avec une approche fonctionnelle différente. L'équation de la conservation de la masse est prise exacte alors que le tourbillon est minimisé (voir le sous-paragraphe 2.1). De plus, afin de contraindre les ondes de chocs à apparaître, une inégalité de saut est introduite (voir (9.5)) dès que les équations gouvernant l'écoulement sont posées. Décrivons les avantages principaux de la méthode proposée. L'équation de la conservation de la masse est résolue exactement. La minimisation du tourbillon peut entraîner qu'il est petit, ce qui est en accord avec les

écoulements ayant des ondes de choc faibles et sont de ce fait faiblement rotationnels en aval de ces ondes. Observons que la minimisation de la divergence du flux de masse impliquerait qu'elle est petite sans garantir exactement la conservation de la masse.

La méthode proposée ici fournit outre les champs de vitesse, de pression, ... , les lignes soniques et les ondes de choc.

Le tourbillon est minimisé par l'introduction d'une fonctionnelle adaptée (voir (28)), cependant il reste une difficulté fondamentale : la fonctionnelle et le domaine associés ne sont pas convexes en général. Ceci est pris en compte en appliquant des arguments de compacité dans le cadre des espaces de Sobolev alors que les principales propriétés des chocs sont maintenues (voir le choix de G après (22)). Finalement on montre (voir le théorème du sous-paragraphe 5.4) que la fonctionnelle atteint sa borne inférieure.

References

1. C. Jacob, *Introduction mathématique à la mécanique des fluides* (Éditions de l'Académie de la République Populaire Roumaine, Bucharest, ed.), Gauthier-Villars, Paris, 1959.
2. P. Germain, *Mécanique*, T. 1 and T. 2, Ellipse, Paris, 1986.
3. J. D. Cole and L. P. Cook, *Transonic aerodynamics*, North-Holland, 1986.
4. M. Pogu and G. Tournemine, *Modélisation et résolution d'équations de la mécanique des milieux continus*, Ellipse, Paris, 1992.
5. J. L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires* (Dunod, ed.), Gauthier-Villars, Paris, 1969.
6. E. Zeidler, *Nonlinear functional analysis and its applications, II/B. Nonlinear monotone operators*, Springer-Verlag, New York-Berlin, 1990.
7. C. S. Morawetz, *Transonic flow and compensated compactness*, Maths Sciences Research, vol. 7, Springer-Verlag, 1987.
8. H. P. Gittel, *Studies on transonic flows problems by nonlinear variational inequations*, Z. Anal. Anwendungen Band **6** (1987).
9. J. Necas, *Écoulements de fluides: compacité par entropie*, RMA: Research Notes in Applied Mathematics, vol. 10, Masson, Paris, 1989.
10. M. Pogu and G. Tournemine, *Une méthode fonctionnelle de résolution approchée d'un problème transsonique*, C.R.A.S. **312, II** (1991).
11. M. Pogu and G. Tournemine, *Contribution à la résolution d'écoulements transsoniques*, Rev. Roum. Sci. Tech. Méc. Appl. **36** (1991).
12. M. Pogu and G. Tournemine, *A functional approach to the solution of the Karman-Guderley equation*, Bull. Pol. Acad. Sci. Tech. Sci. **40** (1992).
13. H. Berger, G. Warnecke and W. Wendland, *Finite elements for transonic flows*, Num. Meth. for P.D.E. (1990).
14. R. Dautray and J. L. Lions, *Analyse mathématique et calcul numérique pour les sciences et les techniques*, vol. 1, Masson, Paris, 1987.
15. V. G. Mazja, *Sobolev spaces*, Springer Series in Soviet Mathematics, Springer-Verlag, Berlin-New York, 1985.
16. S. H. Chew and Q. Zheng, *Integral Global Optimization*, Lecture Notes in Economics and Mathematical Systems, vol. 298, Springer-Verlag, Berlin, 1988.

PROFESSEUR M. POGU
DÉPARTEMENT D'INFORMATIQUE ET MATHÉMATIQUE
ÉCOLE CENTRALE DE NANTES
1, RUE DE LA NOË
44072 NANTES, FRANCE

PROFESSEUR G. TOURNEMINE
UNIVERSITÉ DE RENNES I
INSTITUT MATHÉMATIQUE DE RENNES
LABORATOIRE D'ANALYSE NUMÉRIQUE ET MÉCANIQUE
CAMPUS DE BEAULIEU
35042 RENNES, CÉDEX, FRANCE