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FINITELY ADDITIVE INTEGRATION: INTEGRAL EXTENSION WITH LOCAL CONVERGENCE

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Introduction. In this paper we present some results concerning integral extension for arbitrary nonnegative linear functionals on function vector lattices. For a ring Ω of sets from an arbitrary set X and $\mu\colon \Omega \to [0,\infty[$ only finitely additive, the space of Riemann μ-integrable functions was presented essentially by Loomis in [13]; for Banach space-valued functions it has been introduced by Dunford-Schwartz in [6], and more generally by Günzler in [10]. Analogue extension processes, without or with weaker continuity conditions on the elementary integral, have been treated by Aumann [2], Loomis [13] and Gould [8].

In [3] the process of a Daniell-Bourbaki integral has been generalized, starting in this case with a nonnegative linear functional I (without any continuity conditions) defined on a vector lattice B of real- valued functions on X. The main object of this paper is to introduce an analogue to "convergence in measure" for sequences, in order to obtain the space of Riemann I-integrable functions. We also obtain Lebesgue's convergence theorems. In §3 we compare our integral with some other integrals and related ideas in the literature; in particular, it is possible to give a unified treatment of Riemann-μ [9], abstract Riemann-Loomis [13], and Dunford-Schwartz [6] integrals.

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0. Preliminaries. The terminology ansd notation used is similar to that of [3, 4, 11].

We extend the usual + in $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, \infty\}$ to $\overline{\mathbb{R}} \times \overline{\mathbb{R}}$ by

(1) $a + b := 0$, $a \dot{+} b := \infty$, if$a = -b \in \{-\infty, \infty\}$; $a - b := a + (-b)$, etc.

If we denote $a \vee b := \max(a,b)$, $a \wedge b := \min(a,b)$, $a \cap b := (a \wedge b) \vee (-b)$, $a^+ := a \vee 0$, $a^- := (-a) \vee 0$, one has for $a, b, c, d \in \overline{\mathbb{R}}$, $t \in \overline{\mathbb{R}}_+ := [0, \infty]$ the following inequalities:

(2) $|a \cap t - b \cap t| \le 2(|a-b| \wedge t)$, $|a \wedge t - b \wedge t| \le |a - b|$ (Birkhoff inequalities).
(3) $\big||a| - |b|\big| \le |a-b| \le |a-c| + |c-b|$, $|(a+b) - (c+d)| \le |a-c| + |b-d|$.
(4) For $a, b, c, d \in \overline{\mathbb{R}}_+$ with $a \le b + c$, $a \wedge d \le b \wedge d + c \wedge d$ (See [2, p. 442], [10, p. 354].)

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Let B be a vector lattice of real-valued functions on a nonempty set X, and $I\colon B \to \mathbb{R}$ a nonnegative linear functional on B, i.e. under the on X pointwise defined +, $\alpha\cdot$ ($\alpha \in \mathbb{R}$), $=$, $\le$, $\vee$, $\wedge$, $|\ |$, B is a real space of functions $f\colon X \to \mathbb{R}$ containing with f, g also $f \wedge g$, $f \vee g$, $|f|$, and $0 \le I(f)$ if $0 \le f \in B$, where $|f|(x) := |f(x)|$ for $x \in X$.

The triple (X, B, I) is called a Loomis system, which we will denote I/B for short.

In the following we assume a Loomis system and the following definitions and results of [3].

A preliminary extension is defined by

$$B^+ = \{f \in \overline{\mathbb{R}}^X; f = \sup g, g \in B, g \le f\} - \{-\infty\}.$$

(5) For any $f \in \overline{\mathbb{R}}^X$, $I^+(f) := \sup\{I(g); g \in B, g \le f\}$, with $\sup\varnothing := -\infty$. Similarly, $B^- := -B^+$, and $I^-(f) := -I^+(-f)$.

Since I^+ is not additive on B^+, it is introduced the class

(6) $B_+ = \{f \in B^+; I^+(f+g) = I^+(f) + I^+(g)$ for all $g \in B^+\}$ and $B_- = -B_+$.

Now using the classes B_+ and B_-, for each $f \in \overline{\mathbb{R}}^X$ the upper and lower integrals $\bar{I}(f)$ and $\underline{I}(f)$ are defined as usual: $\bar{I}(f) := \inf\{I^+(h); f \le h, h \in B_+\}$, with $\inf\varnothing := \infty$, and $\underline{I}(f) := -\bar{I}(-f)$.

(7) For $f \in \overline{\mathbb{R}}^X$, $I^+(f) \le \underline{I}(f) \le \bar{I}(f) \le I^-(f)$.

With (1), $\bar{I}$ and I^- are subadditive on $\overline{\mathbb{R}}^X$; and I^+, $\bar{I}$, $\underline{I}$ and I^- are monotone increasing on $\overline{\mathbb{R}}^X$.

(8) The elements of $\overline{B} := \{f \in \overline{\mathbb{R}}^X; \underline{I}(f) = \bar{I}(f) \in \mathbb{R}\}$ (= B_0 in [3]) are called I-summable).

