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# REPRESENTATION-FINITE ITERATED TILTED ALGEBRAS OF TYPE $\widetilde{\mathbb{D}}_n$

MICHAŁ WENDERLICH

Throughout the paper $k$ denotes a fixed algebraically closed field. We use the term algebra to mean a finite dimensional $k$–algebra and the term module to mean a finite dimensional right module.

Following [HR, B], a module $T$ over an algebra $A$ is called a *tilting module* provided the following conditions are satisfied

(T1) $\operatorname{Ext}^2_A(T,-)=0$
(T2) $\operatorname{Ext}^1_A(T,T)=0$
(T3) The number of nonisomorphic indecomposable direct summands of $T$ equals the rank of the Grothendieck group $K_0(A)$ of $A$.

Given a finite quiver $\Delta$ without oriented cycles, an algebra $A$ is called an *iterated tilted algebra of type* $\Delta$, see [AH], if there exists a sequence of algebras $A=A_0$, $A_1, \dots, A_m$, where $A_m$ is the path algebra of $\Delta$, and a sequence of tilting modules $T^i_{A_i}$,$(0\le i<m)$ such that $A_{i+1}=\operatorname{End}(T^i_{A_i})$ and every indecomposable $A_i$–module $M$ satisfies either $\operatorname{Hom}_{A_i}(T^i,M)=0$ or $\operatorname{Ext}^1_{A_i}(T^i,M)=0$. If $m\le 1$, $A$ is called a tilted algebra of type $\Delta$, see [HR].

The representation theory of iterated tilted algebras was proved to be related to that of self-injective algebras, see [AHR, ANS, BLR, H, HW, S]. They were also shown to arise naturally in the study of the derived category of bounded complexes of finite dimensional modules, see [H, HRS, AS2]. Iterated tilted algebras of type $\Delta$ where the underlying graph of $\Delta$ is a Dynkin diagram, were studied in [AH, AS3, H1, K], and the iterated tilted algebras of Euclidean type $\widetilde{\mathbb{A}}_m$ $(m\ge 1)$ were classified in [AS1]. Further, a complete description of the representation-infinite iterated tilted algebras of Euclidean type was given in [AS2]. It was also shown in [AS5] that a representation-finite algebra is an iterated tilted algebra of Euclidean type $\widetilde{\mathbb{D}}_n$ or $\widetilde{\mathbb{E}}_p$ if and only if it is simply connected and its (homological) quadratic form is positive semi-definite of corank one. Moreover an algebra is an iterated tilted algebra of Dynkin type if and only if it is simply connected and its quadratic form is positive definite (see [AS5]).

The purpose of this article is to give a complete classification, in terms of their bound quivers, of the representation-finite iterated tilted algebras of Euclidean type $\widetilde{\mathbb{D}}_n$ in a spirit similar to the classification of the iterated tilted algebras of type $\mathbb{D}_n$ presented in [AS3]. This completes the classification of the iterated tilted algebras of Dynkin and

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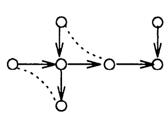
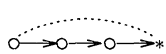
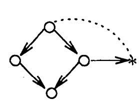
**1. The main result.** In this chapter we shall present a complete classification of the representation-finite iterated tilted algebras of type $\widetilde{\mathbb{D}}_n$. After the main theorem there are given two lists of algebras. The algebras on the left side of both lists will be called frames. By algebra of type (1) (respectively (2)) we mean an algebra of the form $A$ or $A^{\mathrm{op}}$ where $A$ is a full convex subcategory of an algebra on the right side of the List 1 (respectively List 2) and containing the frame on the left side of this list. Observe that, by [AS3, K], the class of iterated tilted algebras of type $\mathbb{D}_n$ coincides with the class of all algebras of type (2). We shall now describe two kinds of glueings of algebras of type (2), called *admissible glueings* of algebras of type (2).

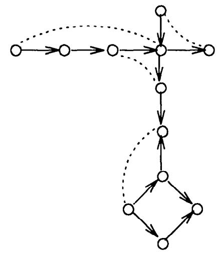
1. Let $A_1$ and $A_2$ be arbitrary algebras of type (2). Assume that $A_1$ and $A_2$ contain starred vertices $i_1$ and $i_2$ respectively, and there are no rooted branches (in the sense of [AS2]) at these vertices. Assume also that $i_1$ and $i_2$ are sinks, that is, $A_1$ contains an arrow $\alpha_1$ ending at $i_1$ and $A_2$ contains an arrow $\alpha_2$ ending at $i_2$. Let now $K$ be an arbitrary branch with at least three vertices and assume there are two different sources, say $j_1$ and $j_2$, which are starting vertices at exactly one arrow, say $\beta_1$ and $\beta_2$, respectively. Then the glueing of $A_1$ and $A_2$ by $K$ using the arrows $\alpha_1$, $\alpha_2$, $\beta_1$, $\beta_2$ is obtained by identifying $\alpha_1$ with $\beta_1$ and $\alpha_2$ with $\beta_2$.

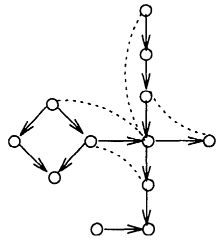
*Remark.* In all examples in this chapter all cycles are bound by a commutativity-relation, and a dotted line means a zero-relation.

**Example.** Let $K$, $A_1$, $A_2$ be the following bound quivers

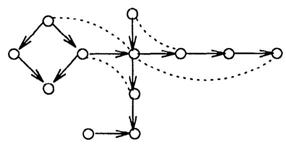
$K$  $A_1$  $A_2$ 

Then the following bound quiver algebras are examples of glueings of $A_1$ and $A_2$ by $K$.



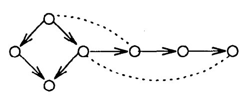


In case $i_1$ and $i_2$ are sources or one of them is a source and the other a sink we can make an analogous glueing. Taking the same algebras we may for example obtain



2. Let $A_1$ and $A_2$ be again of type (2), $A_1$ (respectively, $A_2$) contains an arrow $\xrightarrow{\alpha_1} *$ ending (respectively, $\xleftarrow{\alpha_2} *$ starting) at a starred vertex and there are no rooted branches at these starred vertices. Then the glueing of $A_1$ and $A_2$ using the arrows $\alpha_1$ and $\alpha_2$ is obtained by identifying $\alpha_1$ with $\alpha_2$.

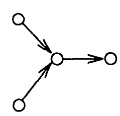
**Example.** If we take $A_1$ and $A_2$ as above then we obtain



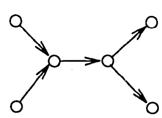
Observe that we may consider 2. as a special case of 1. when $K$ is just an arrow. Observe also that by these glueings we can obtain an algebra which contains a tame concealed algebra as a full subcategory.

**Example.** For

$A_1 = A_2$



we have a representation-infinite glueing



A representation-finite algebra obtained from algebras of type (2) by one of the glueings described above will be called an admissible glueing. Now we may formulate the main result of this paper.

**Theorem.** *Let $A$ be a representation-finite, finite dimensional, basic, connected algebra over an algebraically closed field. Then $A$ is an iterated tilted algebra of type $\widetilde{\mathbb{D}}_n$ if and only if $A$ is an algebra of type* (1) *or is an admissible glueing of two algebras of type* (2).

Let us note a few remarks concerning lists below. We assume again that any cycle is commutative and its number of vertices is greater than 3. A dotted line means a zero-relation. We also assume that we may root any extension (coextension) branch to any extension (coextension) vertex which is marked by a star. Observe also that 1.1′ and 1.4′ are special cases of 1.1 and 1.4 respectively.

**2. Preliminaries for the proof.** Recall that a quiver $Q$ is defined by its set of vertices $Q_0$ and its set of arrows $Q_1$. A *relation* from a vertex $x$ to a vertex $y$ is a linear combination $\rho = \sum_{j=1}^{m} \lambda_j w_j$, where, for each $1 \leq j \leq m$, $\lambda_j$ is a nonzero scalar, and $w_j$ is a path of length at least two from $x$ to $y$. The relation $\rho$ is called a *zero-relation* (respectively a *commutativity relation*) whenever $m = 1$ (respectively $m = 2$). The set of all relations on $Q$ generates an ideal $I$ in the path algebra $k$Q of $Q$. The pair $(Q, I)$ is called a *bound quiver*.

We shall usually assume that an algebra $A$ is basic and connected. In this case, there exists a connected bound quiver $(Q_A, I)$ and an isomorphism $A \simeq kQ/I$, see [G]. We shall denote by mod $A$ the category of finite dimensional right $A$–modules. For a vertex $i$ belonging to $Q_A$, we denote by $e_i$ the corresponding primitive idempotent of $A$, by $S(i)$ the corresponding simple $A$-module, and by $P(i)$ (respectively $I(i)$) the projective cover (respectively the injective hull) of $S(i)$. Also, we recall from [BG] that a bound quiver algebra $A \simeq kQ/I$ can equivalently be considered as a $k$–category of which the object class $A_0$ is the set $Q_0$, and the set of morphisms $A(x, y)$ from $x$ to $y$ is the quotient of the vector space $kQ(x, y)$ of all linear combinations of paths in $Q$ from $x$ to $y$ by the subspace $I(x, y) = I \cap kQ(x, y)$. A full subcategory $C$ of $A$ is called *convex* if any path in $A$ with source and target in $C$ lies entirely in $C$. Finally, a $k$–category $A$ is *Schurian*, if, for each pair $x, y \in A_0$, $\dim_k A(x, y) \leq 1$.

An algebra $A$ is called *simply connected* if it is triangular (that is, its ordinary quiver has no oriented cycles) and, for any presentation $A \simeq kQ/I$ of $A$ as a bound quiver algebra, the fundamental group of the bound quiver $(Q, I)$ (in the sense of [MP]) is trivial; see [AS4]. A representation-finite algebra is simply connected if and only if it is simply connected in the sense of [BG]. It was shown in [A] that iterated tilted algebras of Dynkin type are simply connected and in [AS4] that an iterated tilted algebra of Euclidean type is simply connected if and only if it is of type $\widetilde{\mathbb{D}}_n$ or $\widetilde{\mathbb{E}}_p$.

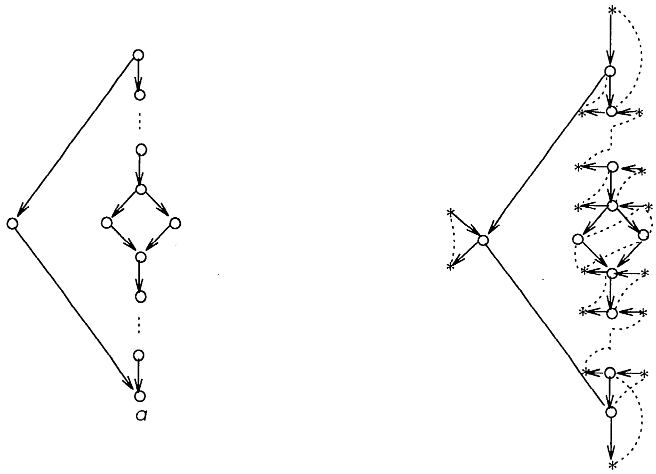
The dual notion of a tilting module is that of cotilting module: a module $T$ is called a *cotilting module* if it satisfies (T2), (T3) and

(T1') $\mathrm{Ext}_A^2(-, T) = 0$.

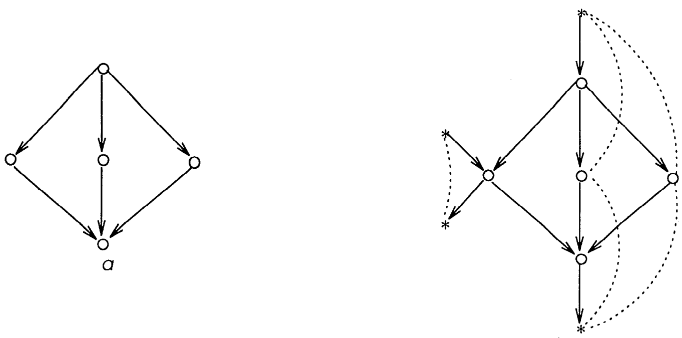
Two algebras $A$ and $B$ are *tilting-cotilting equivalent* if there exists a sequence of algebras $A = A_0, A_1, \ldots, A = B$ and a sequence of modules $T_{A_i}^i$ $(0 \leq i < m)$ such that $A_{i+1} = \mathrm{End}(T_{A_i}^i)$ and $T_{A_i}^i$ is either a tilting or a cotilting module. It follows from [HRS] that an algebra $A$ is iterated tilted of type $\Delta$ if and only if $A$ and $k\Delta$ are tilting-cotilting equivalent.

LIST 1.

