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GENERATING FUNCTIONS FOR IRREDUCIBLE CHARACTERS OF S_n INDEXED WITH MULTIPLE HOOKS

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RÉSUMÉ. Nous présentons de nouvelles expressions pour des fonctions génératrices de caractères irréductibles du groupe symétrique S_n. Ces fonctions génératrices sont de la forme $\sum_\lambda^* \chi_\mu^\lambda q^\lambda$ où la somme est prise sur les partages λ de formes restreintes tels des équerres, des double-équerres…, etc. Pour faire les preuves de nos résultats, nous utilisons la théorie des λ-anneaux pour les fonctions symétriques.

ABSTRACT. We present new expressions for generating functions of irreducible characters of the symmetric group S_n. These generating functions are of the form $\sum_\lambda^* \chi_\mu^\lambda q^\lambda$ where the sum is over partitions λ of restricted shapes such as hooks and double hooks. We use the λ-ring theory for symmetric functions to demonstrate our statements.

1. Introduction. For two partitions $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k)$ and $\mu = (\mu_1 \geq \mu_2 \geq \cdots \geq \mu_r)$ of the integer n, denote by χ_μ^λ the irreducible character of the symmetric group S_n indexed by λ and evaluated on the conjugacy class C_μ. The generating function

$$(1) \qquad \sum_{k=0}^{n-1} \chi_\mu^{1^k n-k} q^k = \frac{\prod_i (1-(-q)^i)^{\beta_i}}{(1+q)}$$

where $\mu = 1^{\beta_1} 2^{\beta_2} \ldots n^{\beta_n}$ first appeared in Littlewood [5] and was used by several authors (see for instance Adrianov [1], Jackson [3], Stanley [7] and Zagier [8]) as a computational tool in representation theory of S_n. The purpose of this note is first, to present a proof of Equation (1) using λ-ring methods and then to use these methods to develop generating functions for irreducible characters indexed by double hooks diagrams $\lambda = 1^{k_1} 2^{k_2} m_1 m_2$ consisting of one hook inside another (see Figure 1). In fact, the method that we are using to develop our generating functions apply to any multiple hook diagrams (see Definition 4) but the expressions become more involved and are not developed in this paper. Our notations for symmetric functions are taken from Macdonald ([5]) and we will attempt to produce a self explanatory exposition. Nevertheless, the reader who wants more details on the theory of λ-rings is referred to the book of Knutson ([4]).

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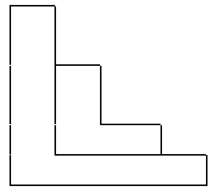

Figure 1
A double hook diagram

2. λ-ring Methods. Let $\Lambda^n(\mathbf{x})$ be the free Z-module of homogeneous symmetric functions of degree n on the set $\mathbf{x} = \{x_1, x_2, \dots\}$ of independent commuting variables so that $\Lambda = \oplus_{n\geq 0}\Lambda^n$ is the graded ring of symmetric functions. In λ-ring theory, Λ is the free λ-ring on one generator. We will be using two linear bases of Λ. The power sum symmetric functions $p_\lambda(\mathbf{x}) = p_{\lambda_1}(\mathbf{x})p_{\lambda_2}(\mathbf{x})\dots$ indexed with partitions $\lambda = (\lambda_1, \lambda_2, \dots)$ where $p_r(\mathbf{x}) = \sum x_i^r$. The Schur symmetric functions $s_\lambda(\mathbf{x})$ can be defined as followed. For a partition $\lambda \vdash n$, let ST_λ be the set of semistandard tableaux of shape λ which are obtained by first declaring $x_1 < x_2 < \cdots$ and then filling the diagram λ with elements from $\mathbf{x}$ by allowing repetitions so that the x_i are nondecreasing in rows and strictly increasing in columns. The *weight* $w(T)$ of a tableau T is the monomial obtained as the product of the x_i in T. Then the Schur function indexed by λ is defined by $s_\lambda(\mathbf{x}) = \sum_{T\in ST_\lambda} w(T)$. The Schur functions are related to the power sum symmetric functions by the character table of S_n:

$$s_\lambda(\mathbf{x}) = \sum_{\mu\vdash n} \frac{\chi_\mu^\lambda}{z_\mu} p_\mu(\mathbf{x}) \tag{2}$$

where $z_\mu = n!/(1^{\beta_1}\beta_1!\cdots n^{\beta_n}\beta_n!)$. A skew diagram λ/μ is obtained by removing the diagram μ from the diagram λ as illustrated in Figure 2 where $\lambda = (5,4,4,1), \mu = (4,3,2)$ and where the nodes containing $\sim$ constitute the diagram λ/μ.

Skew Schur functions $s_{\lambda/\mu}(\mathbf{x})$ are symmetric functions of degree $|\lambda| - |\mu|$ indexed with skew diagrams λ/μ and defined by $s_{\lambda/\mu}(\mathbf{x}) = \sum_{T\in ST_{\lambda/\mu}} w(T)$ where $ST_{\lambda/\mu}$ is the set of semistandard tableaux of skew shape λ/μ.

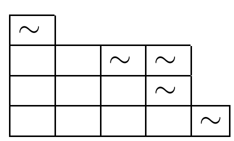

Figure 2
The skew diagram (5,4,4,1)/(4,3,2)

Definition 1. Let $A = \{a_1, \dots, a_n\}$ be an alphabet on the letters $a_1, \dots, a_n$. Then we define

$$\Omega(A) = \prod_{a\in A} \frac{1}{(1-a)}. \tag{3}$$

A fundamental idea in λ-ring theory is to consider polynomials with integer coefficients as multisets of their monomial components. Let

$$p(\mathbf{x}) = \sum_{(q_1,\dots,q_k)\in N^k} c_{(q_1,\dots,q_k)} x_1^{q_1}\cdots x_k^{q_k} = \sum_{q=(q_1,\dots,q_k)} c_q \mathbf{x}^q$$

be a polynomial in the variables $x_i \in \mathbf{x}$ with positive integer coefficients c_q. If we associate to each polynomial $p(\mathbf{x})$ a multiset P containing each monomial $\mathbf{x}^q$ of $p(\mathbf{x})$ c_q times, we may apply the operator Ω on the multiset P (and on $p(\mathbf{x})$) as follows:

$$\Omega(P) = \prod_{\mathbf{x}^q\in P} \frac{1}{(1-\mathbf{x}^q)^{c_q}}\,.$$

For example, if $p(x_1, x_2) = 3x_1^2 + x_1x_2 + 2x_2$, then its corresponding multiset is $P = \{x_1^2, x_1^2, x_1^2, x_1x_2, x_2, x_2\}$ and $\Omega(P) = 1/((1-x_1^2)^3(1-x_1x_2)(1-x_2)^2)$. Our next step is to extend the operator Ω to polynomials with integer coefficients c_q possibly negatives. For that purpose, we need to recall the following facts and definitions.

Definition 2. Let $p(\mathbf{x})$ and $q(\mathbf{y})$ be two polynomials on the sets of variables $\mathbf{x} = \{x_1, x_2, \dots\}$; $\mathbf{y} = \{y_1, y_2, \dots\}$, P and Q their corresponding multisets and let t be any independent variable. Then $P + Q$, PQ and tP are the multisets associated with the polynomials $p(\mathbf{x}) + q(\mathbf{y})$, $p(\mathbf{x})q(\mathbf{y})$ and $tp(\mathbf{x})$ respectively.