We have that $\overline{B}$ is a lattice, and if f, $g \in \overline{B}$ then $|f|$, $\alpha f \in \overline{B}$ ($\alpha \in \mathbb{R}$), besides $h \in \overline{B}$, where $h(x) + f(x) = g(x)$ only for those $x \in X$ for which $f(x)$, $g(x) \in \mathbb{R}$ (see [3, Theorem 5.2]).

(9) B is dense in $\overline{B}$ with respect to the integral seminorm defined by $\|f\|_I = \bar{I}(|f|)$, ([3, Theorem 5.6]): The necessary and sufficient condition that $f \in \overline{B}$ is that, given any $\varepsilon > 0$, there exists $h \in B$ such that $\bar{I}(|f - g|) < \varepsilon$.

(10) $\overline{B}$ is $\bar{I}$-closed, i.e.: For $(f_n)_n \subset \overline{B}$, and $f \in \overline{\mathbb{R}}^X$ such that $\bar{I}(|f_n - f|) \to \infty$ then $f \in \overline{B}$, [11, Corollary 3].

1. Local convergence and integrable functions. For any $T\colon \overline{\mathbb{R}}_+^X \to \overline{\mathbb{R}}_+$ monotone functional, an appropriate "convergence in measure" for sequences is introduced.

Definition 1.1. Let f, $(f_n)_n \subset \overline{\mathbb{R}}^X$, $(f_n)_n \to f$ (T) means that for each fixed $0 \le h \in B$ one has $T(|f_n - f| \wedge h) \to 0$, as $n \to \infty$, (where e.g. $\infty - \infty = 0$ by (1)).

In particular for $T = I^-$, with (5) and (6), we get

(11) $(f_n)_n \to f$ (I^-) iff for any $\varepsilon > 0$ and $0 \le h \in B$, there exist $n_0(\varepsilon, h) \in \mathbb{N}$, $k_n \in B$, such that $|f_n - f| \wedge h \le k_n$ and $I(k_n) \le \varepsilon$, for all $n \ge n_0$.

$(f_n)_n \subset \overline{\mathbb{R}}^X$ is called a *T-Cauchy sequence* if $T(|f_n - f_m|) \to 0$, as $n, m \to \infty$.

Note that if $(f_n)_n \to f$ (T) and $(g_n)_n \to g$ (T), then $(f_n + g_n) \to f + g$ (T), $(f_n \dotplus g_n) \to f \dotplus g$ (T), $(\alpha f_n)_n \to \alpha f$ (T), $\alpha \in \mathbb{R}$.

Lemma 1.2. *Let $(f_n)_n \subset \overline{B}$ and $\bar{I}$-Cauchy sequence such that $(f_n)_n \to 0\ (\bar{I})$, then $\bar{I}(|f_n|)_n \to 0$, as $n \to \infty$.*

Proof. It is clear that one may assume $0 \leq f_n \to 0\ (\bar{I})$. For every $\varepsilon > 0$, $\bar{I}(|f_n - f_m|) < \varepsilon$, if $n, m \geq n_1$. By (9) and (3), $\bar{I}(|f_{n_1} - h_\varepsilon|) < \varepsilon$, for any $0 \leq h_\varepsilon \in B$.

Now, from the estimation

$$|f_n| \leq |f_n \wedge h_\varepsilon| + |h_\varepsilon - f_{n_1}| + |f_n - f_{n_1}| + |f_{n_1} - f_n|,$$

and (9) the result follows. □

Corollary 1.3. *Let $(f_n)_n$, $(g_n)_n \subset B$ I-Cauchy sequences and $f \in \overline{\mathbb{R}}^X$, such that $(f_n)_n \to f\ (I^-)$ and $(g_n)_n \to f\ (I^-)$. Then, $\lim I(f_n) = \lim I(g_n)$, as $n \to \infty$.*

Proof. Since $B \subset \overline{B}$ and $\bar{I} \leq I^-$, From Lemma 1.2, $|I(f_n) - I(g_n)| = |I(f_n - g_n)| \leq \bar{I}(|f_n - g_n)| \to 0$, as $n \to \infty$. □

Lemma 1.4. *Let f, $(f_n)_n \subset \overline{\mathbb{R}}^X$ such that $|f_n - f| \leq g \in \overline{B}$, for all $n \in \mathbb{N}$. Then $(f_n)_n \to f\ (\bar{I})$ if and only if $\bar{I}(|f_n - f|) \to 0$, as $n \to \infty$.*

Proof. If $g \in \overline{B}$, given any $\varepsilon > 0$ there exists $t \in B_+$, with $0 \leq g \leq t$ and $I^+(t) \leq \bar{I}(g) + \varepsilon < \infty$. For $\varepsilon > 0$ and $0 \leq t \in B_+$, there exists $h \in B$, such that $h \leq t$ and $I^+(t) \leq I(h) + \varepsilon$.

