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ISOTONE PROJECTION CONES IN EUCLIDEAN SPACES

GEORGE ISAC AND ALEXANDRE B. NÉMETH

RÉSUMÉ. Soit **K** un cône convexe fermé dans l'espace euclidien R^n. On note par $P_{\mathbf{K}}$ la projection sur K. Dans ce papier on caractérise les cônes convexes fermés **K** qui engendrent l'espace R^n et qui ont la propriété que $P_{\mathbf{K}}$ est isotone par rapport à l'ordre défini par **K**. Voir le résumé substantiel en français à la fin de l'article.

ABSTRACT. Let **K** be a closed convex cone in the Euclidean space R^n. We denote by $P_{\mathbf{K}}$ the projection onto **K**. In this paper we characterize the generating closed convex cones such that $P_{\mathbf{K}}$ is isotone with respect to the ordering defined by **K**.

0. Introduction. The metric projections on closed convex sets in Hilbert or Banach spaces have been deeply investigated (see for instance the monograph [19] and the papers [4–6, 13–16].

A special case is the metric projection on a closed convex cone in a Hilbert space.

Although this subject was much studied by Zarantonello in [19], it seems that the relation between the projection operator and the ordering defined by a cone was first considered in our paper [7].

The cited paper as well as [8–11] are concern with various characterization of a cone **K** in a Hilbert space having the property that the metric projection $P_{\mathbf{K}}$ is isotone with respect to the order defined by K (called in this case *isotone projection cone*).

Besides its theoretical importance this property has interesting applications to the study and the solvability of the Complementarity Problem (important in Optimization, Mechanics, Game Theory, *etc.*) [8–11, 13–15].

The aim of this paper is to place our investigations on isotone projection cones in Euclidean spaces, in the recent literature which investigates some related problems.

More precisely, we intend to exploit from this point of view some recent results of Barker, Laidacker and Poole [2] to complete the existent characterizations of isotone projection cones with new ones, and finally, to simplify some earlier proofs and to present them in a concise and independent exposition.

1. Preliminaries and the main result. For the following basic facts about cones we refer the reader to the book [17].

A subset **K** in the Euclidean space R^n is a *cone* if

(i) $\mathbf{K} + \mathbf{K} \subseteq \mathbf{K}$,

(ii) $\lambda\mathbf{K} \subseteq \mathbf{K}$ whenever $\lambda \in R_+$ and

(iii) $\mathbf{K} \cap (-\mathbf{K}) = \{0\}$.

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A cone is a convex set. We say that $\mathbf{K}$ is *generating* if $R^n = \mathbf{K} - \mathbf{K}$. A cone in R^n is generating if and only if its interior is nonempty. The set

$$\mathbf{K}^0 = \{x \in R^n \mid \langle x, y\rangle \leq 0, \forall y \in \mathbf{K}\}$$

(where $\langle \cdot, \cdot \rangle$ is the inner product) is called the *polar* of $\mathbf{K}$. If $\mathbf{K}$ is generating, then $\mathbf{K}^0$ is a closed cone. If $\mathbf{K}$ is closed then $\mathbf{K} = (\mathbf{K}^0)^0$.

If we put $x \leq y$ whenever $y - x \in \mathbf{K}$, then we obtain an order relation (that is a reflexive, transitive and antisymmetric relation) compatible with the vector structure of R^n. We say in this case that $(R^n, \mathbf{K})$ is an *ordered vector space* and $\mathbf{K}$ is its *positive cone*. The order defined by $\mathbf{K}$ is called the *order induced* by $\mathbf{K}$.

An upper bound of a set $A \subset R^n$ is an element $b \in R^n$ such that $a \leq b$ for every $a \in A$.

If there exists a least upper bound for A, it will be called the *supremum* of A and will be denoted by $\sup A$. Lower bounds and infima can be defined similarly.

If for any two elements x, $y \in R^n$ there exists $\sup\{x, y\}$ (which will be denoted by $x \vee y$), then the ordered vector space is called a *vector lattice* and its positive cone $\mathbf{K}$ is said to be *latticial* (or minihedral).

We say that a subset F of the cone $\mathbf{K}$ is a *face* if it is a cone and if it satisfies the condition: from $x \in F$, $y \in \mathbf{K}$ and $y \leq x$ it follows that $y \in F$.

A closed half-space of R^n with boundary point 0 is a subset of R^n of the form $\{x \in R^n \mid \langle x, p\rangle \leq 0\}$ where $p \in R^n$, $p \neq 0$.

A *polyhedral cone* in R^n is the intersection of finitely many closed half-spaces of R^n with boundary point 0.

A closed cone $\mathbf{K} \subset R^n$ is a polyhedral cone if and only if $\mathbf{K}$ is a finitely generated cone, that is there exists a finite subset $\{a_1, a_2, \dots, .a_k\}$ of R^n, called a *set of generators* for $\mathbf{K}$ such that,

$$\mathbf{K} = \{\lambda_1 a_1 + \lambda_2 a_2 + \cdots + \lambda_k a_k \mid \lambda_1, \lambda_2, \dots, \lambda_k \geq 0\}.$$

A closed generating cone $\mathbf{K} \subset R^n$ is polyhedral if it has a finite number of proper faces having codimension one in R^n and every proper face of $\mathbf{K}$ is contained in some such face.