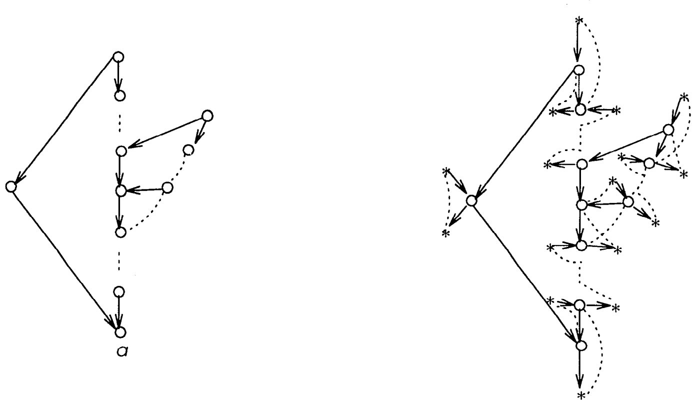
1.1



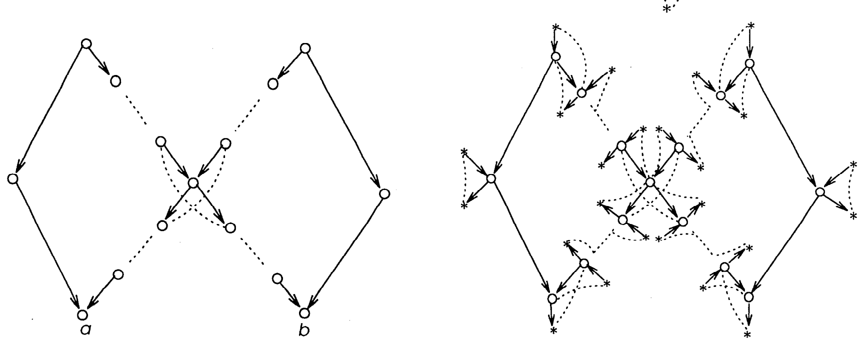

1.1′



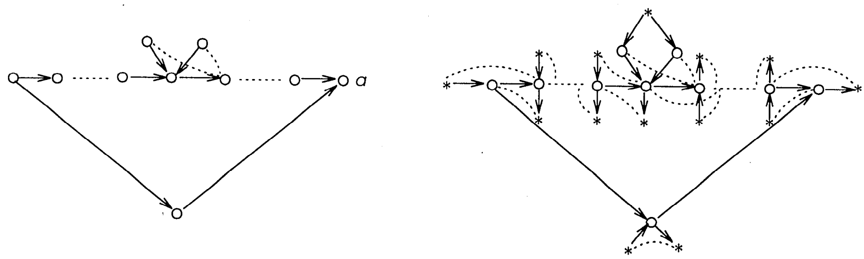

1.2



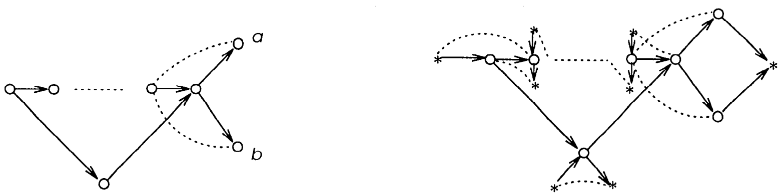

1.3



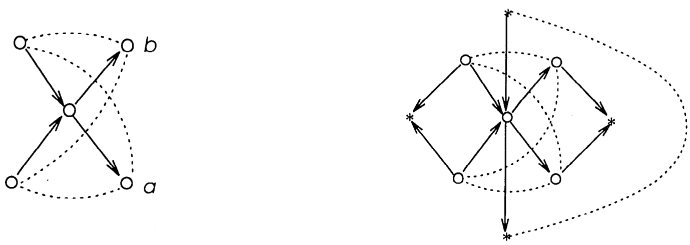
1.4



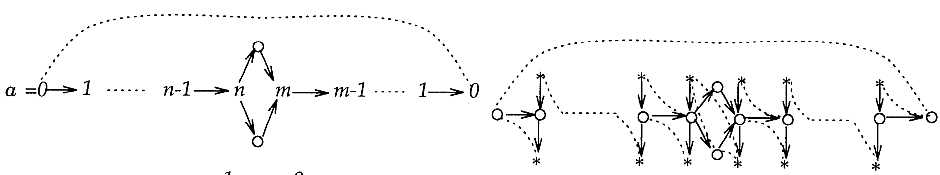

1.4′




1.5

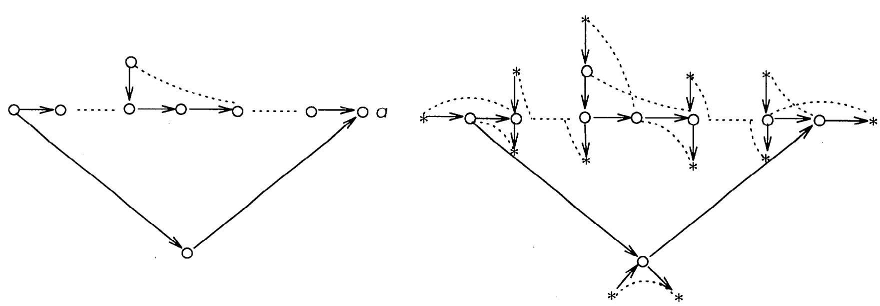



1.6

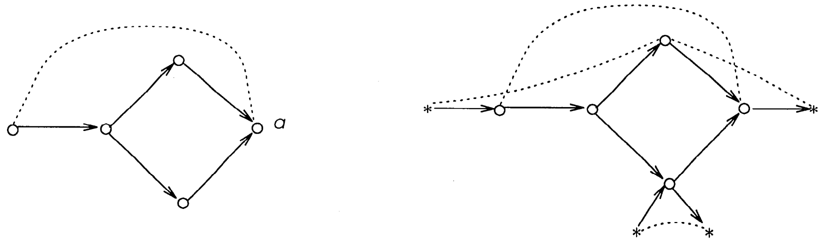



$n \geq 1,\ m \geq 0$

1.7

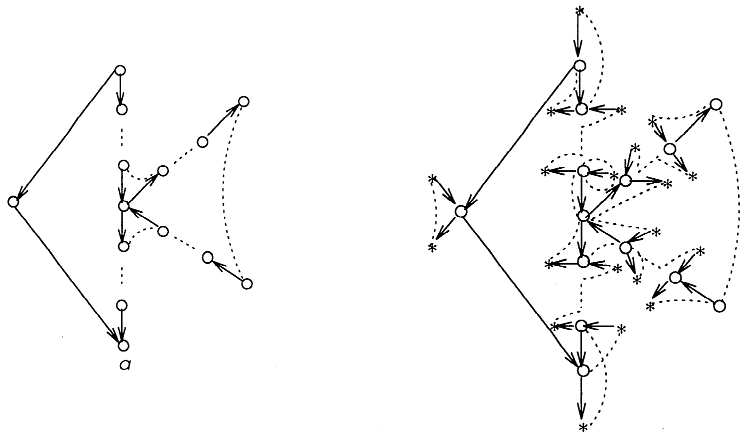



1.8

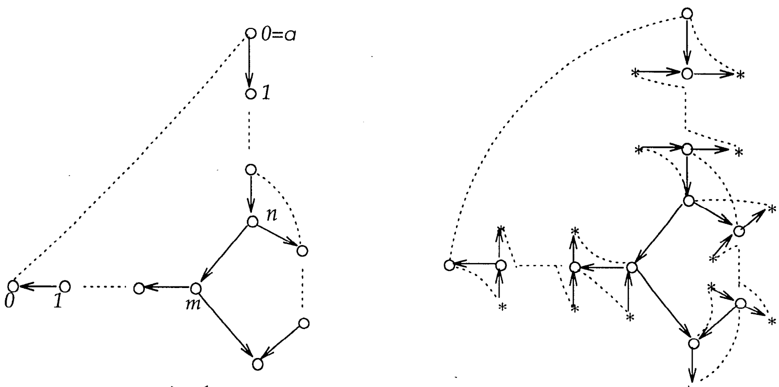


1.9


1.10



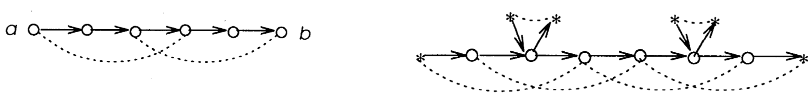

1.11



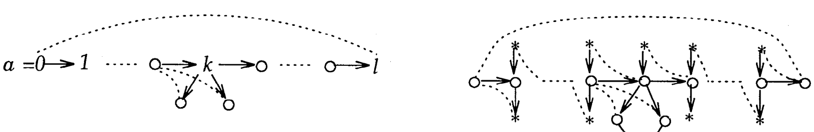

$m, n \geq 1$

1.12


1.13

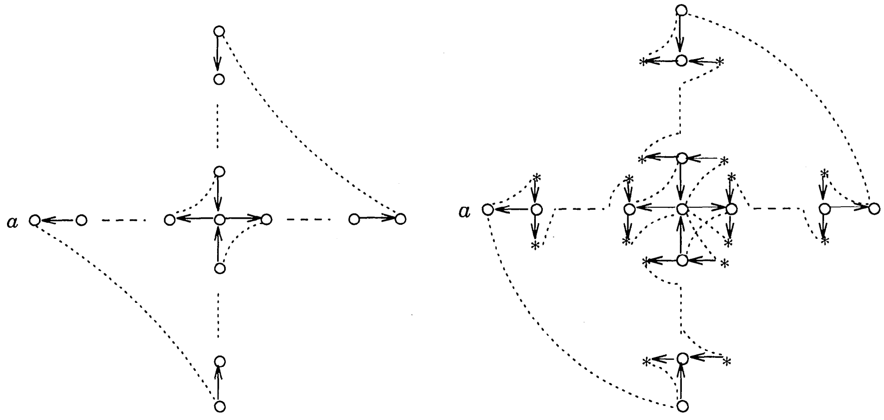



1.14

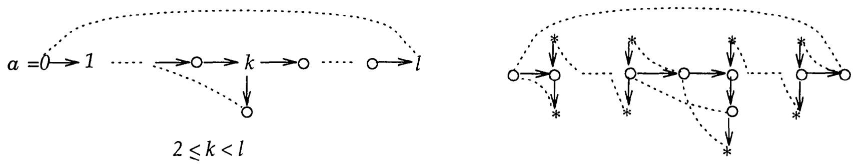



$1 \leq k < l$

1.15




1.16




Let $A$ be an algebra and $M$ be an $A$-module. The one-point extension of $A$ by $M$ is the matrix algebra

$$A[M] = \begin{bmatrix} A & 0 \\ M & k \end{bmatrix}$$

with the usual addition and multiplication of matrices. The quiver of $A[M]$ contains $Q_A$ as a full subquiver and there is an additional (extension) vertex which is a source. Dually, the one-point coextension of $A$ by $M$ is the algebra

$$[M]A = \begin{bmatrix} k & 0 \\ DM & A \end{bmatrix}.$$

Its quiver contains $Q_A$ as a full subquiver and there is an additional (coextension) vertex which is a sink.

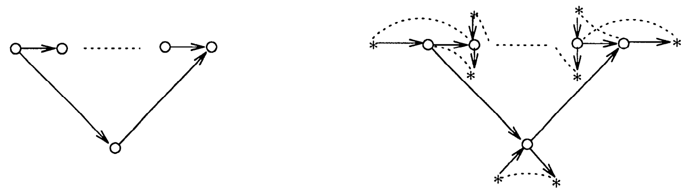
Let A be a triangular algebra, and $i$ be a sink in $Q_A$ .The reflection $S_i^+A$ of $A$ at $i$ is the quotient of the one-point extension $A[I(i)]$ by the two-sided ideal generated by $e_i$. It is shown in [TW] that $A$ and $S_i^+A$ are tilting-cotilting equivalent. The sink $i$ of $Q_A$ is replaced in the quiver $\sigma_i^+Q_A$ of $S_i^+A$ by a source. A reflection sequence of sinks $i_1, \dots, i_p$ is a sequence of vertices of $Q_A$ such that $i_p$ is a sink in $\sigma_{i_{p-1}}^+ \dots \sigma_{i_1}^+ Q_A$ for $1 \leq p \leq m$. Dually starting with a source in $Q_A$ we define the reflection $S_j^- A$ of $A$ at the source $j$. It follows immediately from[ANS, 3.4] and the main theorem of [AS4] that if $A$ is a representation-finite iterated tilted algebra of type $\widetilde{\mathbb{D}}_n$, then there exists a reflection sequence of sinks $i_1, \dots, i_m$ such that $S_{i_{m-1}}^+ \dots S_{i_1}^+ Q_A$ is a representation-finite iterated tilted algebra of type $\widetilde{\mathbb{D}}_n$ and $S_{i_m}^+ \dots S_{i_1}^+ Q_A$ is a representation-infinite algebra of the same type.

We shall need the following lemma.

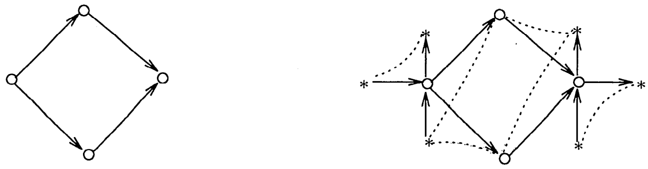
**Lemma.** *Let $A$ be a representation-finite simply connected algebra and $i$ be a sink in its quiver. Then $S_i^+A$ is Schurian.*

LIST 2.

2.1



2.2



2.3


2.4

2.5

2.6

2.7

2.8

*Proof.* [AS5, 1.4]. □

For the next notion we need, that is, the notion of branch enlargement, we refer the reader to [AS2] (see also [ANS, AS3, AS5]).

**Proposition.** *Let $A$ be an algebra. Then $A$ is a representation-infinite iterated tilted of Euclidean type $\Delta$ if and only if there exists a (unique ) tame concealed full convex subcategory $C$ of $A$ such that $A$ is a branch enlargement of $C$ and its tubular type $n_A$ is one of the following types $(p,q)$, $p \le q$, $(2,2,r)$ $2 \le r$, $(2,3,3)$, $(2,3,4)$ or $(2,3,5)$. Moreover, in this case, $n_A$ equals the tubular type $n_{k\Delta}$ of the hereditary algebra $k\Delta$.*

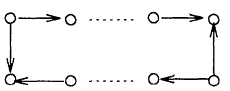
*Proof.* See [AS2]. □

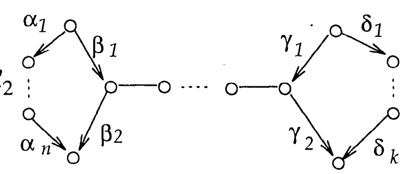
Thus we have that an algebra A is representation-infinite iterated tilted of type $\widetilde{\mathbb{D}}_{r+2}$, $2 < r$, if and only if there exists a tame concealed full convex subcategory $C$ of $A$ such that $A$ is a branch enlargement of $C$ of type $n_A = (2,2,r)$.

An application of Bongartz' criterion (see [B]) of representation-finiteness is the following corollary.

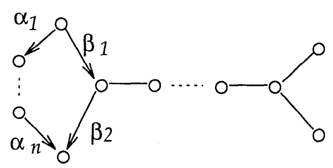
**Corollary.** *Let $A$ be an iterated tilted algebra of type $\widetilde{\mathbb{D}}_n$. Then $A$ is representation-finite if and only if $A$ does not contain a tame concealed algebra as a full convex subcategory.*

It follows from the list of tame concealed algebras, given in [HV], that if $A$ is a representation-infinite Schurian iterated tilted algebra of type $\widetilde{\mathbb{D}}_n$ then its unique tame concealed full convex subcategory is one of the following

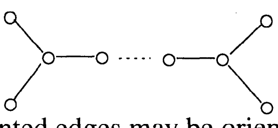
$C_1$


$C_2$



bound by $\alpha_1 \cdots \alpha_n = \beta_1\beta_2$, $\delta_1 \cdots \delta_k = \gamma_1\gamma_2$

$C_3$



bound by $\alpha_1 \cdots \alpha_n = \beta_1\beta_2$

$C_4$


where unoriented edges may be oriented arbitrarily.

We have also the following consequence of the above corollary and the classifications of the iterated tilted algebras of types $\mathbb{A}_n$, see [AH], $\mathbb{D}_n$, see [AS3,K] and $\widetilde{\mathbb{A}}_n$, see [AS1].

**Corollary.** *Let $A$ be a representation-infinite iterated tilted algebra of type $\widetilde{\mathbb{D}}_n$ which has one of the above tame concealed algebra as a connected full subcategory and let $B$ be a connected full subcategory of $A$. Then $B$ is an iterated tilted algebra of one of the following types: $\widetilde{\mathbb{D}}_m$, $\widetilde{\mathbb{A}}_m$, $\mathbb{D}_m$ or $\mathbb{A}_m$ where $m \le n$.*

**Lemma.** *Let $A$ and $B$ be as in the above corollary and let $i$ be a sink in $A$. Then*

(i) *If $i$ does not belong to $B$ then $B$ is also a full subcategory of the algebra $S_i^+A$.*
(ii) *If $i$ belongs to $B$, then the full subcategory of $S_i^+A$ formed by all objects of $B$ except $i$, and the new extension vertex, is isomorphic to $S_i^+B$.*

*Proof.* The first part is obvious, the second follows directly from the construction of the algebras $S_i^+A$ and $S_i^+B$. □

Clearly we have a similar statement for a source $j$ and algebra $S_j^-A$. Combining all these results we obtain

**Corollary.** *Let $A$ be a representation-finite iterated tilted algebra of type $\widetilde{\mathbb{D}}_n$ and $B$ a connected full subcategory of $A$. Then $B$ is an iterated tilted algebra of one of the following types: $\widetilde{\mathbb{D}}_m$, $\widetilde{\mathbb{A}}_m$, $\mathbb{D}_m$ or $\mathbb{A}_m$, where $m \leq n$.*

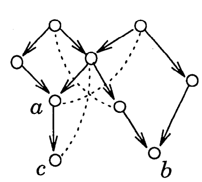
These corollaries are used in the necessity part of the proof of the main theorem to eliminate bound quivers which contain full subcategories which are iterated tilted of a type different from $\widetilde{\mathbb{D}}_m$, $\widetilde{\mathbb{A}}_m$, $\mathbb{D}_m$ and $\mathbb{A}_m$ (such as $\mathbb{E}_6$).

