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RESONANCE AT TWO CONSECUTIVE EIGENVALUES FOR SEMILINEAR ELLIPTIC PROBLEM: A VARIATIONAL APPROACH

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RÉSUMÉ. Ce travail traite de l'existence de solutions d'un problème elliptique semi-linéaire satisfaisant certaines conditions de résonnance de Ahmad-Lazer-Paul.

ABSTRACT. This paper deals with the existence of solutions for a semilinear elliptic problem in some resonance conditions of Ahmad-Lazer-Paul occur.

1. Introduction. Let Ω be a bounded domain in $\mathbb{R}^n$, and let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a non-linear function satisfying the Carathéodory conditions. We consider the Dirichlet problem

$$\begin{cases} -\Delta u = \lambda_k u + f(x,u) + h(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1}$$

where $h \in L^p(\Omega)$, for some suitable $p \geq 2$, is given, and $\lambda_k, k = 1, 2, \ldots$, denote the (order distinct) eigenvalues of problem $-\Delta u = \lambda u$ in Ω, $u = 0$ on $\partial\Omega$.

Let us denote by $F(x,s)$ the primitive $\int_0^s f(x,t)\,dt$, and write

$$l_\pm(x) = \liminf_{s\to\pm\infty} \frac{f(x,s)}{s}, \qquad k_\pm(x) = \limsup_{s\to\pm\infty} \frac{f(x,s)}{s}$$

$$L_\pm(x) = \liminf_{s\to\pm\infty} \frac{2F(x,s)}{s^2}, \qquad K_\pm(x) = \limsup_{s\to\pm\infty} \frac{2F(x,s)}{s^2}$$

with, for an autonomous non-linearity $f(x,s) = f(s)$, $l_\pm$ instead of $l_\pm(x)$. Assume that

$$0 \leq l_\pm(x) \leq k_\pm(x) \leq \lambda_{k+1} - \lambda_k \tag{2}$$

uniformly for a.e. $x \in \Omega$.

There exists a rich literature devoted to this kind of problems, starting from Berestycki and DeFigueiredo [3], Dancer [6], Mawhin [9] and references therein.

In [7], DeFigueiredo and Gossez considered (2) (in the autonomous case), where a so called positive density condition was introduced. Roughly speaking, this condition imposes to $f(s)/s$ to remain greater than 0 and less than $\lambda_{k+1} - \lambda_k$ for sufficiently

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many values of s as $s \to \pm\infty$. They showed that (1) is solvable for any h. After that, in [5], Costa and Oliveira proved an existence result when (2) occurs and

$$0 \leq L_{\pm}(x) \leq K_{\pm}(x) \leq \lambda_{k+1} - \lambda_k \tag{3}$$

uniformly for a.e. $x \in \Omega$, with strict inequalities $\lambda_k < L_{\pm}(x), K_{\pm}(x) < \lambda_{k+1}$ holding on subsets of Ω of positive measure.

Beside the above considered non-resonance conditions, many papers have been devoted to resonance ones (see e.g. [1, 8, 10, 11]).

In the present paper, we will extend the cited above non-resonance conditions to resonance ones. First, we prove the existence of solutions for (1) in some situations of (2), on one side of (3) we will have a classical resonance conditions of Ahmad-Lazer-Paul and, on the other hand side, we will impose the following condition:

$$\int_{z>0} (\alpha - k_+) z^2 \, dx + \int_{z<0} (\alpha - k_-) z^2 \, dx > 0,$$

for every $z \in E_{k+1}$.

To state our main result, let us denote by E_j the λ_j-eigenspace and for each real valued function u defined on Ω, we define $\Omega^-(u) = \{x \in \Omega : u(x) < 0\}$ and $\Omega^+(u) = \{x \in \Omega : u(x) > 0\}$.

Theorem 1.1. *Assume that,*

F_0) $\sup_{|s| \leq R} |f(x,s)| \in L^2(\Omega)$, *for all* $R > 0$,

F_1) $0 \leq f(x,s)/s$, *for* $|s| \geq r > 0$ *and a.e.* $x \in \Omega$,

$$k_{\pm}(x) = \limsup_{s \to \pm\infty} \frac{f(x,s)}{s} \leq \lambda_{k+1} - \lambda_k = \alpha, \quad \textit{uniformly on } \Omega,$$

F_2) $\lim_{\|u^0\| \to \infty, u^0 \in E_k} \int F(x, u^0(x)) \, dx = \infty$,

F_3)

$$\int_{z>0} (\alpha - k_+) z^2 \, dx + \int_{z<0} (\alpha - k_-) z^2 \, dx > 0,$$

for every $z \in E_{k+1}$.

F_4) $h \in (E_k)^{\perp}$.

Then (1) has at least one solution.

Remark 1. F_3) occurs if F verified the following condition:

$$\begin{cases} \text{there exists a subset } \Omega' \text{ of } \Omega \text{ such that } \operatorname{mes}(\Omega' \cap \Omega^+(v)) > 0, \\ (\text{resp. } \operatorname{mes}(\Omega' \cap \Omega^-(v)) < 0) \quad \text{for every } v \in E_{k+1} \text{ and } \limsup\limits_{s \to \infty} \dfrac{2F(x,s)}{s^2}, \\ \left(\text{resp. } \limsup\limits_{s \to -\infty} \dfrac{2F(x,s)}{s^2} \right) < \lambda_{k+1} - \lambda_k, \ \text{a.e. in } \Omega'. \end{cases}$$

Second, we prove the weak solvability when $f(x,s)/s$ stays between 0 and $\lambda_{k+1} - \lambda_k$ for large values of $|s|$ and we will replace (3) by Ahmad-Lazer-Paul conditions, precisely we will prove the following.