We shall use this last characterization for polyhedral cones.

If $\mathbf{C}$ is a closed convex set in R^n, then for each $x \in R^n$ there exists a unique point in $\mathbf{C}$ denoted by $P_{\mathbf{C}}(x)$ such that $\|x - P_{\mathbf{C}}(x)\| \leq \|x - y\|$, $\forall y \in \mathbf{C}$. The operator $P_{\mathbf{C}}$ is called the *projection* (or metric projection) on $\mathbf{C}$ [17].

The cone $\mathbf{K} \subset R^n$ is called *correct* if for each of its face F we have that $P_{\mathrm{sp}\,F}(\mathbf{K}) \subset F$, where $\mathrm{sp}\,F$ denotes the linear span of the set F. Correct cones are called projectionally exposed by Borwein and Wolkowicz [3] and orthogonally projectionally exposed cones by Barker, Laidacker and Poole [2].

We have independently introduced this notion and called it correct by some analogy with the notion of perfect cones in which occur the additional condition $\mathbf{K} = \mathbf{K}^*$, where $\mathbf{K}^* = -\mathbf{K}^0$ (see [1, 12]).

We maintain this term here to be in keeping with our terminology in [8, 9, 11].

The closed cone $\mathbf{K} \subset R^n$ is called an *isotone projection* cone if from $y - x \in \mathbf{K}$ it follows that $P_{\mathbf{K}}(y) - P_{\mathbf{K}}(x) \in \mathbf{K}$, for every x, $y \in R^n$

By using the order relation defined by $\mathbf{K}$, this condition can be written in the form: $x \leq y \implies P_{\mathbf{K}}(x) \leq P_{\mathbf{K}}(y)$.

We are now ready to give our main result.

Theorem. *Let* $\mathbf{K}$ *be a closed generating cone in* R^n. *Then the following assertions are equivalent:*

(i) $\mathbf{K}$ *is an isotone projection cone,*
(ii) $\mathbf{K}$ *is correct and latticial,*
(iii) $\mathbf{K}$ *is polyhedral and correct,*
(iv) *there exists a set of vectors* $\{u_i \mid i \in I\}$ *with the property that* $\langle u_i, u_j\rangle \le 0$, $\forall i, j \in I$, $i \neq j$ *and such that* $\mathbf{K} = \left(\{u_i \mid i \in I\}\right)^0$,
(v) $\mathbf{K}$ *is latticial and* $P_{\mathbf{K}}(x) \le x^+$ *for every* $x \in R^n$, *where* $x^+ = x \vee 0$.

The equivalence (i) $\Longleftrightarrow$ (iv) was proved in [7]. The equivalence (ii) $\Longleftrightarrow$ (iv) was independently established in [2, 8] while (ii) $\Longleftrightarrow$ (iii) was established in [2].

In [8] was proved (i) $\Longrightarrow$ (ii) for a general Hilbert space.

We shall give in the sequel a complete proof of this theorem witch we shall make as self contained as possible. The only facts we shall use apart from the ones in this section are the theorem of Youdine on latticial cones and some properties of the projection operator including Moreau's decomposition theorem with respect to mutually polar cones. The most part of the proofs are new.

The proof of (i) $\Longrightarrow$ (ii) is a simplified version of the similar result for Hilbert spaces proved in [8]. The most difficult steps are those which imply the operator $P_{\mathbf{K}}$.

Hence one of the main reaches of the paper is the proof of (ii) $\Longrightarrow$ (i) presented in Section 4 and which is much simpler than that of (iv) $\Longrightarrow$ (i) in [7].

Condition (v) constitutes a new characterization of the isotone projection cones in R^n.

2. Preliminary results. The following result of Youdine [18] will be used often in our proofs.

Theorem (Youdine). *The cone* $\mathbf{K} \subset R^n$ *is latticial if and only if there exist* n *vectors linearly independent in* R^n, $u_1, u_2, \dots, u_n$ *such that*

$$\mathbf{K} = \{x \in R^n \mid \langle x, u_i\rangle \le 0, i = 1, 2, \dots, n\}. \tag{2.1}$$

That is, $\mathbf{K}$ is latticial if and only if it is of form $\mathbf{K} = \left(\{u_i \mid i = 1, 2, \dots, n\}\right)^0$, where $u_1, u_2, \dots, u_n$ are linearly independent vectors.

Several technical corollaries follow from this result.

Let $A \subset R^n$. The *affine hull* $\operatorname{aff}(A)$ of A is the smallest affine subset of R^n containing A. The *relative interior*, $\operatorname{rint}(A)$ of A is defined as the interior of A regarded as a subset of $\operatorname{aff}(A)$ (with the relative topology).

We remark that if $A \subset R^n$ is nonempty and convex then $\operatorname{rint}(A)$ is nonempty and $\dim(\operatorname{rint}(A)) = \dim(A)$.

Lemma 1. *If* $\mathbf{K}$ *is of form* (2.1) *with* $u_1, u_2, \dots, u_n$ *linearly independent then for every subset* $\{i_1, \dots, i_k\} \subset \{1, 2, \dots, n\}$ *the set* $F_{i_1, \dots, i_k} = \{x \in \mathbf{K} \mid \langle x, u_{i_j}\rangle = 0, j = 1, \dots, k\}$ *is a face of* $\mathbf{K}$. *If* $i_h \neq i_l$ *whenever* $h \neq l$, *then both* $F_{i_1, \dots, i_k}$ *and*

$$\operatorname{rint}(F_{i_1, \dots, i_k}) = \left\{x \in F_{i_1, \dots, i_k} \mid \langle x, u_j\rangle < 0, j \in \{1, \dots, n\} \setminus \{i_1, \dots, i_k\}\right\}$$

are for $k < n$ *nonempty sets in* R^n *of codimension* $n - k$.

Every face of $\mathbf{K}$ *is of form* $F_{i_1,\dots,i_k}$ *with some set* $\{i_1,\dots,i_k\} \subset \{1,2,\dots,n\}$.

Proof. The assertion that $F_{i_1,\dots,i_k}$ and $\operatorname{rint}(F_{i_1,\dots,i_k})$ are nonempty and of codimension $n-k$ if $k<n$ is a routine exercise of linear algebra.

Suppose that $x \in F_{i_1,\dots,i_k}$, $y \in \mathbf{K}$ and $y \le x$.

Then $\langle x-y, u_{i_j}\rangle = -\langle y, u_{i_j}\rangle \le 0$, $j = 1, 2, \dots, k$ since $x-y \in \mathbf{K}$.

Hence $\langle y, u_{i_j}\rangle = 0$, $j = 1, 2, \dots, k$ because $y \in \mathbf{K}$ and we know that $\langle y, u_j\rangle \le 0$, $j = 1, 2, \dots, n$. Thus $y \in F_{i_1,\dots,i_k}$ and this set is a face of $\mathbf{K}$.

Suppose that F is an arbitrary proper face of $\mathbf{K}$.

If for some $x \in F$ we would have that $\langle x, u_j\rangle < 0$, $j = 1, 2, \dots, n$ then for arbitrary $y \in \mathbf{K}$ there exist some positive scalar t such that $\langle x - ty, u_j\rangle \le 0$, $j = 1, 2, \dots, n$.

But then $x - ty \in \mathbf{K}$, that is $ty \le x$ and $ty \in \mathbf{K}$ whence $ty \in F$ by the definition of F. Now, since F is a cone, it follows that $y \in F$ and y being arbitrary in $\mathbf{K}$ we obtain that $\mathbf{K} \subset F$ contradicting the hypothesis that F is a proper face of $\mathbf{K}$. Hence there exists some minimal set $\{i_1,\dots,i_k\} \subset \{1,2,\dots,n\}$, $k \ge 1$ so that $\langle x, u_{i_j}\rangle = 0$, $j = 1, 2, \dots, k$ for every $x \in F$. By the first part of the proof we have $F = F_{i_1,\dots,i_k}$. □

Lemma 2. *If* $\mathbf{K}$ *is a latticial cone given by* (2.1), *then for* $y, z \in R^n$ *the supremum* $y \vee z$ *is the solution of the following system in* x:

$$\langle x, u_i\rangle = \min\{\langle y, u_i\rangle, \langle z, u_i\rangle\} \qquad i = 1,2,\dots,n \tag{2.2}$$

In particular, if $v \in R^n$ *and* $\langle v, u_j\rangle = 0$ *for some* $j \in \{1,2,\dots,n\}$ *then* $\langle v^+, u_j\rangle = 0$ *where* $v^+ = v \vee 0$.

Proof. Since $u_1, u_2, \dots, u_n$ are linearly independent vectors, the system (2.2) has a unique solution x_0. Let us see that $x_0 = z \vee y$. From the definition of x_0 we have,

$$\langle x_0 - y, u_i\rangle = \langle x_0, u_i\rangle - \langle y, u_i\rangle = \min\{\langle y, u_i\rangle, \langle z, u_i\rangle\} - \langle y, u_i\rangle \le 0, \quad i = 1,\dots,n.$$

Hence $x_0 - y \in \mathbf{K}$, that is $y \le x_0$ Similarly we deduce that $z \le x_0$.

Suppose now that for some $x \in R^n$, $y \le x$ and $z \le x$ Then by the definition of $\mathbf{K}$, $\langle x-y, u_i\rangle \le 0$ and $\langle x-z, u_i\rangle \le 0$, $i = 1, 2, \dots, n$ which imply

$$\langle x, u_i\rangle \le \min\{\langle y, u_i\rangle, \langle z, u_i\rangle\} = \langle x_0, u_i\rangle, \quad i = 1,2,\dots,n.$$

Using again the definition of $\mathbf{K}$ we conclude that $x - x_0 \in \mathbf{K}$, *i.e.*, $x_0 \le x$. Thus we have $x_0 = y \vee z$. If for some $v \in R^n$ and some $j \in \{1,2,\dots,n\}$ one has $\langle v, u_j\rangle = 0$ we get $\langle v^+, u_j\rangle = \min\langle v, u_j\rangle, 0 = 0$, since $v = v \vee 0$ is the solution of the system:

$$\langle x, u_i\rangle = \min\{\langle v, u_i\rangle, \langle 0, u_i\rangle\}, \qquad i = 1,2,\dots,n. \quad \square$$

Lemma 3. *Suppose that* $\mathbf{K}$ *is a latticial cone given by* (2.1). *Then there exists the linearly independent vectors* $e_1, e_2, \dots, e_n \in R^n$ *with* $\langle e_i, u_j\rangle = 0$ *if* $i \ne j$ *and* $\langle e_i, u_j\rangle < 0$, i, $j = 1, 2, \dots, n$, *such that*

$$\mathbf{K} = \operatorname{cone}\{e_1,\dots,e_n\} \quad \left(= \left\{\sum_{i=1}^{n} \lambda_i e_i \,\middle|\, \lambda \ge 0, i = 1,2,\dots,n\right\}\right). \tag{2.3}$$

In particular, $\mathbf{K}^0 = \text{cone}\{u_1, u_2, \dots, u_n\}$ *and every latticial cone has a representation of form* (2.3) *with some linearly independent vectors* $e_1, e_2, \dots, e_n$.

Since $e_1, e_2, \dots, e_n$ *are linearly independent then every* $y \in R^n$ *can be uniquely represented in the form,* $y = c_1 e_1 + c_2 e_2 + \cdots + c_n e_n$; $c_1, c_2, \dots, c_n \in R$.

If for another vector $z \in R^n$ *we have* $z = d_1 e_1 + d_2 e_2 + \cdots + d_n e_n$; $d_1, d_2, \dots, d_n \in R$ *then* $z \leq y$ *is equivalent with* $d_i \leq c_i$, $i = 1, 2, \dots, n$.

Proof. Since $u_1, u_2, \dots, u_n$ are linearly independent, then $u_1, \dots, u_{j-1}, u_{j+1}, \dots, u_n$ span a hyperplane in R^n. If e is a normal vector to this hyperplane then, since $u_j \notin \text{sp}\{u_1, \dots, u_{j-1}, u_{j+1}, \dots, u_n\}$ it follows that $\langle e, u_j \rangle \neq 0$. Choose a normal e_j to this hyperplane so that $\langle e_j, u_j \rangle < 0$. Obviously $\langle e_j, u_i \rangle = 0$ if $i \neq j$ and hence $e_j \in \mathbf{K}$.

Take $j = 1, 2, \dots, n$ in order to obtain $e_1, e_2, \dots, e_n$. By the biorthogonality of the systems $e_1, e_2, \dots, e_n$ and $u_1, u_2, \dots, u_n$, it can be easily deduced that $e_1, e_2, \dots, e_n$ are linearly independent. We have obviously $\text{cone}\{e_1, e_2, \dots, e_n\} \subset \mathbf{K}$. To show the converse inclusion take $x = c_1 e_1 + \cdots + c_n e_n$ with $c_j < 0$. By scalar multiplication with u_j it follows that $\langle x, u_j \rangle = c_j \langle e_j, u_j \rangle > 0$ and hence $x \notin \mathbf{K}$.

The last assertion of the lemma follows directly from the representation (2.3) of $\mathbf{K}$. □

The next result is true for a well based closed convex cone in a reflexive Banach space but because in this paper $\mathbf{K}$ is in R^n we give this result with an elementary proof.

Lemma 4. *If* $\mathbf{K}$ *is a closed cone in* R^n *then every* $\mathbf{K}$*-increasing,* $\mathbf{K}$*-order bounded sequence in* R^n *converges to its* $\mathbf{K}$*-supremum.*

Proof. Since $\mathbf{K}$ is a closed cone, we have $\mathbf{K} = (\mathbf{K}^0)^0$.

Hence $\mathbf{K}^0$ must be generating, since if $\mathbf{K}^0$ would be contained in some subspace of codimension one, then the orthogonal complement of this last space would be in $(\mathbf{K}^0)^0 = \mathbf{K}$, contradicting the definition of $\mathbf{K}$.

Let $u_1, u_2, \dots, u_n$ be a linearly independent vectors in $\mathbf{K}^0$. Then $\text{cone}\{u_1, u_2, \dots, u_n\} \subset \mathbf{K}^0$ and hence $\mathbf{K} \subset \mathbf{K}_0$, where $\mathbf{K}_0 = (\{u_1, \dots, u_n\})^0$.

By Lemma 3, $\mathbf{K}_0$ can be represented in the form, $\mathbf{K}_0 = \text{cone}\{e_1, e_2, \dots, e_n\}$, $e_1, e_2, \dots, e_n$ being linearly independent vectors in R^n.

Consider now the sequence $\{x_m\}_{m \in N}$ in R^n such that,

$$x_1 \leq_{\mathbf{K}} x_2 \leq_{\mathbf{K}} \cdots \leq_{\mathbf{K}} x_m \leq_{\mathbf{K}} \cdots \leq_{\mathbf{K}} u$$

for some $u \in R^n$. Since $\mathbf{K} \subseteq \mathbf{K}_0$ we have also

$$x_1 \leq_{\mathbf{K}_0} x_2 \leq_{\mathbf{K}_0} \cdots \leq_{\mathbf{K}_0} x_m \leq_{\mathbf{K}_0} \cdots \leq_{\mathbf{K}_0} u \tag{2.4}$$

Let us take the representations

$$x_m = c_1^m e_1 + \cdots + c_n^m e_n, \quad m = 1, 2, \dots$$
$$u = c_1 e_1 + \cdots + c_n e_n$$

where c_j^m, $c_j \in R$, $j = 1, 2, \dots, n$. Then according (2.4) and Lemma 3, every sequence of real numbers $\{c_j^m\}_{m \in N}$ ($j = 1, 2, \dots, n$) is monotonically increasing and bounded by c_j, hence convergent. Denote

$$c_j^0 = \lim_{m \to \infty} c_j^m, \qquad j = 1, 2, \dots, n. \tag{2.5}$$

Then $\{x_m\}_{m \in N}$ is convergent to $x_0 = c_1^0 e_1 + \cdots + c_n^0 e_n$.

From relations $x_p - x_q \in \mathbf{K}$ for $q \leq p$ and $u - x_p \in \mathbf{K}$ for each p, passing to the limit with $p \to \infty$ and taking into account that $\mathbf{K}$ is closed, we deduce that $x_q \leq_{\mathbf{K}} x_0$ for each q and $x_0 \leq_{\mathbf{K}} u$, witch completes the proof of the lemma. □

Before passing to some facts concerning correct cones, let us remember some results on projections maps. First of all we have that $P_{\mathbf{C}}(x)$ is the nearest element in the closed convex set $\mathbf{C} \subset R^n$ to $x \in R^n$, if and only if we have:

$$\langle x - P_{\mathbf{C}}(x), P_C(x) - y \rangle \geq 0, \qquad \forall y \in \mathbf{C}. \tag{2.6}$$

(see [19, Lemma 1.1])

We shall also use the fact that for any x and y in R^n and for every closed convex set $\mathbf{C} \subset R^n$ the following holds

$$\|P_{\mathbf{C}}(x) - P_{\mathbf{C}}(y)\| \leq \|x - y\|. \tag{2.7}$$

that is, $P_{\mathbf{C}}$ is nonexpansive and hence also continuous (see [19, formula (1.8)]).

The characterization of projections on a cone and its polar is the object of the following result.

Theorem (Moreau). *If* $\mathbf{K}$ *is a closed convex cone in* R^n *then the following assertions are equivalent:*

(i) $x = u + v$, $u \in \mathbf{K}$, $v \in \mathbf{K}^0$ *and* $\langle u, v \rangle = 0$
(ii) $u = P_{\mathbf{K}}(x)$, $v = P_{\mathbf{K}^0}(x)$.

Lemma 5. *If* $\mathbf{K} \subset R^n$ *is a correct cone and if* F *is a face of* $\mathbf{K}$, *then for every* $x \in \operatorname{sp} F$ *one has* $P_{\mathbf{K}}(x) = P_F(x)$.

Proof. Assume the contrary, that is, there exists some x in $\operatorname{sp} F$ such that $P_{\mathbf{K}}(x) \notin F$.

Since $P_{\operatorname{sp} F}$ is nonexpansive (see (2.7)) we have

$$\|x - P_{\operatorname{sp} F}(P_{\mathbf{K}}(x)\| = \|P_{\operatorname{sp} F}(x) - P_{\operatorname{sp} F}(P_{\mathbf{K}}(x))\| \leq \|x - P_{\mathbf{K}}(x)\|. \tag{2.8}$$

Since $P_{\operatorname{sp} F}(\mathbf{K}) \subset F$, by the correctness of $\mathbf{K}$ we have $P_{\operatorname{sp} F}(P_{\mathbf{K}}(x)) \in F \subset \mathbf{K}$.

By the uniqueness of the nearest element, we have by (2.8) that $P_{\operatorname{sp} F}(P_{\mathbf{K}}(x)) = P_{\mathbf{K}}(x)$, whence $P_{\mathbf{K}}(x) \in (\operatorname{sp} F) \cap \mathbf{K} = F$ which is impossible and the lemma is proved. □

Let v be in $\mathbf{K}^0$ and consider the set $F_v = \{x \in \mathbf{K} \mid \langle x, v \rangle = 0\}$.

Then a straightforward verification shows that F_v is a face of $\mathbf{K}$.

Faces of the above kind are called *exposed faces* [19].

The vector v is said a *normal* to the face F_v.

Lemma 6. *Let* $\mathbf{K}$ *be a correct cone in* R^n *and let* F *be an exposed face of* $\mathbf{K}$ *with codimension one in* R^n.

If v *is normal to* F, *then for any other normal* v' *to any other exposed face* F' *of* $\mathbf{K}$, *not contained in* F, *we have* $\langle v, v' \rangle \leq 0$.

Proof. Suppose the contrary. So, we suppose that for some such normal v' we have $\langle v, v' \rangle > 0$. Let $x \in F' \setminus F$

Hence $\langle v, x\rangle < 0$ (since $v \in \mathbf{K}^0$) and we can determine a positive scalar t such that $\langle x + tv', v\rangle = 0$.

But from Moreau's theorem we have $P_{\mathbf{K}}(x + tv') = x$. since F is of codimension one, its normal is v and $\langle x + tv', v\rangle = 0$, necessarily we have $x + tv' \in \operatorname{sp} F$ and we have got a contradiction with Lemma 5. □

Proof of the principal Theorem

3. Proof of the implication (i) $\Longrightarrow$ (ii). In proving that the isotone projection cone $\mathbf{K} \subset R^n$ is latticial we shall use the following assertion:

(a). Let $\mathbf{K}$ *be a closed and generating cone in* R^n *and* u, v *two elements of* R^n. *If there exist* $a \in u + \mathbf{K}$, $b \in v + \mathbf{K}$ *with the properties*

$$a = P_{u+\mathbf{K}}(b) \quad \textit{and} \quad b = P_{v+\mathbf{K}}(a), \quad \textit{then} \quad a = b \in (u + \mathbf{K}) \cap (v + \mathbf{K})$$

Indeed, since $\mathbf{K}$ is generating the set $(u + \mathbf{K}) \cap (v + \mathbf{K})$ is nonempty, that is, there exists an element w such that $u \leq w$ and $v \leq w$. This follows by writing $u = u_1 - u_2$, $v = v_1 - v_2$, where u_1, u_2, v_1, $v_2 \in \mathbf{K}$ and observing that we can consider $w = u_1 + v_1$.

We have from the characterization (2.6) of the metric projections that,

$$\begin{aligned} \langle a - P_{v+\mathbf{K}}(a), P_{v+\mathbf{K}}(a) - w\rangle \geq 0 \\ \langle b - P_{u+\mathbf{K}}(b), P_{u+\mathbf{K}}(b) - w\rangle \geq 0 \end{aligned} \tag{3.1}$$

Using the conditions in the assertion (a) the second relation becomes,

$$\langle P_{v+\mathbf{K}}(a) - a, a - w\rangle \geq 0. \tag{3.2}$$

On the other hand we have

$$\begin{aligned} \langle P_{v+\mathbf{K}}(a) - a, a - w\rangle &= \langle P_{v+\mathbf{K}}(a) - a, (a - P_{v+\mathbf{K}}(a)) + (P_{v+\mathbf{K}}(a) - w)\rangle \\ &= -\left(\|P_{v+\mathbf{K}}(a) - a\|^2 + \langle a - P_{v+\mathbf{K}}(a), P_{v+\mathbf{K}}(a) - w\rangle\right) \end{aligned}$$

whence, taking into account (3.1) and (3.2) it follows that,

$$\|P_{v+\mathbf{K}}(a) - a\| = \|b - a\| = 0,$$

and the assertion (a) is proved.

(b). Let us pass to the proof of the latticiality of $\mathbf{K}$.

Consider the arbitrary elements u and v in R^n. We shall show, using the isotone projection property of $\mathbf{K}$, that they admit a least upper bound $u \vee v$ by constructing effectively this element.

We can assume that u and v are not comparable.

Let w be an arbitrary upper bound of the set $\{u, v\}$, *i.e.* an arbitrary element of the set $(u + \mathbf{K}) \cap (v + \mathbf{K})$ which is not empty since $\mathbf{K}$ is generating by hypothesis.

Let us note next that if $P_{\mathbf{K}}$ is isotone, then for an arbitrary element y in R^n the operator $P_{y+\mathbf{K}}$ is isotone too.

This follows from the relation $P_{y+\mathbf{K}}(x) = P_{\mathbf{K}}(x-y)+y$ which holds for an arbitrary x in R^n and which can be directly verified by using (2.6). Hence $P_{u+\mathbf{K}}$ and $P_{v+\mathbf{K}}$ are both isotone. Since no one of the convex sets $u + \mathbf{K}$ and $v + \mathbf{K}$ is contained in the other, using assertion (a) we see that there cannot hold simultaneously the relations $u = P_{u+\mathbf{K}}(v)$ and $v = P_{v+\mathbf{K}}(u)$.

Suppose that $u \neq P_{u+\mathbf{K}}(v) \in u + \mathbf{K}$

Then $u \leq P_{u+\mathbf{K}}(v) \leq P_{u+\mathbf{K}}(w) = w$, since $P_{u+\mathbf{K}}$ is isotone and $w \in u + \mathbf{K}$.

Let us consider the operators $Q = P_{v+\mathbf{K}} \circ P_{u+\mathbf{K}}$ and $R = P_{u+\mathbf{K}} \circ P_{v+\mathbf{K}}$. They are isotone since $P_{v+\mathbf{K}}$ and $P_{u+\mathbf{K}}$ are. Put $v_n = Q^n(v)$, $u_1 = P_{u+\mathbf{K}}(v)$ and $u_n = R^{n-1}(u)$. Then we have the following relations:

$$v \leq v_1 \leq \cdots \leq v_n \leq \cdots \leq w$$
$$u \leq u_1 \leq \cdots \leq u_n \leq \cdots \leq w$$

since $u \leq u_1$, $v \leq v_1$, since $P_{u+\mathbf{K}}$, Q and R are isotone, and since $P_{u+\mathbf{K}}(w) = Q(w) = R(w) = w$. Obviously $P_{v+\mathbf{K}} \circ P_{u+\mathbf{K}}(v) \in v + \mathbf{K}$, hence $v \leq P_{v+\mathbf{K}} \circ P_{u+\mathbf{K}}(v) = Q(v) = v_1$ and $u_1 = P_{u+\mathbf{K}}(v) \leq P_{u+\mathbf{K}} \circ P_{v+\mathbf{K}} \circ P_{u+\mathbf{K}}(v)$, that is, $u_1 \leq R(u_1) = u_2$ *etc.* We have further

$$\begin{aligned} v_n &= Q^n(v) = (P_{v+\mathbf{K}} \circ P_{u+\mathbf{K}})^n(v) \\ &= P_{v+\mathbf{K}} \circ (P_{u+\mathbf{K}} \circ P_{v+\mathbf{K}})^{n-1} \circ P_{u+\mathbf{K}}(v) = P_{v+\mathbf{K}} \circ R^{n-1}(u_1) = P_{v+\mathbf{K}}(u_n) \end{aligned} \tag{3.3}$$

and

$$u_{n+1} = R(u_n) = P_{u+\mathbf{K}} \circ P_{v+\mathbf{K}}(u_n) = P_{u+\mathbf{K}}(v_n). \tag{3.4}$$

Since the sequences $\{u_n\}$ and $\{v_n\}$ are increasing and bounded above by w, we have (using Lemma 4) the following relations:

$$u_0 = \lim_{n\to\infty} u_n \quad \text{and} \quad v_0 = \lim_{n\to\infty} v_n \tag{3.5}$$

as well as

$$u \leq u_0 \leq w \quad \text{and} \quad v \leq v_0 \leq w. \tag{3.6}$$

From the continuity of the metric projections (see relation (2.7))the formulas (3.3), (3.4) and (3.5) yield

$$v_0 = P_{v+\mathbf{K}}(u_0) \quad \text{and} \quad u_0 = P + v + \mathbf{K}(v_0).$$

Using assertion (a) again we deduce that

$$u_0 = v_0 \in (u + \mathbf{K}) \cap (v + \mathbf{K}).$$

Since the upper bound u was arbitrary, from the relation (3.6) we obtain that indeed $u_0 = v_0 = u \vee v$ and the latticiality of $\mathbf{K}$ is proved.

To prove the correctness of $\mathbf{K}$ we begin by proving the following assertion:

(c). For every face F of the generating isotone projection cone $\mathbf{K}$ in R^n the subspace $\operatorname{sp} F$ *projects onto F by $P_{\mathbf{K}}$ and F is an isotone projection cone in the space* $\operatorname{sp} F$.

Consider $z \in \operatorname{sp} F$. Then $z = x - y$ with $x, y \in F \subset \mathbf{K}$ whence $z \leq x$.

Since $P_{\mathbf{K}}$ is isotone, one follows $0 \leq P_{\mathbf{K}}(z) \leq P_{\mathbf{K}}(x) = x \in F$. Hence $P_{\mathbf{K}}(z) \in F$.

This relation shows that $P_F(z) = P_{\mathbf{K}}(z)$ and implicitly that $P_F|_{\operatorname{sp} F}$ is isotone projection in $\operatorname{sp} F$ and (c) is proved.

(d). We pass to the proof of correctness of the isotone projection cone $\mathbf{K}$ by assuming the contrary, that is, we suppose that there exists a face F of $\mathbf{K}$ and an element k of $\mathbf{K}$ such that $z = P_{\operatorname{sp} F}(k) \notin F$.

Put $z_0 = P_{\mathbf{K}}(z)$. Since $z \in \operatorname{sp} F$, it follows from the assertion (c) that $z_0 \in F$.

We shall show first that one can find a real number $t \in (0,1)$ such that the element w given by

$$w = tk + (1-t)z_0 \tag{3.7}$$

satisfies the relation

$$\langle z - w, k - z_0 \rangle = 0 \tag{3.8}$$

Indeed, we have

$$\begin{aligned}\langle z - tk - (1-t)z_0, k - z_0 \rangle &= \langle z - k + (1-t)(k - z_0), k - z_0 \rangle \\ &= \langle z - k, k - z_0 \rangle + (1-t)\|z_0 - k\|^2 \\ &= \langle z - k, k - z + z - z_0 \rangle + (1-t)\|z_0 - k\|^2 \\ &= -\|z - k\|^2 + (1-t)\|z_0 - k\|^2,\end{aligned}$$

since $\langle z - k, z - z_0 \rangle = 0$ ($z - z_0 \in \operatorname{sp} F$ and $z - k$ is orthogonal to $\operatorname{sp} F$).

Since $\|z - k\| < \|z_0 - k\|$ by the definition of z and z_0, then putting

$$1 - t = \frac{\|z - k\|^2}{\|z_0 - k\|^2} < 1,$$

we have (3.8) for w determined by (3.7).

Using the characterization (2.6) of the metric projections, we have

$$\langle z - z_0, z_0 - k \rangle = \langle z - P_{\mathbf{K}}(z), P_{\mathbf{K}}(z) - k \rangle \geq 0. \tag{3.9}$$

From the definition of w it follows on the other hand that

$$\begin{aligned}\langle z - z_0, z_0 - k \rangle &= \langle z - w + w - z_0, z_0 - k \rangle = \langle w - z_0, z_0 - k \rangle \\ &= \langle tk + (1-t)z_0 - z_0, z_0 - k \rangle = t\,\langle k - z_0, z_0 - k \rangle < 0.\end{aligned}$$

This relation contradicts (3.9) and shows that our hypothesis that $\mathbf{K}$ is not correct, is false.

4. Proof of the implications (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (i). Obviously, the implication (ii) $\Longrightarrow$ (iii) is a consequence of Youdine's Theorem.

We shall prove (iii) $\Longrightarrow$ (i) by induction with respect to the dimension of the space.

For dimension one we have nothing to prove. We shall do the induction step for the sake of simplicity as follows.

Suppose that the implication

$$z \leq_F y \Longrightarrow P_F(z) \leq_F P_F(y), \qquad y, z \in \operatorname{sp} F \tag{4.1}$$

holds for every face F of codimension one of $\mathbf{K}$ in R^n and prove it for F replaced by $\mathbf{K}$. (Observe that the hypothesis in (iii) hold for faces too since correctness and polyhedrality are both hereditary for faces).

Since $\mathbf{K}$ is polyhedral, there exists a finite set of unit vectors $\{u_i\}_{i=1}^m$, the normals to the maximal proper faces of $\mathbf{K}$, such that $\mathbf{K} = \left(\{u_i\}_{i=1}^m\right)^0$ and $F_i = \mathbf{K} \cap \ker u_i$ is a face of codimension one for each i.

(a). Consider the elements y, z in R^n such that $z \leq y$. Let u_i be the normal to the face F of codimension one of $\mathbf{K}$.

Then $\ker u_i = \operatorname{sp} F$ and let us denote $p = P_{\operatorname{sp} F}$. Since u_i is a unit vector we have, $p(y) = y - \langle y, u_i \rangle u_i$ and $p(z) = z - \langle z, u_i \rangle u_i$. Let us show that

$$p(z) \leq p(y). \tag{4.2}$$

We have obviously $\langle p(y) - p(z), u_i \rangle = 0$

Using the above expressions for $p(y)$ and $p(z)$ we have for $j \neq i$:

$$\langle p(y) - p(z), u_j \rangle = \langle y - z - \langle y - z, u_i \rangle u_i, u_j \rangle = \langle y - z, u_j \rangle - \langle y - z, u_i \rangle \langle u_i, u_j \rangle .$$

The first term in the last sum and the factor $\langle y - z, u_i \rangle$ in the second term are both nonpositive since $y - z \in \mathbf{K}$.

The correctness of $\mathbf{K}$ implies via Lemma 6 that $\langle u_i, u_j \rangle \leq 0$, whence the second term in the last sum of the above formula is also nonpositive.

According to the definition of $\mathbf{K}$ as $\left(\{u_j\}_{j=1}^m\right)^0$ the above conclusions prove (4.2), which can be written also in the form,

$$p(z) \leq_F p(y) \tag{4.3}$$

since $p(z)$, $p(y) \in \operatorname{sp} F$ and $F = \operatorname{sp} F \cap \mathbf{K}$.

(b). Let us show next that, if condition (iii) is satisfied then for every $x \in R^n$ such that $\langle x, u_i \rangle \geq 0$ for some i, one has

$$P_{\mathbf{K}}(x) = P_F\big(p(x)\big) \tag{4.4}$$

with $F = (\ker u_i) \cap \mathbf{K}$ and $p = P_{\operatorname{sp} F}$.

Indeed, since $\mathbf{K}$ is correct, Lemma 5 implies,

$$P_F\big((x)\big) = P_{\mathbf{K}}\big(p(x)\big).$$

Hence, for an arbitrary $w \in R^n$ we have

$$\begin{aligned}&\langle x - P_F(p(x)), P_F(p(x)) - w\rangle \\ &\qquad = \langle x - p(x), P_{\mathbf{K}}(p(x)) - w\rangle + \langle p(x) - P_{\mathbf{K}}(p(x)), P_{\mathbf{K}}(p(x)) - w\rangle\end{aligned}$$

Let now w be an arbitrary element of $\mathbf{K}$.

Then the second term in the last sum is nonnegative according to the characterization (2.6) of the projection maps.

If $\langle x, u_i\rangle = 0$, then $x = p(x)$ and the first term in the above sum is zero.

If $\langle x, u_i\rangle > 0$, then $x - p(x)$ is orthogonal to $\operatorname{sp} F = \ker u_i$. