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ON THE NUMBER OF CONVEX POLYOMINOES

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RÉSUMÉ. Lin et Chang ont donné la série génératrice du nombre de polyominos convexes qui s'inscrivent dans un rectangle minimal de format $m+1$ par $n+1$. Nous montrons que ce résultat entraîne que le nombre de tels polyominos est égal à

$$\frac{m+n+mn}{m+n}\binom{2m+2n}{2m} - 2(m+n)\binom{m+n-1}{m}\binom{m+n-1}{n}.$$

ABSTRACT. Lin and Chang gave a generating function for the number of convex polyominoes with an $m+1$ by $n+1$ minimal bounding rectangle. We show that their result implies that the number of such polyominoes is

$$\frac{m+n+mn}{m+n}\binom{2m+2n}{2m} - 2(m+n)\binom{m+n-1}{m}\binom{m+n-1}{n}.$$

A *polyomino* is a connected union of squares in the plane whose vertices are lattice points. A polyomino is called *convex* if its intersection with any horizontal or vertical line is either empty or a line segment. (Note that a convex polyomino is generally not a convex polygon in the usual sense.) Any convex polyomino has a minimal bounding rectangle whose perimeter is the same as that of the polyomino. Delest and Viennot [2] found a generating function for counting convex polyominoes by perimeter and showed that the number of convex polyominoes with perimeter $2n+8$, for $n \geq 0$, is $(2n+11)4^n - 4(2n+1)\binom{2n}{n}$. Another proof of Delest and Viennot's formula was given by Kim [4].

Delest and Viennot's generating function was independently discovered empirically by Guttmann and Enting [3], and verified by Lin and Chang [5], who showed more generally that the number of convex polyominoes with an $(m+1)\times(n+1)$ minimal bounding rectangle is the coefficient of $x^{2m}y^{2n}$ in

$$P(x,y) = A(x,y) - 4x^2y^2\Delta(x,y)^{-3/2},$$

where

$$A(x,y) = \big(1 - 3x^2 - 3y^2 + 3x^4 + 3y^4 + 5x^2y^2 - x^6 - y^6 \\ -x^4y^2 - y^4x^2 - x^2y^2(x^2-y^2)^2\big)/\Delta(x,y)^2$$

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and

$$\Delta(x,y) = 1 - 2x^2 - 2y^2 + (x^2 - y^2)^2$$
$$= (1+x+y)(1+x-y)(1-x+y)(1-x-y).$$

Another proof of Lin and Chang's generating function was given by Bousquet-Mélou and Guttman [1].

We show here that the coefficient of $x^{2m}y^{2n}$ in $P(x,y)$ is given by the simple explicit formula

$$\frac{m+n+mn}{m+n}\binom{2m+2n}{2m} - 2(m+n)\binom{m+n-1}{m}\binom{m+n-1}{n}, \tag{1}$$

which is easily seen to give a refinement of Delest and Viennot's formula.

We will need the case $\alpha = 3/2$ of the formula

$$\frac{1}{(1-2x-2y+(x-y)^2)^\alpha} = \sum_{i,j\geq 0} \frac{(\alpha+1/2)_{i+j}(2\alpha)_{i+j}}{i!\,j!\,(\alpha+1/2)_i(\alpha+1/2)_j} x^i y^j, \tag{2}$$

where $(u)_n = u(u+1)\cdots(u+n-1)$. This formula is easily proved by expanding

$$\frac{1}{(1-2x-2y+(x-y)^2)^\alpha} = (1-x-y)^{-2\alpha}\left(1 - \frac{4xy}{(1-x-y)^2}\right)^{-\alpha}$$
$$= \sum_{k=0}^{\infty}\binom{\alpha+k-1}{k}\frac{(4xy)^k}{(1-x-y)^{2k+2\alpha}}$$

by the binomial theorem, extracting the coefficient of $x^i y^j$, and evaluating the resulting sum by Vandermonde's theorem. As pointed out by Strehl [7, p. 180], (2) is a consequence of classical formulas for Gegenbauer polynomials. (Replace x with $(x+y)/(x-y)$ and t with $x-y$ in equation (1), p. 276, and equation (17), p. 279 of Rainville [6].)

It follows immediately from (2) that

$$\Delta(x,y)^{-3/2} = \sum_{m,n\geq 0}\frac{m+n+2}{2}\binom{m+n+1}{m}\binom{m+n+1}{n}x^{2m}y^{2n}. \tag{3}$$

We could use the case $\alpha = 2$ of (2) to find the coefficients of $A(x,y)$, but a different approach, in which we derive the generating function from the explicit formula, is much easier.

We start with

$$\frac{1}{1-x-y} = \sum_{i,j\geq 0}\binom{i+j}{i}x^i y^j. \tag{4}$$

Differentiating (4) with respect to x and multiplying by xy we obtain

$$\frac{xy}{(1-x-y)^2} = \sum_{i,j\geq 1}\frac{ij}{i+j}\binom{i+j}{i}x^i y^j. \tag{5}$$

Now let

$$f(x) = \frac{1}{1-x-y} + \frac{1}{2}\frac{xy}{(1-x-y)^2}.$$

Then from (4) and (5) follows

$$f(x,y) = \sum_{i,j\ge 0} \frac{i+j+ij/2}{i+j}\binom{i+j}{i} x^i y^j,$$

where the summand is taken to be 1 for $i = j = 0$. We can extract the terms in f with only even powers of x and y by bisecting twice:

$$\frac{1}{4}\big(f(x,y) + f(-x,y) + f(x,-y) + f(-x,-y)\big) \\ = \sum_{m,n\ge 0} \frac{m+n+mn}{m+n}\binom{2m+2n}{2m} x^{2m}y^{2n}. \qquad (6)$$

It is straightforward to verify that the left side of (6) is equal to $A(x,y)$. Then formula (1) follows from (3) and (6).

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Résumé substantiel en français. Un *polyomino* est une réunion connexe de carrés dans le plan dont les sommets sont des points du réseau $\mathbb{Z}\times\mathbb{Z}$. Un polyomino est dit *convexe* si son intersection avec toute droite horizontale ou verticale est soit vide, soit un segment de droite. Tout polyomino convexe s'inscrit dans un rectangle minimal dont le périmètre est celui du polyomino. Delest et Viennot [2] ont trouvé une série génératrice pour les polyominos convexes selon le périmètre et ont montré que le nombre de polyominos convexes de périmètre $2n+8$ est $(2n+11)4^n - 4(2n+1)\binom{2n}{n}$, pour $n \ge 0$.

Lin et Chang [5] ont montré plus généralement que le nombre de polyominos convexes dont le rectangle minimal est de format $(m+1)\times(n+1)$ est égal au coefficient de $x^{2m}y^{2n}$ dans

$$P(x,y) = A(x,y) - 4x^2y^2\Delta(x,y)^{-3/2},$$

où

$$A(x,y) = \big(1 - 3x^2 - 3y^2 + 3x^4 + 3y^4 + 5x^2y^2 - x^6 - y^6 \\ - x^4y^2 - y^4x^2 - x^2y^2(x^2-y^2)^2\big)/\Delta(x,y)^2$$

et

$$\Delta(x,y) = 1 - 2x^2 - 2y^2 + (x^2-y^2)^2 \\ = (1+x+y)(1+x-y)(1-x+y)(1-x-y).$$

Nous montrons ici, à partir de la fonction génératrice de Lin et Chang, que ce coefficient est donné par la formule explicite

$$\frac{m+n+mn}{m+n}\binom{2m+2n}{2m} - 2(m+n)\binom{m+n-1}{m}\binom{m+n-1}{n}.$$

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