Theorem 1.2. *Assume F_0), F_3), and*

F_5) $0 \leq f(x,s)/s \leq \lambda_{k+1} - \lambda_k$ *for* $|s| \geq r > 0$ *and a.e.* $x \in \Omega$,

F_6) $\lim_{\|u^0\| \to \infty, u^0 \in E_{k+1}} \int [\frac{1}{2}(\lambda_{k+1} - \lambda_k) u^{0^2}(x) - F(x, u^0(x))]\, dx = \infty$,

F_7) $h \in (E_k)^\perp \cap (E_{k+1})^\perp \cap L^2(\Omega)$.

Then (1) has at least one solution.

Next, some variants of Theorem 1.1 and Theorem 1.2 will be given.

Our approach to theorems 1.1 and 1.2, is variational and uses the general minimax theorem proved by Bartolo *et al.* in [2]. The proofs of theorems 4.2 and 4.3 (see section 4) use the preceding results and an approximation argument.

We also mention that our approach can be adapted to study higher order self adjoint elliptic partial differential equations.

2. Preliminaries.

2.1. A compactness condition. We denote by $\|\cdot\|$ the norm in $H_0^1(\Omega)$ induced by the inner product

$$\langle u, v \rangle = \int_\Omega \nabla u \nabla v, \quad u, v \in H_0^1(\Omega)$$

and by $\|\cdot\|_{H^{-1}}$, the norm in $H^{-1}(\Omega)$, the dual space of $H_0^1(\Omega)$.

In this section, we start by recalling a compactness condition of the Palais-Smale type which was introduced by Cerami in [4] and which allows rather general minimax results of [2].

A functional $\Phi \in C^1(E, R)$, E being a real Banach space, is said to satisfy condition (*C*) at the level $c \in \mathbb{R}$ if the following holds:

$(C)_c$ i) any bounded sequence $(u_n) \subset E$ such that $\Phi(u_n) \to c$ and $\Phi'(u_n) \to 0$ possesses a convergent subsequence;

ii) there exist constants δ, R, $\alpha > 0$ such that

$$\|\Phi'(u)\| \|u\| \geq \alpha \text{ for any } u \in \Phi^{-1}([c-\delta, c+\delta]) \text{ with } \|u\| \geq R.$$

It was shown in [2] that condition (*C*) actually suffices to get a deformation theorem and then, by standard minimax arguments (see [2]), the following result was proved.

Theorem 2.1. *Suppose that $\Phi \in C^1(E, \mathbb{R})$, E being a real Banach space, satisfies condition $(C)_c$ $\forall c \in \mathbb{R}$ and that there exist a closed subset $S \subset E$ and $Q \subset E$ with boundary ∂Q verifying the following conditions:*

i) $\sup_{u \in \partial Q} \Phi(u) \leq \alpha < \beta \leq \inf_{u \in S} \Phi(u)$ *for some* $0 \leq \alpha < \beta$;

ii) *S and ∂Q link;*

iii) $\sup_{u \in Q} \Phi(u) < \infty$.

Then Φ possesses a critical value $c \geq \beta$.

2.2. Generalities and technical lemma. Since we are going to apply the variational characterisation of the eigenvalues, we shall decompose the space $H_0^1(\Omega)$ as following $E = E_- \oplus E_k \oplus E_{k+1} \oplus E_+$, where E_- is the subspace spanned by the λ_j-eigenfunctions with $j < k$ and E_j is the eigenspace generated by the λ_j-eigenfunctions and E_+ is the orthogonal complement of $E_- \oplus E_k \oplus E_{k+1}$ in $H_0^1(\Omega)$ and we shall decompose for any $u \in H_0^1(\Omega)$ as following $u = u^- + u^k + u^{k+1} + u^+$ where $u^- \in E_-$, $u^k \in E_k$, $u^{k+1} \in E_{k+1}$ and $u^+ \in E_+$.

We verify easily

$$\int |\nabla u|^2 \, dx - \lambda_k \int |u|^2 \, dx \geq \delta \|u\|^2 \quad \forall u \in E_+ \oplus E_{k+1} \tag{4}$$

$$\int |\nabla u|^2 \, dx - \lambda_k \int |u|^2 \, dx \leq -\delta \|u\|^2 \quad \forall u \in E_- \oplus E_k \tag{5}$$

where

$$\delta = \min \left\{ 1 - \frac{\lambda_k}{\lambda_{k+1}}, \frac{\lambda_k}{\lambda_{k-1}} - 1 \right\}. \tag{6}$$

The idea to establish results of existence of solution to problem (1) is to use Theorem 2.1. For this purpose, we shall consider the functional $\Phi : E \to \mathbb{R}$ defined by

$$\Phi(u) = \frac{1}{2} \int |\nabla u|^2 dx - \lambda_k \int |u|^2 dx - \int F(x, u) - \int hu$$

where hereafter the integrals are over Ω.

We have the following important lemma.

Lemma 2.1. *Let $(u_n) \subset H_0^1(\Omega)$ and $(p_n) \subset L^\infty(\Omega)$ be sequences, and let A a non-negative constant such that*

$$0 \leq p_n(x) \leq A \quad \text{a.e. on } \Omega \text{ and for all } n \in \mathbb{N}$$

and $p_n \rightharpoonup 0$ in the weak topology of L^∞, as $n \to \infty$. Then, there are subsequences $(u_n), (p_n)$ satisfying the above conditions and, there is a positive integer n_0 such that for all $n \geq n_0$, one has*

$$\int p_n u_n((u_n^- + u_n^k) - (u_n^{k+1} + u_n^+)) \, dx \geq \frac{-\delta}{2} \|u_n^+ + u_n^{k+1}\|^2 \tag{7}$$

where $\delta > 0$ is given in (6).

Proof. Since $p_n \geq 0$ a.e. in Ω, we see that

$$\begin{aligned} \int p_n u_n((u_n^- + u_n^k) - (u_n^{k+1} + u_n^+)) &\geq - \int p_n (u_n^+ + u_n^{k+1})^2 \, dx \\ &\geq - \left[\int p_n \left(\frac{u_n^+ + u_n^{k+1}}{\|u_n^+ + u_n^{k+1}\|} \right)^2 dx \right] \left\| u_n^+ + u_n^{k+1} \right\|^2. \end{aligned} \tag{8}$$

Moreover, by the compact imbedding of $H_0^1(\Omega)$ into $L^2(\Omega)$ and $p_n \rightharpoonup 0$ in the weak* topology of L^∞, when $n \to \infty$, then there are subsequences relabelled (u_n), (p_n) such that

$$\int p_n \left(\frac{u_n^+ + u_n^{k+1}}{\|u_n^+ + u_n^{k+1}\|} \right)^2 \, dx \to 0.$$

Therefore, there exists $n_0 \in \mathbb{N}$ such that for $n \geq n_0$ we have

$$\int p_n \left(\frac{u_n^+ + u_n^{k+1}}{\|u_n^+ + u_n^{k+1}\|} \right)^2 dx \leq \frac{\delta}{2}. \tag{9}$$

Combining inequalities (8) and (9), we get inequality (7). $\square$

3. Proof of theorems. We begin by proving the following lemmas.

Lemma 3.1. Φ *satisfies the (C) condition on* H_0^1.

Proof. Let $(u_n)_n \subset H_0^1$ be a (*C*) sequence, i.e.

$$\|\Phi(u_n)\| \leq A \tag{10}$$

$$\|u_n\| \left\langle \Phi'(u_n), v \right\rangle_{H_0^1, H^{-1}} \leq \varepsilon_n \|v\| \quad \forall v \in H_0^1, \tag{11}$$

where A is a constant and $\varepsilon_n \to 0$.

It clearly suffices to shows that $(u_n)_n$ remains bounded in H_0^1. Assume by contradiction and defining $z_n = u_n/\|u_n\|$, we have $\|z_n\| = 1$ and, passing if necessary to a subsequence, we may assume that $z_n \rightharpoonup z$ weakly in H_0^1, $z_n \to z$ strongly in $L^2(\Omega)$ and $z_n(x) \to z(x)$ a.e. in Ω.

We consider $\big(f(\cdot, u_n(\cdot))/\|u_n\|\big)$ which, by the linear growth of f, remains bounded in L^2. Thus, for a subsequence $\big(f(\cdot, u_n(\cdot))/\|u_n\|\big)$ converges weakly in L^2 to some $\tilde{f} \in L^2$ and by standard arguments based on assumptions F_0), F_1), $\tilde{f}$ can be written as

$$\tilde{f}(x) = m(x) z(x)$$

where the L^∞-function m satisfies

$$0 \leq m(x) \leq \lambda_{k+1} - \lambda_k \text{ a.e. in } \Omega$$

(cf. e.g. [5]).

Moreover, divide (11) by $\|u_n\|^2$ and go to the limit to get

$$\int \nabla z \nabla v - \lambda_k \int z v \, dx - \int m(x) z v \, dx = 0 \text{ for all } v \in H_0^1 \tag{12}$$

and we verify easily that $z \not\equiv 0$.

We now distinguish three cases :

i) $m(x) \equiv 0$;

ii) $0 < m(x)$ and $m(x) < \lambda_{k+1} - \lambda_k$ on subsets of positive measure;

iii) $m(x) \equiv \lambda_{k+1} - \lambda_k$.

Case ii). Since $z \not\equiv 0$, this case cannot occur in view of lemma 4 in [3].

Case i). In this case, it follows from (12) that z is a λ_k-eigenfunction.

On the other hand, by F_1), for $\varepsilon > 0$ there exists a constant $r_\varepsilon > r$ such that

$$0 \leq \frac{f(x,s)}{s} \leq \lambda_{k+1} - \lambda_k + \varepsilon \quad \forall \, |s| \geq r_\varepsilon \tag{13}$$

Let $f_n(x) = f(x, u_n(x))/u_n(x)\chi_{[|u_n(x)|\geq r_\varepsilon]}$, which remains bounded in L^∞, converges weakly in L^∞ to some $l \in L^\infty$.