## 3. Proof of the sufficiency.

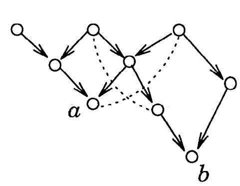
We shall start with a useful remark. Let $C$ be a tame concealed algebra of type $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$, $\mathcal{C}_4$. Let $M$ be a simple regular $C$-module, $K$ an extension branch (which may be empty) and $L$ a nonempty coextension branch. Let $B$ be a branch enlargement of $C$ by branches $K$ and $L$ using the module $M$. Then by [ANS] there exists a reflection sequence of sinks $i_1, \dots, i_k$ such that $S_{i_k}^+ \dots S_{i_1}^+ B$ is still a branch enlargement of $C$ using the same module $M$ but a coextension branch is empty. A dual result also holds.

We may apply this fact in our situation, that is, we may assume that vertices in List 1 marked by letters $a$ and $b$ are sinks or sources (if this is not the case, then there exists a reflection sequence of sources or sinks such that after applying reflection at them the frame will remain unchanged, vertices $a$ and $b$ will become sinks or sources and the algebra will be of the same type). Then after applying the reflections at these vertices we obtain a representation-infinite iterated tilted algebra of type $\widetilde{\mathbb{D}}_n$.

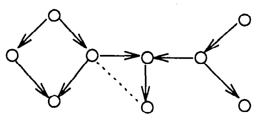
**Example.** If $A$ is of the form (again the cycles are commutative)




then $A' = S_c^+A$ is the following



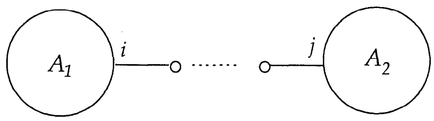
and $S_b^+ S_a^+ A$ is of the form



Hence $A$ is tilting-cotilting equivalent to a representation-infinite iterated tilted algebra of type $\widetilde{\mathbb{D}}_n$ and does not contain a tame concealed algebra as a full convex subcategory and consequently it is a representation-finite iterated tilted algebra of type $\widetilde{\mathbb{D}}_n$.

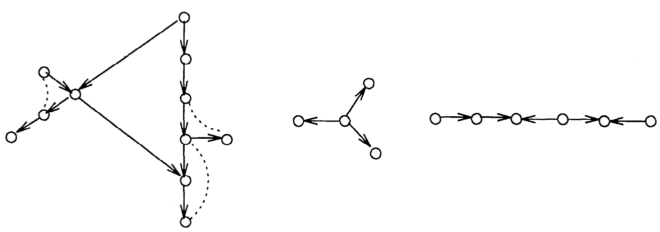
Let now $A$ be obtained by an admissible glueing of two algebras $A_1$ and $A_2$ of type (2). We shall consider two cases

*(i).* $A$ is either obtained by a glueing of the second kind or of the first kind but the branch $K$ is a line (not bound by any relation). Then it is easy to check that $A$ is representation-finite iterated tilted of type $\widetilde{\mathbb{D}}_n$. Observe that $A$ is of the following form

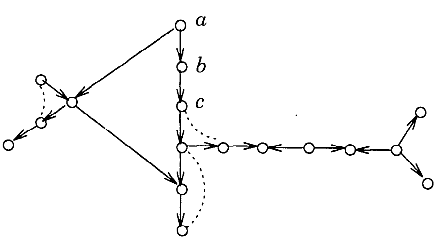



where $A_1$ and $A_2$ are of type (2) and possibly $i = j$. Consequently there exists a reflection sequence of sinks and sources belonging to $A_1$ or $A_2$ such that after applying reflections at them changes only one of $A_1$ and $A_2$, and $K$ remains unchanged. It is enough to check that there exists such a reflection sequence of sinks and sources such that after applying reflections at them we obtain a representation-infinite iterated tilted algebra of type $\widetilde{\mathbb{D}}_n$. It is just a quite easy and routine verification. We shall illustrate it on an example.

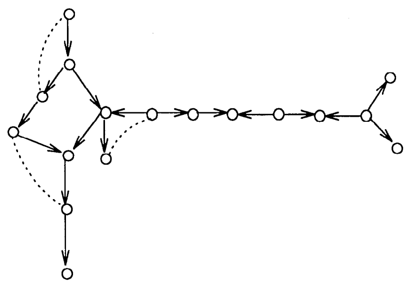
**Example.** Let $A_1$, $A_2$ and $K$ be the following (with the cycle commutative)



and A be of the form



then $S_c^- S_b^- S_a^- A$ is the following



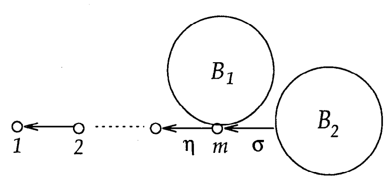
and it is obviously a representation-infinite iterated tilted algebra of type $\widetilde{\mathbb{D}}_{16}$.

In the same way we check that all the remaining algebras are representation-finite iterated tilted of type $\widetilde{\mathbb{D}}_n$.

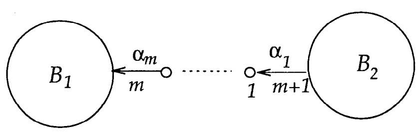
*(ii).* $A$ is obtained from $A_1$ and $A_2$ by a branch $K$ bound by at least one zero-relation. We shall reduce this case to the previous one.

First recall that for any algebra $D$ and a sink $i$ of $D$ we may consider the APR-tilting module corresponding to $i$ $T_D = \tau^{-1}(e_i D) \oplus (1 - e_i)D$ (see [APR]) and the algebra $B = \mathrm{End}_D(T_D)$ called an *APR-tilt* of $D$. The precise description of the bound quiver $(Q_B, I_B)$ is given in [AS3, Lemma 7.4], and we shall use this description several times. Moreover some of the combinatorics we shall do can also be done by using Lemma 7.4 of [AS3]. For the convenience of the reader we recall now from [AS2, 2.4] a procedure of making a line from a branch, called linearizing.

Let $B$ be a branch enlargement of a tame concealed algebra $C$ such that at least one branch has a zero-relation. By passing, if necessary, to the opposite algebra, we may assume that there exists a sink in this branch so that the bound quiver of $B$ has the following form

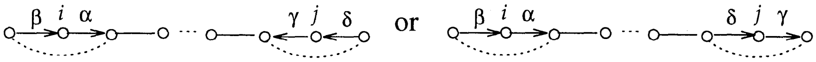



where $\sigma\eta = 0$, one of $B_1$ and $B_2$ is a branch, while the other contains the quiver of $C$. Then it was shown in [AS2, 2.4] that $B$ is tilting-cotilting equivalent to $D$ which has the following form

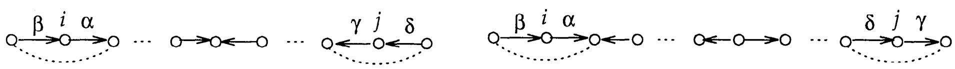



where $B_1$ and $B_2$ remains unchanged. Moreover the linear subquiver $m+1 \to 1 \to \cdots \to m-1 \to m$ is not bound, and there exists a relation of the form $\nu\alpha_1$ in $D$ if and only if there exists the corresponding relation $\nu\sigma = 0$ in $B$.

We may apply the same procedure in our situation, when $B = A$, $B_1 = A_1 \cup K_1$, $B_2 = A_2 \cup K_2$, where $K_1$ and $K_2$ are disjoint parts of $K$, $K_3 = 1 \leftarrow 2 \leftarrow \cdots \leftarrow m-1 \xleftarrow[\eta]{} m \xleftarrow[\sigma]{} m+1$ is also a part of $K$ and $K = K_1 \cup K_2 \cup K_3$. It is clear that after linearizing of $A$ we obtain a new algebra $A'$ (tilting-cotilting equivalent to $A$) consisting of $A_1$, $A_2$ and a line $K'$ (possibly bound by some zero-relations) connecting $A_1$ and $A_2$. Assume now that the vertices $i, j$ belong $K'$, $i \neq j$ and there exist arrows $\alpha, \beta, \gamma, \delta$ such that $i$ is a source of $\alpha$ and a sink of $\beta$, $j$ is a source of $\gamma$ and a sink of $\delta$, $\alpha\beta = 0 = \gamma\delta$ and there are no other arrows between $i$ and j involved in a zero-relation, that is, it is of one of the forms

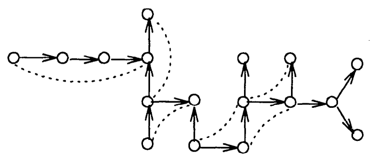



where unoriented edges may be oriented arbitrarily. Then applying APR-tilting modules at vertices between $i$ and $j$, the part of $K'$ lying between $i$ and $j$ takes one of one of the forms

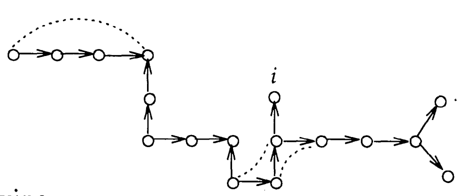



Observe that the two parts of $A'$ out of the part between $i$ and $j$ remain unchanged. After applying the above process to $K'$ we obtain a new line $K''$ called an ordered line.

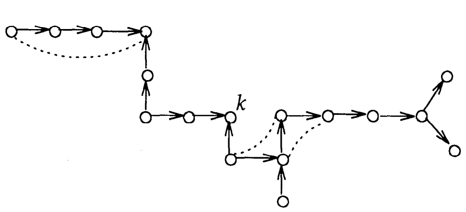
**Example.** Let $A$ be given by the following bound quiver



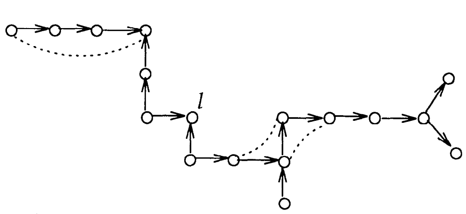
After linearizing we obtain $A'$ of the form



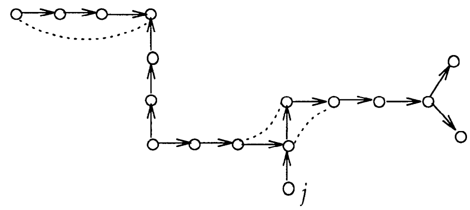

Then $S_i^+ A'$ is the following




Applying the APR-tilting module at $k$ we obtain

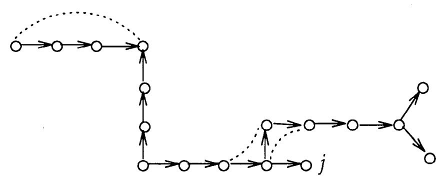


and now we apply the APR-tilting module at $l$ obtaining

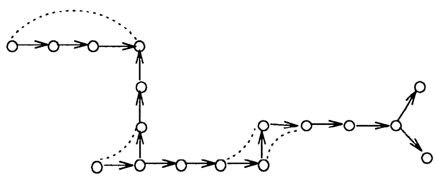



Then after applying the APR-tilting module at $j$ we obtain

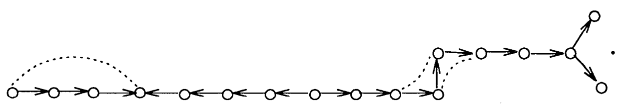
$A''$



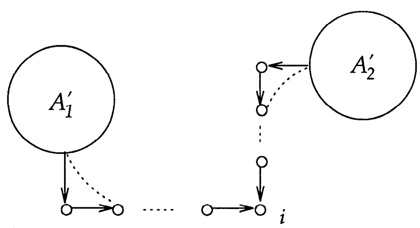

Further $S_j^+ A''$ is of the form



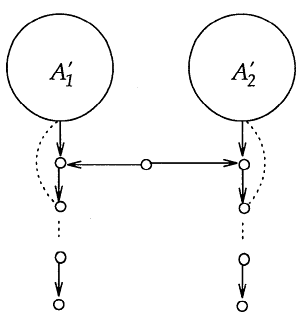
Applying again the linearizing we obtain that $S_j^+ A''$ is tilting-cotilting equivalent to A of the form



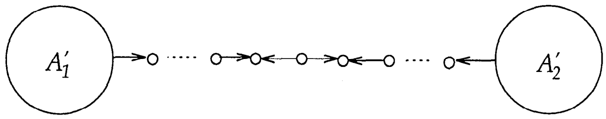
Suppose now that after applying APR-tilting modules and linearizing we have obtained an algebra consisting of $A_1$ and $A_2$ (which are of type (2)) and an ordered line $K''$. Suppose that we have a situation



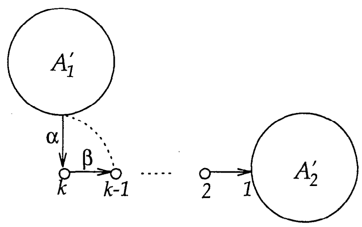

where $A_i'$ contains $A_i$, for $i = 1, 2$. Then $S_i^+ A$ has the form



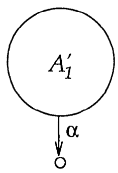
and after linearizing we obtain the following algebra



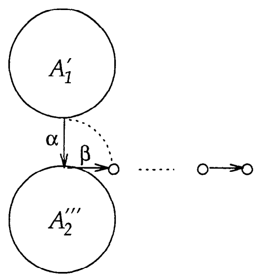

We may obviously do the same with $A^{\mathrm{op}}$. Let now $A$ be the following



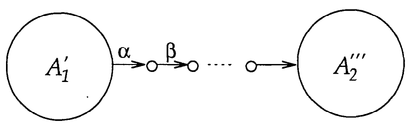

where $A'_1$ contains $A_1$ and $A'_2 = A_2$. Observe first that because of $\beta\alpha = 0$ then after applying any reflection to $A$ at a vertex belonging to $A'_2$ the part




remains unchanged. Let $A''_2$ be a full subcategory of $A$ with vertex set consisting of those of $A'_2$ and 1, 2, ..., $k$. Then $A''_2$ is an algebra of type (2). Let $i_1, \dots, i_s$ be a reflection sequence of sinks belonging to $A'_2$ such that $i_s = 1$. Then $A' = S^+_{i_1} \dots S^+_{i_s} A$ has the following form

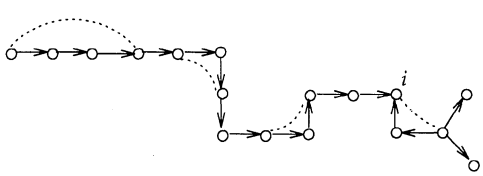



where $A_2$ is of type (2). After linearizing we obtain the algebra

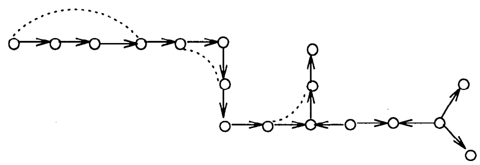



Applying the procedures described above we may remove all zero-relations from $K''$. We shall illustrate this process on an example.

**Example.** Let $A$ be the following

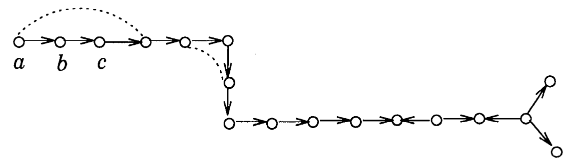


then $S_i^+ A$ has the form

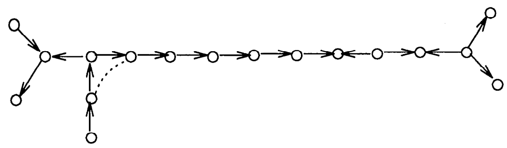


and it is tilting-cotilting equivalent to

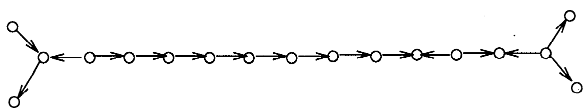
$A'$




But $S_c^- S_b^- S_a^- A'$ is the following



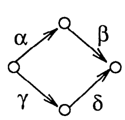
and after linearizing we obtain an algebra of type $\widetilde{\mathbb{D}}_{17}$.

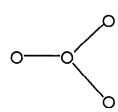
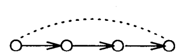
 .

In the same way we can show that all admissible glueings of algebras of type (2) are representation-finite iterated tilted of type $\widetilde{\mathbb{D}}_n$. This finishes the sufficiency part of the proof.

**4. Proof of the necessity.** In order to prove the necessity part of the theorem we need three lemmas.

**Lemma 1.** *Let $A$ be an algebra of type* (1) *consisting of one frame from the left side of List* 1, *one starred vertex $i$ and an arrow connecting $i$ to the frame. Let $B$ be obtained from $A$ and one of the algebras*

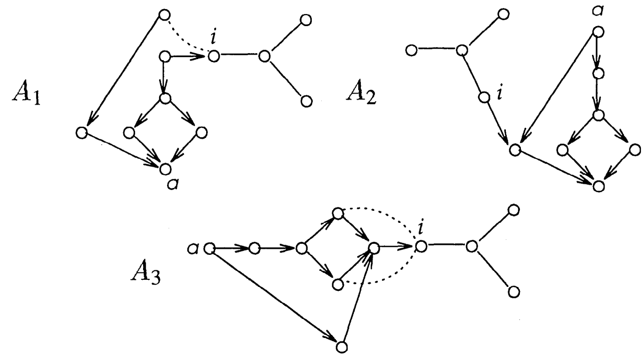


*bound by $\alpha\beta = \gamma\delta$,*  

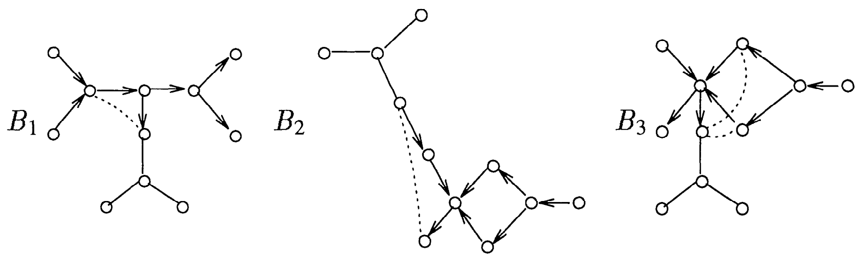
*by identifying $i$ with one of the vertices of these algebras. Then $B$ is not iterated tilted of type $\widetilde{\mathbb{D}}_n$.*

*Proof.* It is just an easy, routine verification. □

**Example.** Consider the following glueings of algebras of type 1.1




(with all cycles commutative). Then $B_1 = S_a^+A_1$, $B_2 = S_a^+A_2$, $B_3 = S_a^+A_3$ are the following

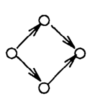



(with all the cycles commutative). They are representation-infinite but they are not iterated tilted of type $\widetilde{\mathbb{D}}_n$. Indeed, each of them contains a tame concealed algebra as a full convex subcategory, but they are not branch enlargements of these algebras.

In the same way we check all other possible cases.

**Lemma 2.** *Let $A$ be a frame from the left side of List* 1. *Assume that $B$ is either*

a) *obtained from $A$ by adding a new vertex $i$, an arrow $\alpha$ connecting $i$ with $A$, possibly some relations involving $\alpha$ and such that $B$ is not of type* (1) *or*
b) *obtained from $A$ and the commutative square*

$D =$ 

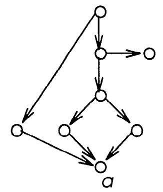
*by identifying a vertex or some arrows of $A$ and $D$, and possibly adding some relations involving arrows from $D$.*

*Then $B$ is not an iterated tilted algebra of type $\widetilde{\mathbb{D}}_n$.*

*Proof.* It is also an easy task. The general idea is the following. First we may assume that the vertices marked by $a$ and $b$ in the algebras from the List 1 are sinks or sources (if this is not the case we may apply some reflections not changing $A$ as long as $a$ and $b$ will not be sinks or sources). Next, applying reflections at these vertices we obtain an algebra which contains a tame concealed algebra as a full convex subcategory but is not a branch enlargement of this algebra of type $(2, 2, r)$, $r \leq 2$. □

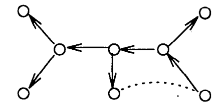
**Example.** If $B$ is the following extension of an algebra of type 1.1

$B$



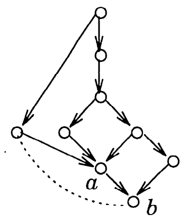

(with both cycles commutative)

then $S_a^+B$ is the following



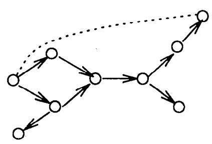
and it is not an iterated tilted algebra of type $\widetilde{\mathbb{D}}_n$ because it is not a branch enlargement of its unique tame concealed subcategory.

If $B$ is of the form




(with all cycles commutative)

then $S_a^+S_b^+B$ is the following



(with the cycle commutative)

and obviously again it is not an iterated tilted algebra of type $\widetilde{\mathbb{D}}_n$.

In the same way we check all other possible cases.

As a consequence we obtain that any representation-finite iterated tilted algebra of type $\widetilde{\mathbb{D}}_n$ containing a frame from the List 1 is of type (1).

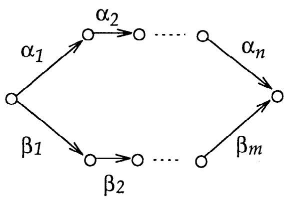
**Lemma 3.** *Let $A$ be an iterated tilted algebra of type $\widetilde{\mathbb{D}}_n$ containing two algebras of type* (2) *and a branch (possibly empty) as full convex subcategories. Then $A$ is an admissible glueing of these algebras.*

*Proof.* The proof is exactly the same as above. □

Observe now that admissible glueings of two frames of type (2) except 2.1 and 2.2 (possibly by branch being a line possibly bound by some zero-relations) and the frames of type (1) are minimal in the sense that any proper full convex subcategory is not an iterated tilted algebra of type $\widetilde{\mathbb{D}}_n$. It may happen that some admissible glueings of an algebra of type 2.1 and some other (but not of type 2.2) are also minimal. Observe also that admissible glueings of a frame of type 2.1 or 2.2 and any other frame of type (2) possibly by a line (with possibly some zero-relations) are also minimal in the sense that no proper full, convex subcategory of this glueing, such that if one vertex of the cycle belongs to this subcategory then all the cycle belongs to it, is an iterated tilted algebra of type $\widetilde{\mathbb{D}}_n$.

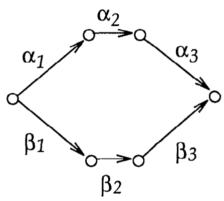
It follows immediately from the above lemmas that it is enough to show that any representation-finite iterated tilted algebra of type $\widetilde{\mathbb{D}}_n$ must contain a minimal algebra (in one of the above senses), that is it must contain a frame of type (1) or an admissible glueing of two frames of type (2). We shall show it in several steps.

1. Assume first that the algebra has a cycle. Since representation-finite iterated tilted algebras of type $\widetilde{\mathbb{D}}_n$ are simply connected this cycle must be of the following form



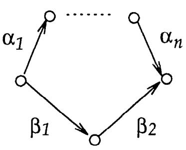

bound by $\alpha_1\alpha_2\ldots\alpha_n = \beta_1\beta_2\ldots\beta_m$.

But the algebra




bound by $\alpha_1\alpha_2\alpha_2 = \beta_1\beta_2\beta_3$

is tilted of type $\mathbb{E}_6$ (see [H1]) so our cycle is of the following form




bound by $\alpha_1\ldots\alpha_n = \beta_1\beta_2$

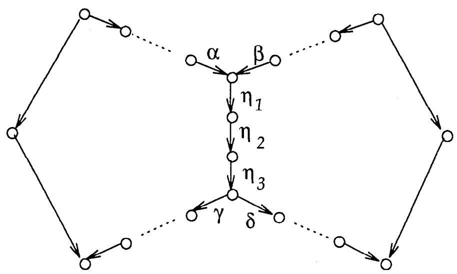
and it is a tilted algebra of type $\mathbb{D}_{n+1}$.

From now on, we shall assume that each such cycle is commutative.

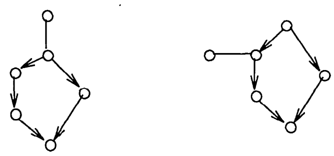
a). We shall now find all algebras which contain two such cycles having at least one common vertex.

*(i).* Observe first that these algebras cannot share three or more arrows. Consider an algebra

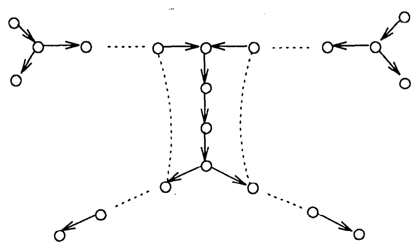
$A$



Because the algebras



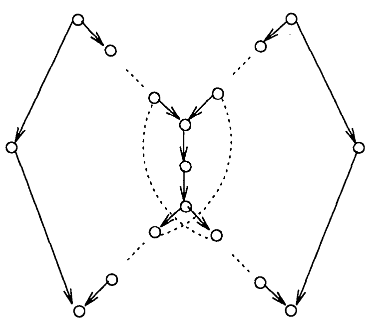
are tilted of type $\mathbb{E}_6$ it follows that in $A$ there are two zero-relations containing paths $\alpha\eta_1\eta_2\eta_3\delta$ and $\beta\eta_1\eta_2\eta_3\gamma$. Then $S_1^+S_2^+A$ is of the form



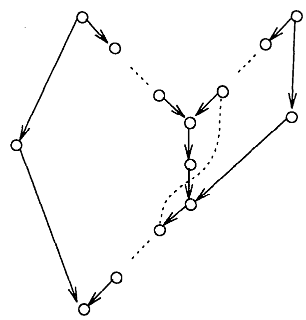
and this algebra contain a wild hereditary algebra as a full convex subcategory, a contradiction.

*(ii).* Assume now that two cycles share 2 arrows. Up to duality we have the following possibilities (algebras must be representation-finite)

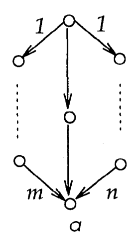
$A_1$




$A_2$

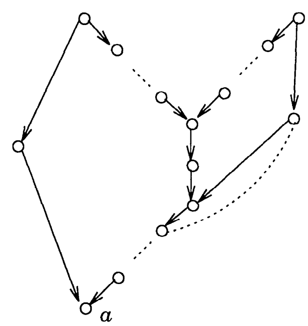



$A_3$

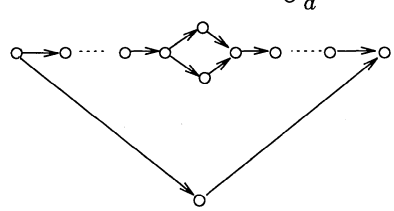



$m \geq 3$ or $n \geq 3$

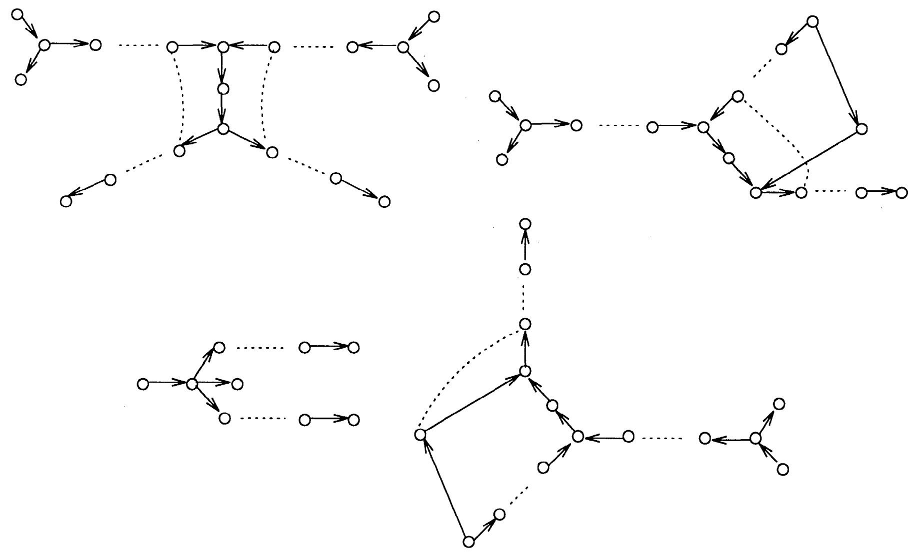
$A_4$




$A_5$



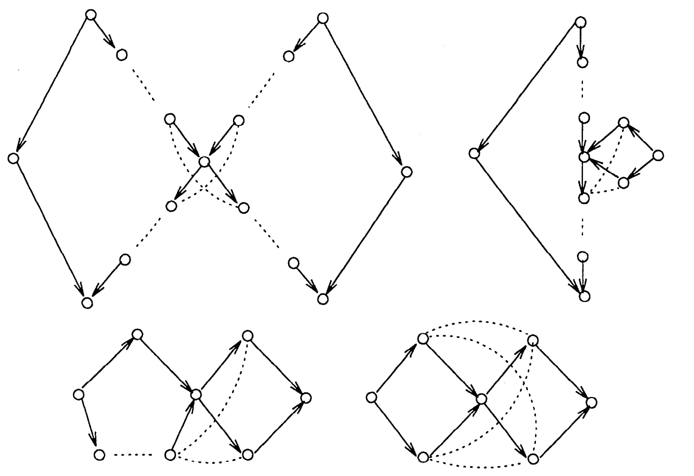
But $S_a^+S_b^+A_1$, $S_a^+A_2$, $S_a^+A_3$, $S_a^+A_4$ are the following



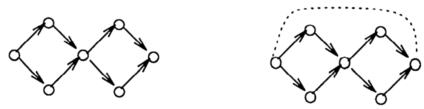
Observe that $S_a^+S_b^+A_1$ and $S_a^+A_3$ contain a wild algebra, $S_a^+A_2$, $S_a^+A_4$ are representation-infinite but not iterated tilted of type $\widetilde{\mathbb{D}}_n$. Thus there remains only $A_5$ which is of type 1.1.

*(iii).* In the same way we prove that 1.2 is the only frame which contains two cycles sharing one arrow.

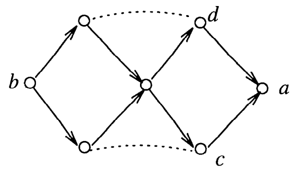
*(iv).* Again making the same analysis we see that if two cycles have exactly one common vertex then they are of one of the forms



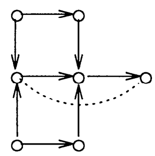
Observe that these algebras are of type 1.3, 1.4, 1.4′ and 1.5 respectively. In order to prove for instance that the fourth frame is the unique frame containing two commutative cycles connected by a sink from the first one and the source from the second one, and not of the form 1.4′ observe that the algebras



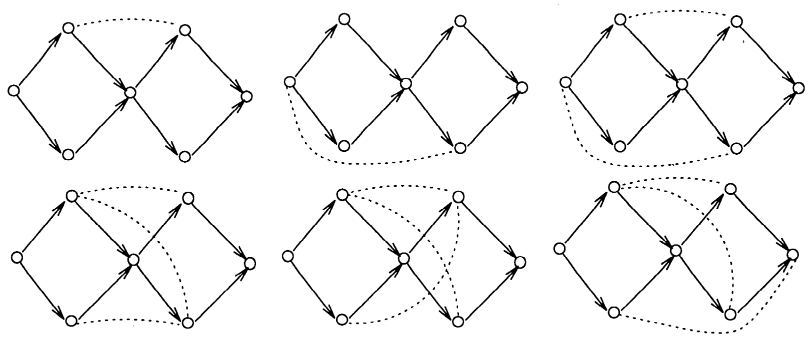
are representation-infinite. If $A$ is of the form




then $S_d^+ S_c^+ S_b^- S_a^+ A$ is the following

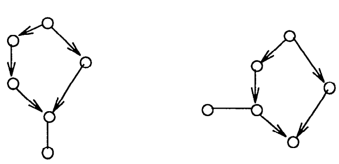


and this algebra contains a tilted algebra of type $\mathbb{E}_6$ as a full convex subcategory. In the same way we prove that the algebras

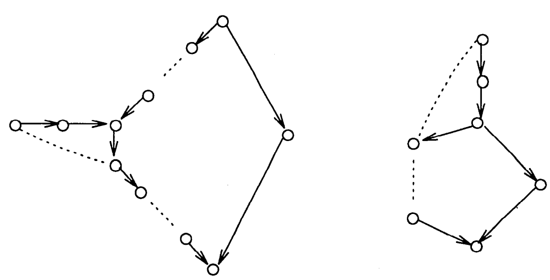


have as a full convex subcategory an algebra which is iterated tilted of type $\mathbb{E}_6$.

b). Let us consider a single cycle having at least 5 vertices. We shall find all iterated tilted algebras of type $\widetilde{\mathbb{D}}_n$ containing this cycle. First observe that the algebras

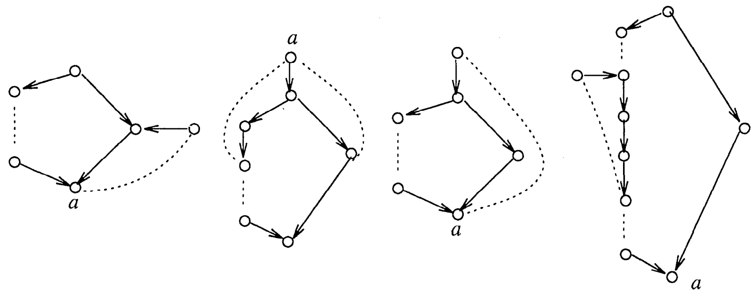


are tilted of type $\mathbb{E}_6$. Thus algebras like

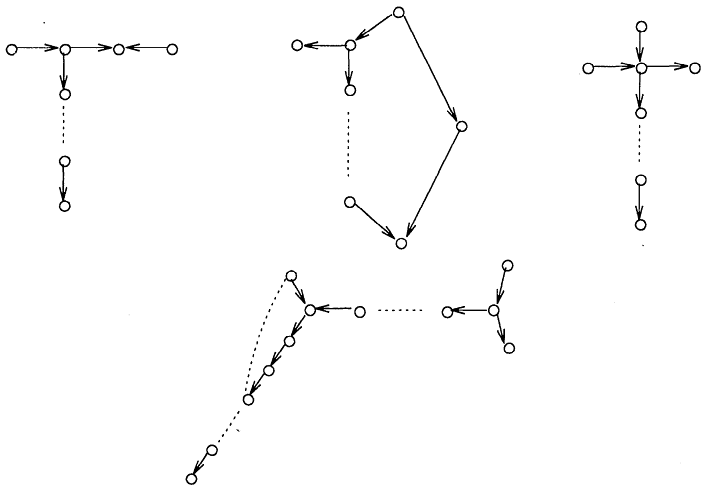


and any other containing one of the above two algebras as a full subcategory are not iterated tilted of type $\widetilde{\mathbb{D}}_n$

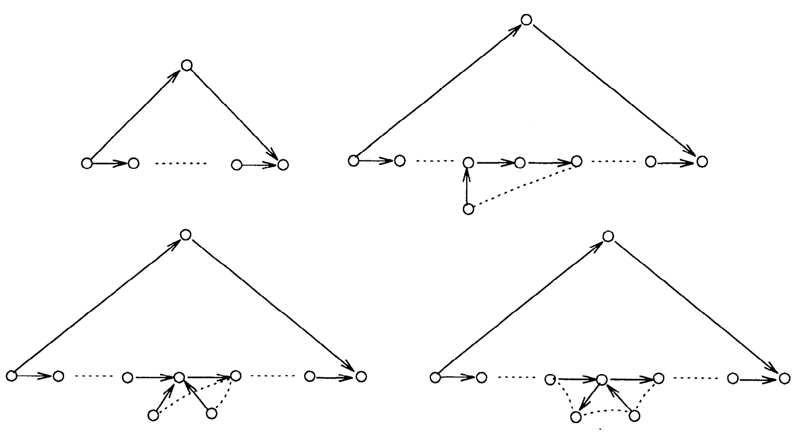
Let $A$, $B$, $C$, $D$ be the following



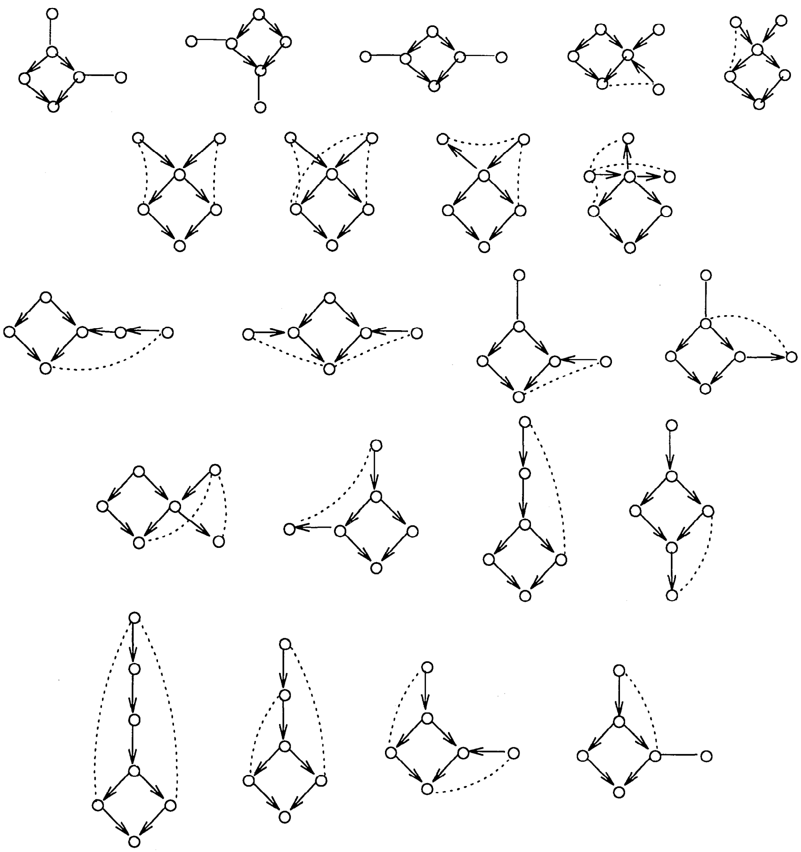

Then $S_a^+A$, $S_a^-B$, $S_a^+C$, $S_a^+D$ are of the respective forms



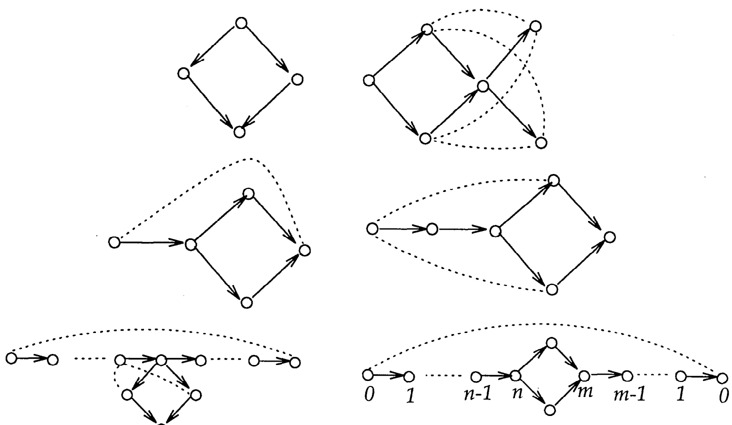
The first and the second algebra contain a tilted algebra of type $\mathbb{E}_6$ as a full subcategory, and the remaining ones are wild. Continuing such an analysis we obtain the following frames



to a wild algebra.



So we have the following frames

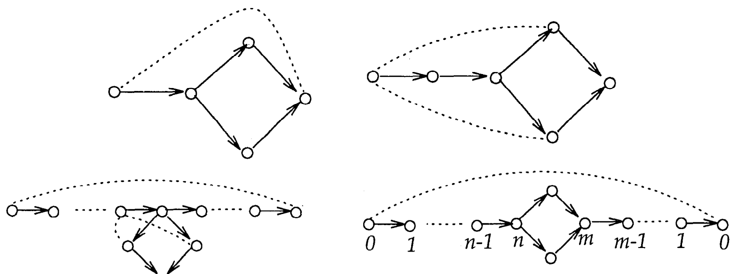



$n \geq 1$ and $m \geq 0$

They are of types 2.2, 1.5, 1.8, 1.9, 1.14, 1.6 respectively.

2. Now we shall find iterated tilted algebras of type $\widetilde{\mathbb{D}}_n$ without a cycle.

a). First consider the situation when the algebra has a zero-relation of length $\geq 4$. It is easy to see (by applying reflections) that the first four among the following algebras

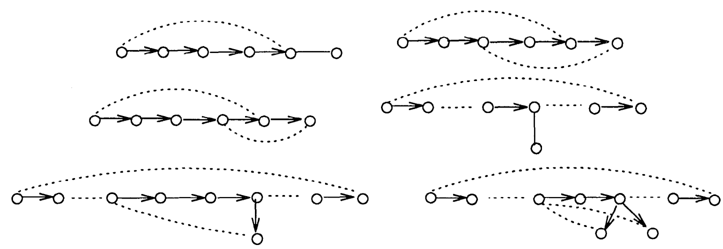



$n \geq 1$ and $m \geq 0$

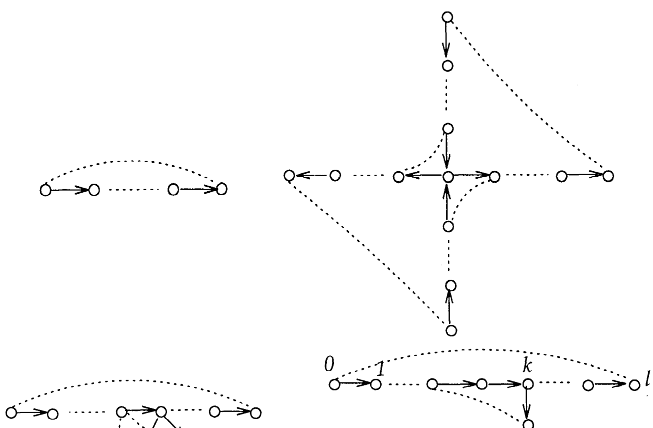
They are of types 2.2, 1.5, 1.8, 1.9, 1.14, 1.6 respectively.

2. Now we shall find iterated tilted algebras of type $\widetilde{\mathbb{D}}_n$ without a cycle.

a). First consider the situation when the algebra has a zero-relation of length $\geq 4$. It is easy to see (by applying reflections) that the first four among the following algebras contain an iterated tilted algebra of type $\mathbb{E}_6$ as a full convex subcategory, and the two remaining ones contain an algebra which is not tilting-cotilting equivalent to a branch enlargement of a tame concealed algebra.



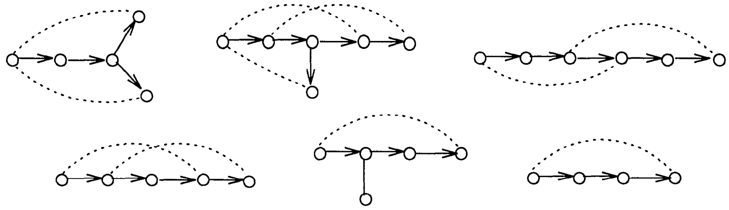
So we have the following frames




$2 \leq k < l$

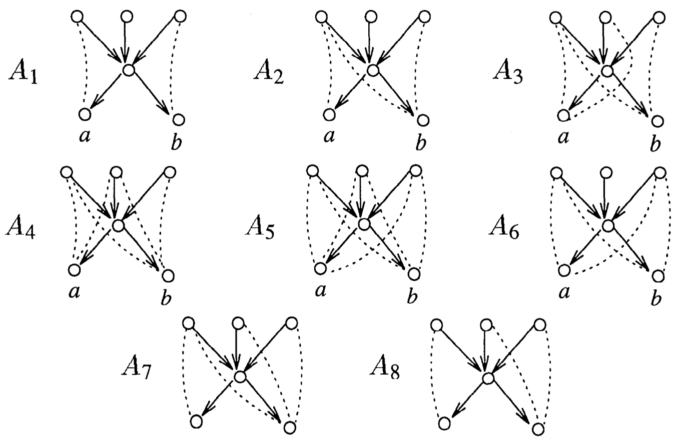
They are of types 2.3, 1.15, 1.14, 1.16 respectively.

b). Consider now algebras with zero-relations of length 3. It is obvious that if we replace any zero-relation of length $\geq 4$ in a) by a zero-relation of length 3 then we shall obtain algebras of the same type. Now we shall find other ones. In the same way as before we obtain the frames

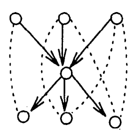


They are of types 1.9, 1.12, 1.13 and 2.4 respectively.

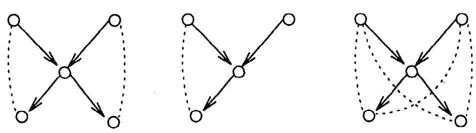
c). We shall describe algebras without cycles and zero-relations of length $\geq 3$. First we shall check in which configurations may occur 5 arrows having one common vertex. Up to duality there are 8 possibilities (note that $\widetilde{\mathbb{D}}_4$ is representation-infinite)




After applying reflections at the vertices $a$ and $b$ we see that $A_1$, $A_2$, $A_3$ are iterated tilted of type $\mathbb{E}_6$, $A_4$, $A_5$ are tilting-cotilting equivalent to wild algebras. Observe also that $A_6$ is of type 1.5, $A_7$ is of type 1.14 and $A_8$ of type 2.5. Using these facts it is easy to see that the only configuration of 6 arrows having one common vertex is the following



and it is of type 1.5. Because of representation-finiteness there are no configurations of seven or more arrows having one common vertex. Observe now that the algebras



are iterated tilted of type $\mathbb{A}_5$, iterated tilted of type $\mathbb{A}_4$ and of type 1.5 respectively, and all the remaining configurations of four or three arrows having one common vertex are algebras of type (2).

Let us recapitulate our analysis of the necessity part of the proof. We have obtained two kinds of algebras: they are frames of type (1) and (2). Algebras containing frames of type (1) are completely described. We shall consider now frames of type (2). As before, applying reflections, it is an easy but rather tedious task to check that two such frames may be connected only by admissible glueings and any connection of three frames does not give us an iterated tilted algebra of type $\widetilde{\mathbb{D}}_n$. This finishes the necessity part of the proof and hence the proof of the theorem.

**Résumé substantiel en français.** Un module $T$ sur une algèbre $A$ est appelé un module inclinant (voir [HR, B]) si les conditions suivantes sont satisfaites :

(T1) $\mathrm{Ext}_A^2(T,-)=0$
(T2) $\mathrm{Ext}_A^1(T,T)=0$
(T3) Le nombre de facteurs direct indécomposables non isomorphes de $T$ est égal au rang du groupe de Grothendieck $K_0(A)$ de $A$.

Étant donné un carquois fini $\Delta$ sans cycles orientés, une algèbre $A$ sera dite *préinclinée de type* $\Delta$ (voir [AH]) s'il existe une suite d'algèbres $A = A_0, A_1, \dots, A_m$, où $A_m$ est l'algèbre des chemins de $\Delta$, et une suite de modules inclinants $T^i_{A_i}$ $(0 \le i < m)$ tels que $A_{i+1} = \mathrm{End}(T^i_{A_i})$ et chaque $A_i$–module indécomposable $M$ satisfait soit $\mathrm{Hom}_{A_i}(T^i, M) = 0$, soit $\mathrm{Ext}^1_{A_i}(T^i, M) = 0$. Si $m \le 1$, on dit que $A$ est une *algèbre inclinée de type* $\Delta$ (voir [HR]).

Soit $A$ une algèbre et $M$ un $A$–module. L'extension ponctuelle de $A$ par $M$ est l'algèbre matricielle

$$A[M] = \begin{bmatrix} A & 0 \\ M & k \end{bmatrix}$$

avec l'addition et la multiplication usuelles des matrices.

Soit $A$ une algèbre triangulaire et $i$ un puits de $Q_A$. La réflexion $S_i^+A$ de $A$ en $i$ est le quotient de l'extension ponctuelle $A[I(i)]$ par l'idéal bilatère engendré par $e_i$. Il est montré dans [TW] que $A$ et $S_i^+A$ sont équivalentes pour les inclinaisons et les coïnclinaisons. Le puits $i$ de $Q_A$ est remplacé dans le carquois $\sigma_i^+A$ de $S_i^+A$ par une source. Dualement, à partir d'une source $j$ de $Q_A$, nous définissons la réflexion $S_j^-A$ de $A$ à la source $j$.

Passons maintenant aux deux listes d'algèbres présentées dans la section 1. Par algèbre de type (1) (respectivement (2)), nous entendons une algèbre de la forme $A$ ou $S_i^+A$, où $A$ est une sous-catégorie pleine et convexe d'une algèbre se trouvant dans la colonne de droite de la liste 1 (respectivement de la liste 2) et contenant le cadre se trouvant dans la colonne de gauche de la même liste. Le théorème principal de cet article est le suivant :

**Théorème.** *Soit $A$ une algèbre de dimension finie sur un corps algébriquement clos, sobre, connexe et de représentationfinie. Alors $A$ est une algèbre préinclinée de type $\widetilde{\mathbb{D}}_n$ si et seulement si $A$ est une algèbre de type* (1) *ou un collage admisssible* (*tel que défini en section* 1) *de deux algèbres de type* (2).

Dans la démonstration, nous utilisons la description des algèbres préinclinées de type $\widetilde{\mathbb{D}}_n$ et de représentation infinie contenue dans [AS2], et le fait qu'après application

d'une suite de réflexions à une algèbre préinclinée de type $\widetilde{\mathbb{D}}_n$ et de représentation infinie, nous obtenons une algèbre de représentation finie et préinclinée du même type. Cela nous permet de prouver la suffisance. Pour la nécessité, nous éliminons les algèbres qui ne sont pas préinclinées de type $\widetilde{\mathbb{D}}_n$ au moyen du corollaire suivant:

**Corollaire.** *Soit $A$ une algèbre préinclinée de type $\widetilde{\mathbb{D}}_n$ et de représentation finie, et $B$ une sous-catégorie pleine et connexe de $A$. Alors $B$ est une algèbre préinclinée de type $\widetilde{\mathbb{D}}_m$, $\widetilde{\mathbb{A}}_m$, $\mathbb{D}_m$ ou $\mathbb{A}_m$, où $m \leq n$.*

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