Then, from the estimation $|f_n - f| \leq |f_n - f| \wedge h + (t - h)$, the result follows.

The converse is evident. □

If with the assumption of Lemma 1.4, additionally $f_m \in \overline{B}$ and $(f_n) \to f\ (I^-)$, then $f \in \overline{B}$ and $I(f_n) \to I(f)$ (Lebesgue's bounded convergence theorem for $\overline{B}$).

In fact $-g \leq \mp(f_n - f)$ yields

$$-\infty < -I(g) = \bar{I}(g) \leq \bar{I}(f_n - f) \leq \bar{I}(f_n) \dot{+} \bar{I}(-f) = I(f_n) - \bar{I}(f)$$

by (1) or $\bar{I}(f) \neq +\infty$; similarly $\bar{I}(f) \neq -\infty$. See [11, Theorem 1 and corollaries].

The usual Monotone convergence theorem is false for $\overline{B}$ with $\to\ (\bar{I})$, (see [11, Exemple 2]); it becomes true for a suitable extension of $\overline{B}$ which will be treated in the following. Thus, using the local convergence we will extend the integral I on B to a larger class of functions such that the usual properties are preserved.

Definition 1.5. A function $f \in \overline{\mathbb{R}}^X$ is said to be I-integrable if there exists an I-Cauchy sequence $(h_n)_n \subset B$ such that $(h_n)_n \to f\ (I^-)$.

The sequence $(h_n)_n$ is called a defining sequence for f. $R_1(B, I)$ denotes the class of all I-integrable functions. If $f \in R_1(B, I)$, $I(f) := \lim_{n\to\infty} I(h_n)$.

By Corollary 1.3 we see that expression $I(f)$ is well defined, that is it does not depend on the particular choice of the sequence $(h_n)_n$ defining the function f.

It is easy to check the following:

(12) If $f \in \overline{\mathbb{R}}^X$, $f_l(x) := \begin{cases} f(x) & \text{if } f(x) \in \mathbb{R}, \\ 0 & \text{otherwise;} \end{cases}$ we have for $f \in R_1(B, I)$, $f_l \in R_1(B, I)$ and $I(f_l) = I(f)$.

The set $R_1(B,I)$ of I-integrable functions is closed with respect $\pm$, $\dot{\pm}$, α ($\alpha \in \mathbb{R}$), $|\,|$, $\vee$, $\wedge$ ($\overline{\mathbb{R}}$-lattice).

Besides, for any $f,\ g \in R_1(B,I)$, $I(\alpha f) = \alpha I(f)$, $I(f+g) = I(f \dot{+} g) = I(f)+I(g)$, $|I(f)| \leq I(|f|)$, and $I(f) \leq I(g)$, if $f \leq g$.

(13) It is easy to see that B is dense in $R_1(B,I)$ with respect to the seminorm $||f||_I := I(|f|)$ for all $f \in R_1(B,I)$.

For $f \in \overline{\mathbb{R}}^X$ we define $f = 0\ (I^-)$ (*I^--nulfunction*) by $f_n := f \to 0\ (I^-)$, i.e. for each $0 \leq h \in B$, $I^-(|f| \wedge h) = 0$.

$N_1(B,I)$ denotes the set of all I^--nulfunctions.

For any $f, g \in \overline{\mathbb{R}}^X$, $f = g\ (I^-)$ (resp. $f \leq g\ (I^-)$) means $f - g \in N_1(B,I)$ (resp. $(g-f)^+ \in N_1(B,I)$).

Using I^--nulfunctions we can establish the following properties.

(14) If $|f| \leq g \in N_1(B,I)$ then $f \in N_1(B,I)$.

With (14) and since $a \leq b \dot{+} (a-b)^+$ for all $a, b \in \overline{\mathbb{R}}$, we have

(15) $f \leq g\ (I^-)$ iff there exists $h \in N_1(B,I)$ such that $f \leq g \dot{+} h$ on X. Note that $=\ (I^-)$ is an equivalence relation in $\overline{\mathbb{R}}^X$, $\leq\ (I^-)$ is transitive, and both are compatible with +, $\dot{+}$, α ($\alpha \in \mathbb{R}_+$).

(16) If $f, g \in R_1(B,I)$, $f \leq g\ (I^-)$, then $I(f) \leq I(g)$.

(17) $f \in N_1(B,I)$ iff $f \in R_1(B,I)$ and $I(|f|) = 0$.

And finally, by (17) and (12), we conclude

(18) $f \in R_1(B,I)$ iff there exists $g \in R_1(B,I) \cap \mathbb{R}^X$ such that $f = g\ (I^-)$.

Let us now consider the class of proper *Riemann-integrable functions*(see, for example [13, 14]).

(19) Let $R_{\text{prop}}(B,I)$ denotes the set of all functions $f \in \mathbb{R}^X$ such that for any $\varepsilon > 0$ there exist $g, h \in B$ with $h \leq f \leq g$ and $I(g-h) \leq \varepsilon$.

$$I(f) := I^-(f) = I^+(f) \quad \text{for all } f \in R_{\text{prop}}(B,I).$$

We have that $R_{\text{prop}}(B,I)$ is the closure of B with respect to the integral-seminorm I^- [2, p. 448].