By (13) this function satisfies

$$0 \leq l(x) \leq \lambda_{k+1} - \lambda_k + \varepsilon.$$

Multiply f_n by z_n^2, integrate on Ω and take the limit to get

$$\int f_n(z_n)^2\,dx \longrightarrow \int l(x)z^2\,dx = \int m(x)z^2(x)\,dx = 0.$$

By the unique continuation property of Δ and $l \geq 0$, we must have $l \equiv 0$ a.e. on Ω.

The contradiction of case ii) will be divided into several steps:

Step 1. We claim that $u_n^k/\|u_n\| \to z$ strongly in H_0^1.

By using the fact that E_k is finite dimensional and the compact imbedding of $(E_k)^\perp$ into $L^2(\Omega)$, it follows that there are $z_1 \in E_k$, $z_2 \in (E_k)^\perp$ such that (by using an appropriate subsequence similarly relabelled if necessary)

$$\frac{u_n^k}{\|u_n\|} \to z_1 \ \text{ strongly in } H_0^1 \text{ and in } L^2,$$

$$\frac{u_n^- + u_n^+ + u_n^{k+1}}{\|u_n\|} \to z_2 \qquad \text{strongly in } L^2.$$

On the other hand, we have $z \in E_k$ and $u_n/\|u_n\| \to z$, which implies that $z = z_1 + z_2$. Thus, $z = z_1$ and $z_2 = 0$. The result follows.

Step 2. We are now ready to prove that the sequence $(\|u_n^- + u_n^+ + u_n^{k+1}\|)_n$ is uniformly bounded in n.

Take $v = (u_n^- + u_n^k) - (u_n^+ + u_n^{k+1})$ in (11) and $p_n(x) = f_n(x)$, we get

$$\Lambda \leq \Gamma \tag{14}$$

where

$$\Lambda = \Bigg\{ -\int |\nabla u_n^-|^2 + \lambda_k \int |u_n^-|^2\,dx + \int |\nabla(u_n^+ + u_n^{k+1})|^2 - \lambda_k \int |u_n^+ + u_n^{k+1}|^2\,dx + \int p_n u_n((u_n^- + u_n^k) - (u_n^{k+1} + u_n^+))\,dx \Bigg\}$$

and

$$\Gamma = \Bigg\{ \varepsilon_n + \int h((u_n^+ + u_n^{k+1}) - u_n^-)\,dx + \int_{|u_n(x)|\leq r_\varepsilon} |f(x, u_n(x)||(u_n^+ + u_n^{k+1}) - (u_n^- + u_n^k)|\,dx \Bigg\}.$$

By the Poincaré inequality and from (4), (5), (7), (14), there exist two constants A_ε, B_ε such that

$$\frac{\delta}{2}\|u_n^- + (u_n^+ + u_n^{k+1})\|^2 \le \varepsilon_n + A_\varepsilon\|u_n^- + (u_n^+ + u_n^{k+1})\| + B_\varepsilon.$$

This gives that $(\|u_n^- + (u_n^+ + u_n^{k+1})\|)_n$ is uniformly bounded in n.
Step 3. We will now reach a contradiction with assumption F_2).

From (10), and Poincaré inequality, we have

$$\int F(x, u_n^k/2)\,dx \le A + \int [F(x, u_n^k/2) - F(x, u_n)]\,dx + \frac{1}{2}\|u_n^+ + u_n^{k+1} + u_n^-\|^2 + \frac{1}{\sqrt{\lambda_1}}\|h\|_{L^2}\|u_n^+ + u_n^{k+1} + u_n^-\|.$$

However, by the mean value theorem, we get for a.e. $x \in \Omega$ and $t = t(x) \in [0,1]$ such that

$$\begin{aligned}\int [F(x, u_n^k/2) - F(x, u_n)]\,dx &= \int f(x, tu_n^k/2 + (1-t)u_n)(u_n^k/2 - u_n)\,dx \\ &= \int_{|tu_n^k/2+(1-t)u_n|\le r\varepsilon} f(x, tu_n^k/2 + (1-t)u_n)\,dx \\ &+ \int_{|tu_n^k/2+(1-t)u_n|\ge r\varepsilon} \frac{f(x, tu_n^k/2 + (1-t)u_n)}{tu_n^k/2 + (1-t)u_n} t(u_n^k/2 - u_n)^2 + (u_n^k/2 - u_n)u_n\,dx.\end{aligned} \tag{16}$$

So that using (16) and the Poincaré inequality again, we have

$$\int [F(x, u_n^k/2) - F(x, u_n)]\,dx \le \frac{2}{\sqrt{\lambda_1}}\|\sup_{|s|\le r\varepsilon}|f(x,s)|\|_{L^2}\|u_n^- + u_n^+ + u_n^{k+1}\| + r_\varepsilon\|\sup_{|s|\le r\varepsilon}|f(x,s)|\|_{L^1} + \frac{\lambda_{k+1} - \lambda_k + \varepsilon}{4\lambda_1}\|u_n^- + u_n^+ + u_n^{k+1}\|^2. \tag{17}$$

From (15) and (17), there exists $M > 0$ such that

$$\int F(x, u_n^k/2)\,dx \le M.$$

This is a contradiction with assumption F_2).

Case iii). If $m(x) \equiv \lambda_{k+1} - \lambda_k = \alpha$

Dividing (11) by $\|u_n\|^2$, then we have

$$\frac{\Phi(u_n)}{\|u_n\|^2} \to 0, \quad \text{as } n \to \infty.$$

Since $z_n \to z$ strongly in $H_0^1(\Omega)$, we get

$$\int \frac{F(x, u_n(x))}{\|u_n\|^2}\,dx \to \frac{1}{2}\left[\int |\nabla z|^2\,dx - \lambda_k \int |z|^2\,dx\right]$$

and using Fatou's lemma, we also have

$$\alpha \int z^2 \leq \int \limsup \frac{2F(x,u_n(x))}{|u_n|^2} \frac{u_n^2}{\|u_n\|^2}\,dx$$
$$\leq \int_{z>0} \limsup \frac{2F(x,u_n(x))}{|u_n|^2} z^2\,dx + \int_{z<0} \limsup \frac{2F(x,u_n(x))}{|u_n|^2} z^2\,dx.$$

Therefore, we obtain

$$\int_{z>0} (\alpha - k_+) z^2\,dx + \int_{z<0} (\alpha - k_-) z^2\,dx > 0.$$

But this gives us once more a contradiction from F_3). This completes the proof. □

Lemma 3.2. *Under hypotheses of Theorem 1.1, the functional* Φ *has the following properties:*

i) $\Phi(w) \to \infty$, *as* $\|w\| \to \infty$, $w \in E_{k+1} \oplus E_+$

ii) $\Phi(v) \to -\infty$, *as* $\|v\| \to \infty$, $v \in E_k \oplus E_-$

Proof. i) Suppose by contradiction that

$$\Phi(w_n) = \frac{1}{2}\left[\int |\nabla w_n|^2\,dx - \lambda_k \int |w_n|^2\,dx\right] - \int F(x,w_n) - \int h w_n\,dx \leq B \quad (18)$$

for some constant B and some sequence $(w_n) \subset E_+$ with $\|w_n\| \to \infty$.

Let $\varepsilon > 0$. From F_1), there exists $B_\varepsilon(x) \in L^1(\Omega)$ such that

$$F(x,s) \leq \alpha \frac{s^2}{2} + \varepsilon s^2 B_\varepsilon(x) \text{ a.e. in } \Omega,\ \forall\, s \in \mathbb{R}. \quad (19)$$

However, by (18) and (19) we get that $\|w_n\|_2 \to \infty$, as $n \to \infty$, otherwise, we would obtain

$$\|w_n\|^2 \leq \lambda_{k+1}\|w_n\|_2^2 + 2\varepsilon\|w_n\|^2 + 2\int B_\varepsilon(x)\,dx + \int |h w_n|\,dx + 2B.$$

If we take $0 < \varepsilon < \frac{1}{2}$, we obtain

$$\|w_n\| \leq cst.$$

Letting $z_n = w_n/\|w_n\|_2$ and dividing (18) by $\|w_n\|_2^2$, we obtain in view of (19) and of the continuous imbedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ that

$$\|z_n\|^2 - \lambda_k \leq \lambda_{k+1} - \lambda_k + 2\varepsilon\|z_n\|^2 + \frac{2\int B_\varepsilon(x)\,dx + 2B}{\|w_n\|_2} + \frac{\int |h z_n|\,dx}{\|w_n\|_2}.$$

As $\|w_n\|_2 \to \infty$, there exist constants $M, N > 0$ such that

$$\|z_n\|^2 - \lambda_k \leq \varepsilon M\|z_n\|^2 + N.$$

If we take $0 < \varepsilon < \min\left\{\frac{1}{2}, \frac{1}{M}\right\}$, we get

$$\|z_n\| \leq cst.$$

Using a subsequence if necessary, we obtain

$$z_n \to z \qquad a.e. \text{ on } \Omega \text{ and in } L^2$$

for some $z \in H_0^1(\Omega)$ with $\|z\|_2 = 1$ (since $\|z_n\|_2 = 1$).

As $z \in E_{k+1} \oplus E_+$ we have necessarily that z is λ_{k+1}-eigenfunction. Dividing (18) by $\|w_n\|_2^2$ and using Fatou's lemma, we get

$$\alpha \int z^2\, dx \leq \int_{z>0} k_+(x) z^2\, dx + \int_{z<0} k_-(x) z^2\, dx > 0.$$

Hence

$$\int_{z>0} (\alpha - k_+) z^2\, dx + \int_{z<0} (\alpha - k_-) z^2\, dx > 0.$$

But this yields us a contradiction.

ii) Assume by contradiction there exist a constant B and a sequence $(v_n) \subset V$ such that

$$B \leq \Phi(v_n) \leq -\delta \|v_n^-\|^2 - \int h v_n^-\, dx.$$

Therefore, $\|v_n^-\|$ is bounded and by a similar argument to that one given in step 3, we obtain

$$\liminf \int F(x, \frac{v_n^k}{2})\, dx \leq cst.$$

This is a contradiction with assumption F_2). □

Proof of Theorem 1.1. In view of Lemmas 3.1 and 3.2, we may apply Theorem 2.1 letting $S = W$ and $Q = \{v \in E_- \oplus E_k : \|v\| \leq R\}$, with $R > 0$ being such that

$$\alpha = \max_{\partial Q} \Phi < \inf_{E_+ \oplus E_{k+1}} \Phi = \beta$$

It follows that the functional Φ has a critical value $c \geq \beta$ and, hence, problem (1) has a solution $u \in H_0^1$. □

In the following result we will be interested in the case when the Ahmad-Lazer-Paul conditions at λ_k and at λ_{k+1} are considered.

Proof of Theorem 1.2. The proof of Theorem 1.2 will be divided into two steps.

Step 1. We prove that Φ satisfies the condition (C) on $H_0^1(\Omega)$.

As in the proof of lemma 3.1, we put

$$f_n(x) = \frac{f(x, u_n(x))}{u_n(x)} \chi_{[|u_n(x)| \geq r]}$$

and $l \in L^\infty$ such that

$$f_n \to l \qquad \text{in the weak* topology of } L^\infty, \quad \text{as } n \to \infty.$$

After dividing (11) by $\|u_n\|^2$ and taking the limit, we get

$$\int \nabla z \nabla v - \lambda_k \int zv\,dx - \int l(x) zv\,dx = 0 \quad for\ all\ v \in H_0^1$$

with $z \not\equiv 0$.

We now separate three cases:

i) $l(x) \equiv 0$. It is similar to case i) of Lemma 3.1 and therefore, this case cannot occur.

ii) $0 < l(x)$ and $l(x) < \lambda_{k+1} - \lambda_k$ on subsets of positive measure. Since $z \not\equiv 0$, this case cannot occur in view of lemma 4 in [3].

iii) $l(x) \equiv \lambda_{k+1} - \lambda_k$. In this case we are just going to follow the same lines of case i) as in Lemma 3.1.

So we write (11) as follows

$$\int \nabla u_n \nabla v - \lambda_{k+1} \int u_n v\,dx + \int [\lambda_{k+1} - \lambda_k) u_n - f(x, u_n)] v - \int hv\,dx \le \varepsilon_n \|v\|. \quad (20)$$

Then let $v = (u_n^- + u_n^k) - (u_n^{k+1} + u_n^+)$ in (19) and

$$p_n(x) = \frac{(\lambda_{k+1} - \lambda_k) u_n(x) - f(x, u_n(x))}{u_n(x)} \chi_{[|u_n(x)| \ge r]}.$$

We obtain that a sequence $(\|(u_n^- + u_n^k + u_n^+\|)_n$ is uniformly bounded in n. In a way similar to the followed in step 3 of case i) of Lemma 3.2, we get

$$\int \left[\frac{\lambda_{k+1} - \lambda_k}{2} (u_n^k/2)^2 - F(x, u_n^k/2) \right] dx \le M.$$

This is a contradiction with assumption F_3).

Step 2. Φ has the following properties:

i) $\Phi(w) \to \infty$, as $\|w\| \to \infty$, $w \in E_{k+1} \oplus E_+$

ii) $\Phi(v) \to -\infty$, as $\|v\| \to \infty$, $v \in E_k \oplus E_-$.

We verify easily as in ii) of Lemma 3.2, that Φ has the preceding properties. Then Theorem 1.2 follows from Theorem 2.1. The proof is complete. $\square$

4. Variant results. When $k > 2$ we can state a "dual" version of Theorem 1.1.

Theorem 4.1. *Suppose* $k > 2, F_0), F_4)$ *and*

F_1') $\dfrac{f(x,s)}{s} \le 0$ *for* $|s| \ge r > 0$ *and a.e.* $x \in \Omega$,

$$\liminf_{s \to \pm\infty} \frac{f(x,s)}{s} \ge \lambda_{k-1} - \lambda_k = \alpha' \quad uniformly\ on\ \Omega$$

F_2') $\displaystyle\lim_{\|u^0\| \to \infty, u^0 \in E_k} \int F(x, u^0(x))\,dx = -\infty,$

F_3') $\displaystyle\int_{z>0} (\alpha' - l_+) z^2\,dx + \int_{z<0} (\alpha' - l_-) z^2\,dx < 0,$ *for every* $z \in E_{k-1}$.

Then (1) has at least one solution.

Remark 2. For some $R > 0$,

$$c = \inf_{g \in H} \max_{u \in B_R} \Phi(g(u))$$

where $B_R = \{u \in E_1 : \|u\| \leq R\}$ and $H = \{g \in C(B_R, H_0^1) : g(u) = u \; if \; u \in \partial B_R\}$ is a negative critical value of Φ when $k = 1$.

Indeed, from theorem 1.1 c is a critical value of Φ. In the other hand, we have $c \leq \max_{u \in B_R} \Phi(u) \leq 0$ because $i : B_R \to E$, $x \mapsto x$ is continuous and $\Phi(u) = -\int F(x, u) \leq 0$ for all $u \in E_1$.

In the next results we will be interested in the case where $k = 1$ and where the Ahmad-Lazer-Paul conditions fails.

Theorem 4.2. *Let Ω a bounded domain in $\mathbb{R}^N (N \geq 2)$ with a boundary of class C^2 and $h \in L^p(\Omega)$ with $p > N$.*

Assume $k = 1, F_0), F_3), F_4)$ and

$F_8)$ $0 \leq$ $\operatorname{sign}(s) f(x, s)$ *for $s \in \mathbb{R}$ and a.e. $x \in \Omega$,*

$$\limsup_{s \to \pm\infty} \frac{f(x, s)}{s} \leq \lambda_2 - \lambda_1 \quad \textit{uniformly on } \Omega.$$

Then (1) has at least one solution.

For the proof of Theorem 4.2, we shall use an approximation argument. So we consider the family of following problems.

$$(\mathcal{P}_n) \qquad \begin{cases} -\Delta u = \lambda_1 u + f_n(x, u) + h(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

with

$$f_n(x, s) = \begin{cases} f(x, s) + \frac{1}{n} & \text{if} s \geq \frac{1}{n} \\ f(x, s) + s & \text{if } \frac{-1}{n} \leq s \leq \frac{1}{n} \\ f(x, s) - \frac{1}{n} & \text{if } s \leq \frac{-1}{n} \end{cases}$$

and the functional associated to our problem $(\mathcal{P}_n)$ is

$$\Phi_n(u) = \frac{1}{2} \int |\nabla u|^2 dx - \lambda_1 \int |u|^2 dx - \int F_n(x, u) - \int hu$$

with $F_n(x, s) = \int_0^s f_n(x, t)\, dt$.

We see that f_n satisfies the conditions of Theorem 1.1, thus the existence of solutions u_n for the problem $(\mathcal{P}_n)$.

Lemma 4.1. *The sequence $(u_n)_n$ is bounded in $H_0^1(\Omega)$.*

Proof. Assume by contradiction that $\|u_n\| \to \infty$, as $n \to \infty$. Letting $z_n = u_n / \|u_n\|$, and take a subsequence such that $z_n \rightharpoonup z$ weakly in H_0^1, $z_n \to z$ strongly in L^2.

Dividing $\langle \Phi_n'(u_n), v\rangle$ by $\|u_n\|$ and by an argument similar to that one given in (12), we obtain

$$\int \nabla z \nabla v - \lambda_1 \int zv\,dx - \int m(x) zv\,dx = 0 \text{ for all } v \in H_0^1 \tag{21}$$

with $0 \le m(x) \le \lambda_2 - \lambda_1$ a.e. in Ω.

We now distinguish three cases:

i) $m(x) \equiv 0$;

ii) $0 < m(x)$ and $m(x) < \lambda_2 - \lambda_1$ on subsets of positive measure;

iii) $m(x) \equiv \lambda_2 - \lambda_1$.

Case i). From (21) we have z is λ_1-eigenfunction.

Dividing the equation

$$-\Delta u_n = \lambda_1 u_n + f_n(x, u_n) + h(x) \tag{22}$$

by $\|u_n\|$, the standard L^p-theory for the Dirichlet problem and the compact imbedding of $W^{2,p}(\Omega)$ in $C^1(\bar{\Omega})$, we can assume (by going if necessary to subsequences) that there exists $z \in W_0^{1,p} \cap W^{2,p}$ such that

$$z_n \to z \quad \text{in } \mathcal{C}^1(\bar{\Omega}),\ \text{as} n \to \infty \tag{23}$$

with $\|z\|_{\mathcal{C}^1} = 1$.

Since $\|z\|_{\mathcal{C}^1} = 1$, we have that either $z > 0$ in Ω and $\partial z/\partial n < 0$ on $\partial\Omega$, or $z < 0$ in Ω and $\partial z/\partial n > 0$ on $\partial\Omega$. Here $\partial/\partial n$ denotes the outward normal derivative. Assuming, for instance, that the first eventuality holds, we deduce from (23), that $u_n > 0$ in Ω, for all large n. Now, multiplying (22) by the positive λ_1-eigenfunction ψ and integrating over Ω, we deduce that, for n sufficiently large

$$\int f_n(x, u_n(x))\psi(x)\,dx = 0.$$

Hence

$$\frac{1}{n}\int_{0 \le u_n(x) \le \frac{1}{n}} u_n(x)\psi(x)\,dx = -\frac{1}{n}\int_{0 \le u_n(x) \le \frac{1}{n}} \psi(x)\,dx. \tag{24}$$

This is a contradiction, since by F_8) and $u_n > 0$ in Ω, we have the first term of equality (24) is positive, but the second term of (24) is not positive.

Case ii). This case does not occur.

Case iii). If $m(x) \equiv \lambda_2 - \lambda_1$, by a simple computation we have

$$\int_{z>0} (\alpha - k_+^n) z^2\,dx + \int_{z<0} (\alpha - k_-^n) z^2\,dx > 0,$$

with $\limsup_{s \to \pm\infty} f_n(x,s)/s = k_\pm^n(x)$.

On the other hand, by the Remark 2, we have $\Phi_n(u_n) \le 0$ and thus similarly to case 3 of proof of Lemma 3.1, we have a contradiction. $\square$

Proof of Theorem 4.2. Since $(u_n)_n$ is bounded in $H_0^1(\Omega)$, we can assume (by going if necessary to a subsequence) that there exists $u \in H_0^1(\Omega)$ such that $u_n \rightharpoonup u$ weakly in $H_0^1(\Omega)$ and $u_n \to u$ strongly in $L^2(\Omega)$.