If we recall the identity

$$\frac{1}{(1-a)} = \exp\left(\sum_{k\geq 1}\frac{a^k}{k}\right), \tag{4}$$

an immediate consequence of (4) is that (3) can be rewritten as

$$\Omega(A) = \prod_{a\in A}\exp\left(\sum_{k\geq 1}\frac{a^k}{k}\right) = \exp\left(\sum_{a\in A}\sum_{k\geq 1}\frac{a^k}{k}\right) = \exp\left(\sum_{k\geq 1}\frac{\psi_k(A)}{k}\right) \tag{5}$$

where $\psi_k(A) = \sum_{a\in A} x_i^k$, $k = 0, 1, 2, \dots$ is the usual power sum symmetric function in the extended sense that it applies to the elements of any multiset A. In λ-ring theory, the expressions $\psi_k(A)$ are called Adams operations and they are linear, multiplicative and additive functions of degree k in the following sense (see [4]):

Proposition 2.1. *Let A, B be any two multisets and t any letter, then*

$$\begin{aligned}
\psi_k(AB) &= \psi_k(A)\psi_k(B), && \textit{(multiplicativity)}\\
\psi_k(tA) &= t^k\psi_k(A), && \textit{(linearity)}\\
\psi_k(A+B) &= \psi_k(A)+\psi_k(B). && \textit{(additivity)}
\end{aligned}$$

Proposition 2.2. *For any polynomials $p(\mathbf{x})$ and $q(\mathbf{y})$ with positive integer coefficients on two distinct sets of variables $\mathbf{x} = \{x_1, x_2, \dots\}$; $\mathbf{y} = \{y_1, y_2, \dots\}$ and associated multisets P and Q, we have*

$$\Omega(P+Q) = \Omega(P)\Omega(Q).$$

Proof. We have

$$\Omega(P+Q) = \exp \sum_{k\geq 1} \frac{\psi_k}{k}(P+Q) = \exp \sum_{k\geq 1} \frac{\psi_k}{k}(P) + \frac{\psi_k}{k}(Q)$$
$$= \Omega(P)\Omega(Q). \qquad \square$$

We are now ready to define the operator Ω on any polynomial $p(\mathbf{x}) \in Z[\mathbf{x}]$ by partitioning its corresponding multiset P into a positive set P^+ and a negative set P^-: $P = P^+ \cup P^-$. The sets P^+ and P^- contain respectively the monomials with positive and negative coefficients of $p(\mathbf{x})$ and we write $p(\mathbf{x}) = p^+(\mathbf{x}) - p^-(\mathbf{x})$.

Definition 3. Let $p(\mathbf{x}) = p^+(\mathbf{x}) - p^-(\mathbf{x}) \in Z[\mathbf{x}]$ be decomposed as above and let $P = P^+ \cup P^-$ be the corresponding multisets. Then

$$\Omega(P) = \Omega(P^+ - P^-) = \prod_{m_i \in P^+} \frac{1}{(1-m_i)} \prod_{m_j \in P^-} (1-m_j).$$

The next expression that we need involves Schur functions $s_\lambda(\mathbf{x})$ and is called *Cauchy's formula* (see [4] or [6] for a proof).

Proposition 2.3. *For the polynomials $p(\mathbf{x}) = x_1 + \cdots + x_n$ and $q(\mathbf{y}) = y_1 + \cdots + y_n$ and their corresponding multisets $P = \{x_1, x_2, \dots\} = \mathbf{x}$, $Q = \{y_1, y_2, \dots\} = \mathbf{y}$, we have*

$$\Omega(PQ) = \prod_{i,j=1}^{n} \frac{1}{1 - x_i y_j} = \sum_{\lambda \vdash n} s_\lambda(P) s_\lambda(Q)\,.$$

The involution ω of Λ satisfies $\omega(s_\lambda(\mathbf{x})) = s_{\lambda^\sim}(\mathbf{x})$, where $\lambda^\sim$ is the *conjugate partition* of λ (see [6]). It acts on polynomials $p(\mathbf{x}) \in Z[\mathbf{x}]$ as follow:

$$\omega(p(\mathbf{x})) = (-1)^{\deg(p(\mathbf{x}))} p(-\mathbf{x}) \tag{6}$$

which implies that

$$s_{\lambda/\mu}(-\mathbf{x}) = (-1)^{|\lambda/\mu|} \omega(s_{\lambda/\mu}(\mathbf{x})) = (-1)^{|\lambda/\mu|} s_{\lambda^\sim/\mu^\sim}(\mathbf{x}) \tag{7}$$

In the next proposition, we develop an expression for $\Omega(tP)$ in terms of power sums.