Observe that $R_{\text{prop}}(B,I) \subset \overline{B}$. Example 1 below shows that $R_{\text{prop}}(B,I) \subsetneqq R_1(B,I)$.

We now can characterize the notion of I-integrability as follows.

Theorem 1.6. *Let $f \in \overline{\mathbb{R}}^X$, then the following conditions are equivalent*:

i) $f \in R_1(B,I)$.

ii) $I^+(|f|) \leq \infty$ *and* $f^{\pm} \wedge g \in R_{\text{prop}}(B,I)$ *for all* $0 \leq g \in B$.

Proof. i) $\Longrightarrow$ ii) Let $0 \leq f \in R_1(B,I)$ with defining sequence $(h_n)_n$; from (11) given $\varepsilon > 0$ and $0 \leq h \in B$, there exist $l_n \in B$ such that $|h_n - f| \wedge h \leq l_n$ and $I(l_n) \leq \varepsilon$, if $n \geq n_0$. Now, with (2) we get $t_n := h_n \wedge h - 2l_n \leq f \wedge h \leq h_n \wedge h + 2l_n =: k_n$, $t_n, k_n \in B$ and $I(k_n - t_n) = 4I(l_n) \leq 4\varepsilon$.

So that, $f \wedge h \in R_{\text{prop}}(B,I)$ for all $0 \leq h \in B$, and $I^+(f) \leq I(f) < \infty$.

ii) $\Longrightarrow$ i) If $f \in |R_+^X$, $I^+(f) < \infty$, by (5) there is an I-Cauchy sequence $(h_n)_n \subset B$ such that $0 \leq h_n \leq h_{n+1} \leq f$ and $I(h_n) \to I^+(f)$, $n \to \infty$.

Set $|h_n - f| \wedge h = f \wedge (h_n + h) - h_n$.

For $t_n := f \wedge (h_n + h) \in R_{\text{prop}}(B, I)$, given $\varepsilon > 0$ there exist $k, l \in B$, $k \leq t_n \leq l$ and $I(l) \leq I^+(f) + \varepsilon$. So that $I(l - h_n) \to 0$, as $n \to \infty$.

Besides, since $|h_n - f| \wedge h = t_n - h_n \leq l - h_n$, $n \in \mathbb{N}$, we conclude that $(h_n)_n \to f$ (I^-).

Using $f = f^+ - f^-$ and $|f| = f^+ + f^-$ the result follows for arbitrary $f \in \mathbb{R}^X$. □

Corollary 1.7. *Let $f \in R_1(B, I)$, then $f \in R_{\text{prop}}(B, I)$ if and only if $|f| \leq h \in B$.*

Corollary 1.8.

$$R_1(B, I) = \{f \in \overline{\mathbb{R}}^X; I^+(|f|) \leq \infty, f^{\pm} \wedge h \in R_{\text{prop}}(B, I), \textit{for all } 0 \leq h \in B\}.$$

2. Convergence theorems. Note that in the finitely additive case (see for example [6, pp. 101–104]), to get convergence theorems it is not sufficient a.e. or the everywhere convergence; thus, in our situation, one has to use a suitable “convergence in measure”, although localized ([11, Definition 1] or Definition 1.1).

The following lemma is basic in the sequel.

Lemma 2.1. *Let $0 \leq f \in R_1(B, I)$, then $f \wedge g \in \overline{B}$ and $\bar{I}(f \wedge g) \leq I(f)$ for all $0 \leq g \in \overline{B}$.*

Proof. We can assume $0 \leq h_n \to f$ (I^-), hence $(h_n \wedge g) \to f \wedge g$ (I^-), with $h_n \wedge g \in \overline{B}$, $|h_n \wedge g| \leq g$. From Lemma 1.4, $\bar{I}(|h_n \wedge g - f \wedge g|) \to 0$ as $n \to \infty$, and by (10) $f \wedge g \in \overline{B}$.

Finally, $\bar{I}(h_n \wedge g) \leq I(h_n) \to i(f)$, as $n \to \infty$, implies $\bar{I}(f \wedge g) \leq \bar{I}(f)$. □

From Lemma 2.1 and [3, Corollary 5.10 a)] we easily get:

(20) If $f \in R_1(B, I)$ and $|f| \leq g \in \overline{B}$, then $f \in \overline{B}$. The condition $|f| \leq g$ can be weakened to $\leq$ (I^-).

Lemma 2.2. *Let $(f_n)_n \subset R_1(B, I)$, with $I(|f_n|) \to 0$, then $(f_n)_n \to 0$ (I^-).*

Proof. From Lemma 2.1 $|f_n| \wedge h \in \overline{B}$ for all $0 \leq h \in B$. Besides, $\bar{I}(|f_n| \wedge h) \leq I(|f_n|)$, $|f_n| \wedge h \in R_1(B, I)$, $n \in \mathbb{N}$, $0 \leq h \in B$. Therefore, $I^-(|f_n| \wedge h) = I^+(|f_n| \wedge h) \leq I(|f_n|) \to 0$. □

The following closedness property of $R_1(B, I)$ holds.

Theorem 2.3. *Let $f \in \overline{\mathbb{R}}^X$, $(f_n)_n \subset R_1(B, I)$ an I-Cauchy sequence with $(f_n)_n \to f$ (I^-). Then $f \in R_1(B, I)$ and $\lim_{n \to \infty} I(f_n) = I(f)$.*

Proof. By (13), given $n \in \mathbb{N}$ there exists $h_n \in B$ such that $I(|f_n - h_n|) \leq \frac{1}{n}$.

Now, using that $|h_n - f| \leq |h_n - f_n| + |f_n - f|$, and Lemma 2.2, we get $(h_n)_n \to f$ (I^-), where $(h_n)_n \subset B$ is an I-Cauchy sequence. □

By [11, Example 2] an analogue for $\overline{B}$ is false with $\to$ $(\bar{I})$.

Theorem 2.3 allows us to derive the convergence theorems.

Theorem 2.4 (Monotone convergence theorem). *Let* $f \in \overline{\mathbb{R}}^X$, $(f_n)_n \subset R_1(B,I)$, $f_n \leq f_{n+1}$ (I^-), $n \in \mathbb{N}$, *with* $(f_n)_n \to f$ (I^-) *and* $\beta := \sup\{I(f_n); n \in \mathbb{N}\} \leq \infty$. *Then*

i) $f \in R_1(B,I)$ *and* $\lim_{n\to\infty} I(f_n) = I(f) = \beta$.
ii) $f_n \leq f$ (I^-), $n \in \mathbb{N}$.

Proof. To prove i), by Theorem 2.3, we have only to show that $(f_n)_n$ is an I-Cauchy sequence.

Using (15) and (17), it is easy to see that $I(|f_n - f_m|) \leq I(f_m) - I(f_n) \leq \infty$, for all $m \geq n$, and the required result follows.

To prove ii), observe that $0 \leq (f_n - f)^+ \leq (f_m - f)^+ \dot{+} u \leq |f_m - f| \dot{+} u$, with $m \geq n$ and $0 \leq u \in N_1(B,I)$. □

Lemma 2.5. *If* $f \in R_1(B,I)$ *and* $f \leq g \in B^+$, *then* $I(f) \leq I^+(g)$.

Proof. a) Let $0 \leq f \in R_1(B,I)$ with defining sequence $0 \leq (h_n)_n \subset B$, by Corollary 1.8, $h_n \wedge f \in R_{\text{prop}}(B,I)$, and $I(h_n \wedge f) \leq I^+(f)$, $n \in \mathbb{N}$.

Now, since $(h_n \wedge f)_n \to f$ (I^-) and $h_n \wedge f \leq g$, we have $I(f) \leq I^+(g)$.

b) For $g \in B^+$ there exists $h \in B$ with $h \leq g$. Put $f - h \leq (f-h)^+ \leq g - h$, where $g - h \in B^+$, therefore, by a), $I(f-h) \leq I^+(g) - I(h)$, and the proof is completed. □

Corollary 2.6. *For any* $f \in R_1(B,I)$, $I^+(f) \leq \underline{I}(f) \leq I(f) \leq \bar{I}(f)$.

Theorem 2.7 (Bounded convergence theorem). *Let* $f \in \overline{\mathbb{R}}^X$, $(f_n)_n \subset R_1(B,I)$, $g \in R_1(B,I)$ *such that* $|f_n| \leq g$ (I^-), $n \in \mathbb{N}$, *and* $(f_n)_n \to f$ (I^-).

Then $f \in R_1(B,I)$ *and* $\lim_{n\to\infty} I(f_n) \to I(f)$.

Proof. Again by Theorem 2.3, we need to prove that $(f_n)_n$ is an I-Cauchy sequence.

Suppose that there is $\varepsilon_0 > 0$ such that given $k \in \mathbb{N}$ there exist n_k, $p_k \in \mathbb{N}$, with $I(|f_{n_k} - f_{p_k}|) \geq \varepsilon_0$.

Put $g_k := |f_{n_k} - f_{p_k}|$, then $g_k \in R_1(B,I)$ and $g_k \leq 2g$ (I^-); by (15), there exists $0 \leq u_k \in N_1(B,I)$ such that $g_k \leq 2g \dot{+} u_k$, $k \in \mathbb{N}$.

Now from the estimation

$$|g_k| \wedge h \leq |f_{n_k} - f| \wedge h + |f - f_{p_k}| \wedge h,$$

for all $0 \leq h \in B$, we get

$$(g_k)_k \to 0 \quad (I^-). \tag{a}$$

On the other hand, applying (13) to $2g \in R_1(B,I)$ with $\frac{1}{2}\varepsilon_0 > 0$, there exists $h \in B$ such that

$$I(2g - 2g \wedge h) \leq I(|2g - h|) \leq \frac{\varepsilon_0}{2}. \tag{b}$$

Note that one can assume $h \geq 0$, since $\big|2g - |h|\big| \leq |2g - h|$.

Also, using (a) and $\bar{I} \leq I^-$ we have $\bar{I}(|g_k \wedge h|) \to 0$, as $k \to \infty$.

Indeed, by Lemma 2.1, $|g_k \wedge h| \in R_1(B,I) \cap \overline{B}$, so that

$$I(|g_k \wedge h|) = \bar{I}(|g_k \wedge h|) \to 0 \quad \text{as } k \to \infty. \tag{c}$$

Now, it is easily verified the estimation

$$0 \leq g_k \leq (g_k \wedge h) + (2g - 2g \wedge h) \dotplus u_k, \quad k \in \mathbb{N}.$$

Finally, from (b), (c) and (d) we conclude that $\varepsilon_0 \leq I(g_k) \leq I(g_k \wedge h) + I(2g - 2g \wedge h) < \varepsilon_0$, which is a contradiction, and the theorem is proved. □

Remarks. This extension process $I/B \longrightarrow I/R_1(B,I)$ is iteration complete, i.e. $R_1(B,i) = R_1(\widetilde{B},\tilde{I})$ with $\widetilde{B} := R_1(B,I) \cap \mathbb{R}^X$, $\tilde{I} := I/B$. Observe that for $(f_n)_n \subset \overline{\mathbb{R}}^X$, $(f_n)_n \to 0\ (\tilde{I}^-)$ implies $(f_n)_n \to 0\ (I^-)$. If $(f_n)_n \subset R_1(B,I)$ the converse holds.

Most the above can be extended to Banach space valued functions, using $a \cap b$ of (1), (see [16]).

3. Applications.

1. If I/B is monotone-net-continuous = Bourbaki's continuity condition, then Daniell $L^1(B,I) =: L^1 \subset \overline{B}$ = Bourbaki extension $L^\tau(B,I) =: L^\tau$, (see [15, 7]). By [3, p. 247], $\overline{B} = L^\tau$. Besides, in this case $B^+ = B_+$, $I^+ = \bar{I}$ (upper Bourbaki integral on B^+); and L^1 is always contained in L^τ.

For example, let $B = C_0(X,\mathbb{R})$ (continuous functions with compact support on X locally compact Hausdorff space) and I: $C_0(X,\mathbb{R}) \to \mathbb{R}$ any nonnegative linear functional, then I/B is automatically monotone-net-continuous.

2. Assume I/B, for any f, $(f_n)_n \subset \mathbb{R}^X$ $(f_n)_n$ is said to converge σ-uniformly to f ($(f_n)_n \Rrightarrow f$), if for any $\varepsilon > 0$, there exist $n_0 \in \mathbb{N}$ and $(g_m)_m \subset B$, $g_m \geq 0$, such that $|f_n - f| \leq \sum_{m\in\mathbb{N}} g_m$ and $\sum_{m\in\mathbb{N}} I(g_m) \leq \varepsilon$, for all $n \geq n_0$.

By means of this type of convergence and following the lines of the Daniell theory, in [12,§5], it is obtained the class of Lebesgue-integrable functions.

Suppose that I/B satisfies the following conditions:

i) Daniell's continuity condition: $0 \leq f_n \in B$, $f \in B$, $f \leq \sum_n f_n \implies I(f) \leq \sum_n I(f_n)$. (See [14, p. 521])

ii) $0 \leq f_n \in B$, $f_n \leq g \in B$, $n \in \mathbb{N}$, $(f_n)_n \uparrow f \implies f \in B$.

For $f \in \mathbb{R}^X$, $(f_n)_n \subset B$, we have that $(f_n)_n \Rrightarrow f$ implies $(f_n)_n \to f\ (I^-)$. In fact, for any $0 \leq h \in B$ one has $|f_n - f| \wedge h \leq \sum_n g_n \wedge h$, by ii) $\sum_{m=1}^n g_m \wedge h \uparrow \sum_{n=1}^\infty g_n \wedge h \in B$ and by i) $I(\sum_{n=1}^\infty g_n \wedge h) \leq \sum_{n=1}^\infty I(g_n) \leq \varepsilon$ for any $\varepsilon > 0$, $n \geq n_0(\varepsilon, h)$.

3. Using Theorem 1.6 and Corollary 2.6, since if $0 \leq f \in R_1(B,I) \cap \mathbb{R}^X$, we have

$$f_s := \sup\{g \in B; 0 \leq g \leq f\} \in \overline{B} \cap R_1(B,I),$$

and $I(f) = I(f_s)$, one can generalize the theorem [4, p. 262] that is: $R_1(B,I) \subset \overline{B}$ modulo I^--nulfunctions. Besides this is the only relation that one has in general between $R_1(B,I)$ and $\overline{B}$ (see examples below and [11, pp. 88–89]).

By [11, Example 3], the inclusion $\overline{B} \subset$ Daniell $L^1(B,I)$ is false.

In the Daniell situation, in [11, Theorem 5] is showed that $\overline{B} \subset L^1(B,I) \cap \overline{B} \cap \mathbb{R} + \overline{B}_{\mathrm{n}}$, with $\overline{B}_{\mathrm{n}} := \{f \in \overline{B}; I(|f|) = 0\}$ and I = Daniell integral on $L^1 \cap \overline{B}$.

4. In [16], Schäfke gives only $R_1(B,I) \cap \mathbb{R}^X$ = Schäfke's local I_B^--closure of B, where $I_B^- := \sup\{I^-(f \wedge |h|); h \in B\}$. In this situation the bounded convergence theorem is applicable only if his condition ② Satz 2.8 holds, which is very restrictive (for example $C_0(X,\mathbb{R})$ does not satisfy ②).

Besides, using Theorm 1.6, the Loomis' extension U ("one-sided-completion") of [13, p. 178] is precisely $R_1(B,I) \cap \mathbb{R}^X$. Also $R_1(B,I) \supsetneqq R_{\text{prop}}(B,I)$ = "two-sided-completion" of Loomis [13, p. 170].

Thus, for Loomis' finitely additive "one-sided-completion" integral a new definition is given and convergence theorems are obtained.

5. In the following the assumption μ/Ω is assumed, which means: Ω a semiring of subsets of X and μ a nonnegative finitely additive measure on Ω.

B_Ω denotes the set of all step-functions $S(\Omega,\mathbb{R})$, and $I_\mu(h) := \int h\,d\mu$, $h \in B_\Omega$, where $S(\Omega,\mathbb{R})$ contains all $h = \sum_{i=1}^n a_i\chi_{A_i}$ with $n \in \mathbb{N}$, $a_i \in \mathbb{R}$, $A_i \in \Omega$ and $\int h\,d\mu = \sum_{i=1}^n a_i\mu(A_i)$. Then (X, B_Ω, I_μ) is a Loomis system. Now starting from B_Ω and I_μ by using the above methods, we obtain an integral extension to the I_μ-integrable functions class $R_1(B_\Omega, I_\mu)$.

For $X - \mathbb{R}$, $\Omega = \{[a,b[\,; -\infty < a \leq b < \infty\}$ and $\mu([a,b[) = b - a$, then $R_{\text{prop}}(B_\Omega, I_\mu)$ gives the classical proper Riemann-integrable functions [10, p. 216].

For μ/Ω one can define μ-local convergence, $(f_n)_n \to f\ (\mu)$, (see [9, p. 172]) and by the lemma in [10, p. 72, A.2.72], one gets using Definition 1.1, (11), and $I_\mu^-(f) := \inf\{I_\mu(h); f \leq h \in B_\Omega\}$: For $(f_n)_n$, $f \subset \overline{\mathbb{R}}^X$

$$(f_n)_n \to f \quad (\mu) \iff (f_n)_n \to f \quad (I_\mu^-).$$

(See [11, Lemma 9]).

Therefore, we have that $R_1(B_\Omega, I_\mu) = R_1(\mu, \overline{\mathbb{R}})$ = abstract Rieman μ-integrable functions of Günzler in [9,10]; and all results of Sections 1–2 are applicable.

In particular, with μ/Ω, the condition ii) given in Theorem 1.6 which characterizes I-integrability is often used as definition of Riemann μ-integrability (see [11, p. 213]).

6. The space $L(\mu,\overline{\mathbb{R}}) = L(X,\Omega,\mu,\overline{\mathbb{R}})$ of μ-integerable functions of Dunford-Schwartz [6, III, 2.17, p. 112] has been generalized to $R_1(\mu,\overline{\mathbb{R}})$ in [10, p. 70 199] (μ-local convergence localizes the convergence in μ-measure of [6, p. 104]). In general, $L(\mu,\overline{\mathbb{R}}) \subsetneqq R_1(B_\Omega, I_\mu)$, and only if $X \in \Omega$ (Ω algebra) and $\mu(X) < \infty$, both concepts coincide (convergence in μ-measure, locally μ-convergence and I_μ^--convergence are equivalent).

If Ω is a σ-ring and $\mu\colon \Omega \to [0,\infty[$ σ-additive, then $R_1 = L_1 =: L^1 + \{f \in \overline{\mathbb{R}}^X; f = 0\ \mu\text{-a.e. on each } A \in \Omega\} \subset \overline{B}$ modulo nullfunctions, and one gets the usual Lebesgue convergence theorems.

In fact with B = step functions B_Ω, $I = I_\mu$, $R_1 \subset \overline{B}$ modulo nullfunctions by [4] or Remark 3, and $L_1 = R_1$ by [10, A146, p. 265].

7. In Aumann [2, p. 78] it is assumed μ/Ω with $X \in \Omega$. Here, we get $N = I_{\mu_L}^-$ and L^* = exactly $R_{\text{prop}}(\mu_L/[a,b],\mathbb{R})$, T^* = classical proper Riemann integral $\int_a^b dx$, $\mu_L(]\alpha,\beta[) := \beta - \alpha$.

4. Examples.

1. Let (X, Ω, μ) be a finitely additive measure space, such that $X \neq X_0 := \bigcup\{A; A \in \Omega, \mu(A) < \infty\}$. Then if $f \in \overline{\mathbb{R}}^X$ is defined as $f(X_0) = \{0\}$ and $f(X - X_0) = \{1\}$, we have $f \in R_1(\mu, \mathbb{R}) \int f\, d\mu = 0$, but f does not belongs to B_Ω, since $\bar{I}_\mu(f) = \infty$.

2. Let $X = \mathbb{R}$, and let Ω be the ring consisting of all the finite unions of disjoint half-open intervals in $\mathbb{R}$, and $\mu :=$ Lebesgue measure on Ω. Let $P := \{r; r \text{ rational}, 0 < r < 1\}$. Then $\chi_P \in \overline{B}_\Omega = L^1(\mu)$ (= usual Lebesgue-integrable functions on $\mathbb{R}$), but χ_P does not belong to $R_1(\mu, \mathbb{R})$.

3. Let $X = \mathbb{R}, \Omega = \{\{x\}; x \in \mathbb{R}\} \cup \{\varnothing\}$ and $\mu \colon \Omega \to \{0,1\}$ σ-additive, $\mu(\{x\}) = 1$ if $x \in X$, $\int f\, d\mu = \sum_{x \in X} f(x)$. We have $R_{\text{prop}}(\mu, \mathbb{R}) = S(\Omega, \mathbb{R}) \subset R_1(\mu, \mathbb{R}) = \mathcal{L}^1(\mathbb{R}, \mathbb{R})$.

4. Let $X = \mathbb{R}^n$, $\mu :=$ Lebesgue measure μ_L^n on $\Omega := \Omega_n := \{\prod_{j=1}^n [a_j, b_j[\,; a_j \leq b_j, a_j, b_j \in \mathbb{R}, n \in \mathbb{N}\}$. Then, we have $R_1(\mu, \overline{\mathbb{R}}) \subset \overline{B}_\Omega$.

Résumé substantiel en français. Considérons un système de Loomis (X, B, I), où B désigne un espace vectoriel réticulé de fonctions réelles définies sur un ensemble abstrait X, et I une fonctionnelle non négative et linéaire sur B

On utilise le processus et la terminologie que nous avons développé dans [3] pour définir la classe des fonctions sommables $\overline{B}$.

De façon précise, on se donne la classe

$$B^+ = \{f \in \overline{\mathbb{R}}^X; f = \sup g, g \in B, g \leq f\} - \{-\infty\}$$

et on considère pour toute $f \in \overline{\mathbb{R}}^X$, $I^+(f) := \sup\{I(g); g \in B, g \leq f\}$. Similairement $B^- := -B^+$, et $I^-(f) := -I^+(-f)$.

On définit ensuite une notion adéquate de convergence en mesure locale séquentielle : f, $(f_n)_n \subset \overline{\mathbb{R}}^X$, $(f_n)_n \to f$ (I^-) si et seulement si pour chaque $0 \leq h \in B$, $I^-(|f_n - f| \wedge h) \to 0$, $n \to \infty$.

Une fonction f de $\overline{\mathbb{R}}^X$ est dite I-intégrable s'il existe une suite $(h_n)_n \subset B$ telle que $I(|h_n - h_m|) \to 0$, n, $m \to \infty$ (suite I-Cauchy) et $(h_n)_n \to f$ (I^-).

On désigne par $R_1(B, I)$ l'ensemble des fonctions I-intégrables. Pour toute $f \in R_1(B, I)$, le nombre $I(f)$ défini par $I(f) = \lim I(h_n)$ $n \to \infty$ est appelé intégrale de f.

$R_1(B, I)$ est un espace vectoriel réticulé et I est linéaire et non négative sur $R_1(B, I)$.

Quelques propriétés importantes de l'intégrale (notamment les théorèmes de convergence monotone et dominée) sont établies.

Soit $f \in \overline{\mathbb{R}}^X$ et $(f_n)_n$ une suite de fonctions de $R_1(B, I)$, telles que $(f_n)_n \to f$ (I^-). On a $f \in R_1(B, I)$ et $\lim I(f_n) = I(f)$, $n \to \infty$, si l'une quelconque des conditions (a), (b) ou (c) suivantes est remplie

(a) $I(|f_n - f_m|) \to 0$, n, $m \to \infty$.
(b) $f_n \leq f_{n+1}$, $n \in \mathbb{N}$ et $\sup\{I(f_n); n \in \mathbb{N}\} < \infty$.
(c) $|f_n| \leq g \in R_1(B, I)$.

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