Letting n tend to ∞ in

$$\int \nabla u_n \nabla v - \lambda_1 \int u_n v\, dx - \int f_n(x, u_n)]v - \int hv\, dx = 0$$

we obtain

$$\int \nabla u \nabla v - \lambda_1 \int uv\, dx - \int f(x, u)v - \int hv\, dx = 0.$$

The proof is completed. $\square$

In a similar way, we have the following result.

Theorem 4.3. *Assume* $k = 1, F_0), F_7)$ *and*

$F_5')$ $0 \leq \operatorname{sign}(s) f(x, s)$ *for* $s \in \mathbb{R}$, *a.e.* $x \in \Omega$ *and*

$$\frac{f(x, s)}{s} \leq \lambda_2 - \lambda_1 \textit{ for } |s| \geq r > 0 \textit{ and a.e. } x \in \Omega$$

$F_6')$

$$\lim_{\|u^0\| \to \infty, u^0 \in E_{k+1}} \int \frac{1}{2}(\lambda_2 - \lambda_1) {u^0}^2(x) - F(x, u^0(x))\, dx = \infty.$$

Then (1) has at least one solution.

Remark 3. Similar results to previous theorems carry over to the problem

$$\begin{cases} -Lu = g(x, u) + h(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where L is given by:

$$Lu = \sum_{i,j=1,\dots,n} \frac{\partial}{\partial x_i}(a_{ij}(x) \frac{\partial u}{\partial x_i}) + a_0(x)u$$

is a symmetric uniformly strongly elliptic second order differential operator, acting on real valued functions u defined on Ω. The coefficients a_{ij} are real valued functions defined on $\bar{\Omega}$, with $a_{ij} \in C^1(\bar{\Omega})$ for $i, j = 1, \dots, n$ and $a_0 \in C(\bar{\Omega})$, $a_0 \geq 0$ in Ω. The unique continuation property which is needed in the proofs, now follows and it is well-known that the first eigenvalue of L in $H_0^1(\Omega)$ is simple and that there exists a corresponding smooth eigenfunction ϕ, with $\phi > 0$ in Ω and $\partial\phi/\partial n < 0$ on $\partial\Omega$.

Résumé substantiel en français. Soit Ω un domaine borné dans $\mathbb{R}^n$, et soit f : $\Omega \times \mathbb{R} \to \mathbb{R}$, une fonction de Carathéodory. Nous montrerons l'existence des solutions du problème semi-linéaire elliptique

$$(\mathcal{P}) \quad \begin{cases} -\Delta u = \lambda_k u + f(x, u) + h(x) & \text{dans } \Omega \\ u = 0 & \text{sur } \partial\Omega \end{cases} \qquad (\mathcal{P})$$

où $h \in L^p(\Omega)$, pour $p \geq 2$.

Dans le premier résultat, nous ferons les hypothèses suivantes :

F_0) $\sup_{|s|\le R} |f(x,s)| \in L^2(\Omega)$ pour tout $R > 0$;

F_1) $\lambda_k \le f(x,s)/s$ pour $|s| \ge r > 0$ et p.p. $x \in \Omega$;

$$k_\pm(x) = \limsup_{s\to\pm\infty} \frac{f(x,s)}{s} \le \lambda_{k+1} - \lambda_k = \alpha \text{ uniformément sur } \Omega;$$

F_2) $\lim_{\|u^0\|\to\infty, u^0\in E_k} \int F(x,u^0(x))\,dx = \infty$;

F_3)

$$\int_{z>0} (\alpha - k_+) z^2\,dx + \int_{z<0} (\alpha - k_-) z^2\,dx > 0$$

pour tout $z \in E_{k+1}$;

F_4) $h \in E_k^\perp$.

Notons par F, la primitive de f, et par λ_k, $k = 1, 2, \dots$, les valeurs propres distinctes du problème $-\Delta u = \lambda u$ dans Ω, $u = 0$ sur $\partial\Omega$, et E_k l'espace propre associé à λ_k.

Le deuxième résultat sera consacré à l'étude du problème $(\mathcal{P})$ sous les hypothèses suivantes, F_0), F_2),

F_1') $0 \le f(x,s)/s \le \lambda_{k+1} - \lambda_k$ pour $|s| \ge r > 0$ et p.p. x dans Ω;

F_3') $\lim_{\|u^0\|\to\infty, u^0\in E_{k+1}} \int \left[\frac{1}{2}(\lambda_{k+1} - \lambda_k)(u^0)^2 - F(x,u^0(x))\right] dx = \infty$;

F_4') $h \in E_k^\perp \cap E_{k+1}^\perp$.

Ensuite nous prouverons certaines variantes des résultats cités plus haut autour de la première valeur propre et sans supposer la condition d'Ahmad, Lazer et Paul.

La méthode utilisée pour montrer l'existence d'une solution du problème $(\mathcal{P})$ est variationnelle et se base sur le théorème de min-max due à Bartolo, Benci et Fortunato.

Pour la preuve des variantes nous approcherons le problème $(\mathcal{P})$ par une suite de problèmes $(\mathcal{P}_n)$ dont la troncature f_n de f vérifie les hypothèses des résultats cités plus haut, puis nous obtiendrons une suite de solutions bornée dans $H_0^1(\Omega)$, qui nous permettra le passage à la limite.

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