Proposition 2.4. *Let $p(\mathbf{x}) \in Z[\mathbf{x}]$ and let P be its associated multiset, then*

$$\Omega(tP) = \sum_{n\geq 0} \frac{t^n}{n!} \sum_{\sigma \in S_n} \psi_{\lambda(\sigma)}(P)$$

where $\lambda(\sigma) = 1^{\alpha_1} \cdots n^{\alpha_n}$ *is the cycle type of the permutation* σ.

Proof. We have by linearity

$$\Omega(tP) = \exp\left(\sum_{k\geq 1} \frac{\psi_k(P)}{k} t^k\right) = \sum_{m\geq 0} \frac{1}{m!}\left(\sum_{k\geq 1} \frac{\psi_k(P)}{k} t^k\right)^m$$

which implies that

$$\Omega(tP)\Big|_{t^n} = \sum_{m=0}^{n} \frac{1}{m!} \sum_{\substack{\alpha_1+\cdots+\alpha_n=m \\ \alpha_1+2\alpha_2+\cdots=n}} \frac{m!}{1^{\alpha_1}\cdots n^{\alpha_n}\alpha_1!\cdots\alpha_n!}\psi_1^{\alpha_1}(P)\cdots\psi_n^{\alpha_n}(P)\Big|_{t^n}$$

and we obtain

$$\begin{aligned}\Omega(tP) &= \sum_{n\geq 0} \frac{t^n}{n!} \sum_{m=0}^{n} \sum_{\substack{\alpha_1+\cdots+\alpha_n=m \\ 1\alpha_1+2\alpha_2+\cdots=n}} |C_{1^{\alpha_1}\cdots n^{\alpha_n}}|\psi_1^{\alpha_1}(P)\cdots\psi_n^{\alpha_n}(P) \\ &= \sum_{n\geq 0} \frac{t^n}{n!} \sum_{\sigma\in S_n} \psi_{\lambda(\sigma)}(P)\end{aligned}$$

where $|C_{1^{\alpha_1}\cdots n^{\alpha_n}}|$ is the cardinality of the conjugacy class indexed by the partition $1^{\alpha_1}\ldots n^{\alpha_n}$. □

If P and Q are any multisets, then we have (see [6])

$$s_\lambda(P+Q) = \sum_{\mu\subseteq\lambda} s_\mu(P)s_{\lambda/\mu}(Q) \tag{8}$$

which implies, using (7), that

$$\begin{aligned}s_\lambda(P^+ - Q^-) &= \sum_{\mu\subseteq\lambda} s_\mu(P^+)s_{\lambda/\mu}(-Q^-) \\ &= \sum_{\mu\subseteq\lambda} (-1)^{|\lambda|-|\mu|} s_\mu(P^+)s_{\lambda^\sim/\mu^\sim}(Q^-)\end{aligned} \tag{9}$$

3. Generating Functions.

Theorem 1. (Littlewood) *Let* $\mu = 1^{\alpha_1}\cdots n^{\alpha_n} \vdash n$ *and let* y *be a formal variable. Then*

$$\sum_{k=0}^{n-1} \chi_\mu^{1^k n-k} y^k = \frac{\prod_{i=1}^n (1-(-y)^i)^{\alpha_i}}{(1+y)}$$

Proof. We start by applying Cauchy's formula to $\Omega(t\mathbf{x}(1-y))$:

$$\Omega(t\mathbf{x}(1-y)) = \sum_{\lambda\vdash n} s_\lambda(t\mathbf{x})s_\lambda(1-y) = \sum_{n\geq 0} t^n \sum_{\lambda\vdash n} s_\lambda(\mathbf{x})s_\lambda(1-y) \tag{10}$$

So now from (9) we obtain

$$s_\lambda(1-y) = \sum_{\mu\subseteq\lambda} s_\mu(1)(-1)^{|\lambda|-|\mu|} s_{\lambda^\sim/\mu^\sim}(y). \tag{11}$$

But since

$$s_\mu(1) = \begin{cases} 1 & \text{if } \mu = (k) \\ 0 & \text{otherwise} \end{cases}$$

for $\mu \vdash k$, Equation (11) becomes

$$s_\lambda(1-y) = \sum_{k=0}^{n-1} (-1)^{|\lambda|-k} s_{\lambda^\sim/1^k}(y). \tag{12}$$

Now for $s_{\lambda^\sim/1^k}(y)$ to be non-zero, the diagram $\lambda^\sim$ has to be a hook $\lambda^\sim = 1^s n-s$ with $s = k$ or $s = k+1$ and the resulting skew shape is called a *horizontal strip*. This gives

$$s_\lambda(1-y) = \begin{cases} (-1)^{s+1}y^{s+1} + (-1)^s y^s = (-y)^s(1-y) & \text{if } \lambda = 1^s n-s \\ 0 & \text{otherwise} \end{cases} \tag{13}$$

and using Proposition 2.4, Equation (10) becomes

$$\Omega(t\mathbf{x}(1-y)) = \sum_{n\geq 0} t^n \sum_{s=0}^{n-1} s_{1^s n-s}(\mathbf{x})(-y)^s(1-y) \tag{14}$$

$$= \sum_{n\geq 0} t^n \sum_{\lambda\vdash n} \frac{\psi_\lambda(\mathbf{x}(1-y))}{z_\lambda}. \tag{15}$$

Comparing (14) and (15) and using multiplicativity we have

$$\sum_{s=0}^{n-1} s_{1^s n-s}(\mathbf{x})(-y)^s(1-y) = \sum_{\lambda\vdash n} \frac{\psi_\lambda(\mathbf{x}(1-y))}{z_\lambda} = \sum_{\lambda\vdash n} \frac{\psi_\lambda(\mathbf{x})\psi_\lambda(1-y))}{z_\lambda}$$

and since by linearity we have $\psi_\lambda(1-y) = \prod_i (\psi_i(1-y))^{\alpha_i} = \prod_i (1-y^i)^{\alpha_i}$ where $\lambda = 1^{\alpha_1}\cdots n^{\alpha_n}$. We obtain for any $\mu \vdash n$

$$\sum_{s=0}^{n-1} s_{1^s n-s}(\mathbf{x})(-y)^s(1-y)\big|_{\psi_\mu(\mathbf{x})} = \sum_{\lambda\vdash n} \psi_\lambda(\mathbf{x}) \frac{\prod_i (1-y^i)^{\alpha_i}}{z_\lambda}\big|_{\psi_\mu(\mathbf{x})}$$

But since $s_{1^s n-s}\big|_{\psi_\mu(\mathbf{x})} = \frac{\chi_\mu^{1^s n-s}}{z_\mu}$, we have

$$\sum_{s=0}^{n-1} \chi_\mu^{1^s n-s}(-y)^s(1-y) = \prod_i (1-y^i)^{\alpha_i}$$

which is equivalent to the desired conclusion when we replace $-y$ by y. □

Theorem 1 and its proof can be generalized to develop generating functions of characters indexed with multiple hooks called (k,m)-hooks.

Definition 4. A partition $\lambda \vdash n$ is a (k,m)-hook or more generally a multiple hook if λ can be written in the form $\lambda = ((k^m)+\mu) \cup \gamma^\sim$ (Figure 3) where μ and γ are partitions of lengths $\leq m$ and $\leq k$ respectively. We denote by $H(k,m,n)$ the set of (k,m)-hooks $\lambda = ((k^m)+\mu) \cup \gamma^\sim$ of weight $|\lambda| = n$.

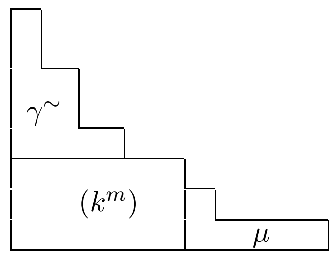

Figure 3
A (k,m)-hook

We need the following generalization of (13) due to Berele and Regev ([2]).

Proposition 3.1. *Let $H(k,m,n)$ be the set of (k,m)-hooks of weight n and let $\mathbf{x} = \{x_1,\dots,x_m\}$, $\mathbf{y} = \{y_1,\dots,y_k\}$, then for any $\lambda \vdash n$ we have*

$$(16) \quad s_\lambda(\mathbf{x}-\mathbf{y}) = \begin{cases} \prod_{i=1}^{k}\prod_{j=1}^{m}(x_i - y_j)s_\mu(\mathbf{x})s_\gamma(\mathbf{y})(-1)^{|\gamma|} & \text{if } \lambda \in H(k,m,n) \\ 0 & \text{otherwise}. \end{cases}$$

Observe that, in view of Equation (7), proposition 3.1 can also be expressed in the form

$$(17) \qquad s_\lambda(\mathbf{x}-\mathbf{y}) = \begin{cases} \prod_{i=1}^{k}\prod_{j=1}^{m}(x_i - y_j)s_\mu(\mathbf{x})s_{\gamma^\sim}(-\mathbf{y}) & \text{if } \lambda \in H(k,m,n) \\ 0 & \text{otherwise} \end{cases}$$

Theorem 2. *Let $\mu = 1^{\alpha_1}\cdots n^{\alpha_n} \vdash n$. Let $m_1 \leq m_2$ be two positive integers and let $t, u, \mathbf{y} = \{y_1, y_2\}$ be formal variables. Then*

$$\text{a)} \sum_{m_1=0}^{\lfloor \frac{n}{2} \rfloor} \chi_\mu^{(m_1, n-m_1)}(u^{m_1} - u^{n-m_1+1}) = (1-u)\prod_{i=1}^{n}(1+u^i)^{\alpha_i},$$

$$\text{b)} \sum_{k=0}^{n-2}\sum_{m_1=0}^{\lfloor \frac{(n-k)}{2} \rfloor} \chi_\mu^{1^k m_1 m_2} u^{m_1-1} y_1^k = \frac{\prod_{i=1}^{n}(1+u^i-(-y_1)^i)^{\alpha_i}(1-u)}{(1-u^{m2-m1+1})(1+y_1)(u+y_1)},$$

$$\text{c)} \sum_{k_1=0}^{n-4}\sum_{k_2=0}^{\lfloor \frac{(n-k_1-4)}{2} \rfloor}\sum_{m_1=2}^{\lfloor \frac{n-k_1-2k_2}{2} \rfloor} \chi_\mu^{1^{k_1}2^{k_2}m_1m_2} u^{m_1-2} y_1^{k_2} y_2^{k_1+k_2}$$

$$= \frac{\prod_{i=1}^{n}(1+u^i-(-y_1)^i-(-y_2)^i)^{\alpha_i}(1-u)(1-y_1/y_2)}{(1-u^{m2-m1+1})(1-(y_1/y_2)^{k1+1})\prod_{i=1,2}(1+y_i)(u+y_i)}.$$

Proof. First observe that for $m_1 \leq m_2$, we have

$$s_{(m_1,m_2)}(u_1,u_2) = u_1^{m_1}u_2^{m_2}\frac{(1-(u_1/u_2)^{m_2-m_1+1})}{(1-u_1/u_2)}$$

$$s_{(m_1,m_2)}(1,u) = u^{m_1}\frac{(1-u^{m_2-m_1+1})}{(1-u)}$$

For the proof of (b), we develop $\Omega(tx(1+u-y_1)$ with Cauchy's formula and $s_\lambda(1+u-y_1)$ with proposition 3.1:

$$\Omega(t\mathbf{x}(1+u-y_1)) = \sum_{\lambda \vdash n} s_\lambda(t\mathbf{x}) s_\lambda(1+u-y_1)\,,$$

$$\begin{aligned} s_\lambda(1+u-y_1) &= \begin{cases} s_{(1^k)}(-y_1) s_{m_1-1 m_2-1}(1,u)(1-y_1)(u-y_1) & \text{if } \lambda = 1^k m_1 m_2 \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} (-y_1)^k \frac{(1-u^{m_2-m_1+1})}{(1-u)} u^{m_1-1}(1-y_1)(u-y_1) & \text{if } \lambda = 1^k m_1 m_2 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Thus we have

$$\Omega(t\mathbf{x}(1+u-y_1))$$

$$= \sum_{n\geq 0} t^n \sum_{k=0}^{n-2} \sum_{m_1=1}^{\lfloor \frac{n-k}{2} \rfloor} s_{1^k m_1, m_2}(\mathbf{x}) \frac{(-y_1)^k u^{m_1-1}(1-u^{m_2-m_1+1})(1-y_1)(u-y_1)}{(1-u)} \tag{18}$$

$$= \sum_{n\geq 0} t^n \sum_{\lambda\vdash n} \frac{\psi_\lambda(\mathbf{x}(1+u-y_1))}{z_\lambda} \tag{19}$$

and observing that $\psi_\lambda(1+u-y_1) = \prod_i (1+u^i-y_1^i)^{\alpha_i}$, if we take the coefficient of $\psi_\mu(x)$ in (18) and (19) and if we replace y_1 by $-y_1$, we obtain Formula (b). To prove Formula (c), we expand similarly the expression $\Omega(t\mathbf{x}(1+u-y_1-y_2))$:

$$\Omega(t\mathbf{x}(1+u-y_1-y_2)) = \sum_{\lambda\vdash n} s_\lambda(t\mathbf{x}) s_\lambda(1+u-y_1-y_2),$$

$$s_\lambda(1+u-y_1-y_2)$$
$$= \begin{cases} (-1)^{k_1} s_{(k_1+k_2,k_2)}(\mathbf{y}) s_{(m_1-2,m_2-2)}(1,u) \times \prod_{i=1,2} (1-y_i)(u-y_i) & \text{if } \lambda = 1^{k_1} 2^{k_2} m_1 m_2 \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} (-y_1)^{k_2} (-y_2)^{k_1+k_2} \frac{(1-(y_1/y_2)^{k_1+1})}{(1-y_1/y_2)} u^{m_1-2} \frac{(1-u^{m_2-m_1+1})}{(1-u)} \times \prod_{i=1,2} (1-y_i)(u-y_i) & \text{if } \lambda = 1^{k_1} 2^{k_2} m_1 m_2 \\ 0 & \text{otherwise} \end{cases}$$

This gives

$$\Omega(t\mathbf{x}(1+u-y_1-y_2))$$

$$(20)\quad = \sum_{n\geq 0} t^n \sum_{k_1=0}^{n-4} \sum_{k_2=0}^{\lfloor \frac{n-k_1-4}{2} \rfloor} \sum_{m_1=2}^{\lfloor \frac{n-k_1-2k_2}{2} \rfloor} s_{1^{k_1}2^{k_2}m_1,m_2}(\mathbf{x})(-y_2)^{k_1+k_2}(-y_1)^{k_2}u^{m_1-2}\times \frac{(1-u^{m_2-m_1+1})(1-(y_1/y_2)^{k_1+1})\prod_{i=1,2}(1-y_i)(u-y_i)}{(1-u)(1-y_1/y_2)}$$

$$(21)\quad = \sum_{n\geq 0} t^n \sum_{\lambda\vdash n} \frac{\psi_\lambda(\mathbf{x}(1+u-y_1-y_2))}{z_\lambda}$$

Observing that $\psi_\lambda(1+u-y_1-y_2) = \prod_i(1+u^i-y_1^i-y_2^i)^{\alpha_i}$, we take the coefficient of $\psi_\mu(x)$ in (20) and (21) and we replace y_i by $-y_i$ and we obtain Formula (c). To obtain Formula (a), we have to develop with the same technique the expression $\Omega(t\mathbf{x}(1+u))$. □

Résumé substantiel en français. Pour deux partages $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k)$ et $\mu = (\mu_1 \geq \mu_2 \geq \cdots \geq \mu_r)$ de l'entier n, notons par χ_μ^λ le caractère de la représentation irréductible du groupe symétrique S_n indexé par le partage λ et évalué sur la classe de conjugaison indexée par le partage μ.

La fonction génératrice des caractères irréductibles indexés par des diagrammes de Ferrers en forme d'équerre

$$(1)\qquad \sum_{k=0}^{n-1} \chi_\mu^{1^k n-k} q^k = \frac{\prod_i(1-(-q)^i)^{\beta_i}}{(1+q)},$$

où $\mu = 1^{\beta_1}2^{\beta_2}\ldots n^{\beta_n}$ est noté multiplicativement, est apparue pour la première fois dans Littlewood [5] et fut depuis utilisée à des fins computationnelles par différents auteurs dont Adrianov [1], Jackson [3], Stanley [7] et Zagier [8]. L'objet de cette note est, tout d'abord, de présenter une preuve de (1) en utilisant les méthodes de la théorie des λ-anneaux. Dans un deuxième temps, nous utilisons ces mêmes méthodes pour construire d'autres séries génératrices pour des caractères irréductibles indexés par des diagrammes en forme de double équerre $\lambda = 1^{k_1}2^{k_2}m_1m_2$ qui sont constitués d'une équerre insérée dans une autre (voir figure 1). Notre résultat principal (théorème 2(c)) est le suivant :

$$\sum_{k_1}\sum_{k_2}\sum_{m_1} \chi_\mu^{1^{k_1}2^{k_2}m_1m_2} u^{m_1-2} y_1^{k_2} y_2^{k_1+k_2} = \frac{\prod_{i=1}^n(1+u^i-(-y_1)^i-(-y_2)^i)^{\alpha_i}(1-u)(1-y_1/y_2)}{(1-u^{m2-m1+1})(1-(y_1/y_2)^{k1+1})\prod_{i=1,2}(1+y_i)(u+y_i)}.$$

Une idée fondamentale de la théorie des λ-anneaux est de considérer un polynôme à coefficients entiers $p(\mathbf{x})$ comme un multi-ensemble P de ses composantes monomiales :

$$p(\mathbf{x}) = \sum_{(q_1,\ldots,q_k)\in N^k} c_{(q_1,\ldots,q_k)} x_1^{q_1}\cdots x_k^{q_k} = \sum_{q=(q_1,\ldots,q_k)} c_q \mathbf{x}^q$$

On peut alors appliquer au multi-ensemble P l'opérateur Ω défini sur un alphabet $A = \{a_1, \ldots, a_n\}$ quelconque comme suit :

$$\Omega(A) = \prod_{a \in A} \frac{1}{(1-a)}.$$

C'est cet opérateur Ω que nous utilisons pour obtenir nos séries génératrices.

En fait, les méthodes que nous utilisons permettent de construire des séries génératrices pour des caractères indexés par des équerres multiples qui peuvent être vues comme une suite d'équerres insérées les unes dans les autres (voir la définition 4), mais les expressions deviennent alors plus élaborées et perdent de leur intérêt.

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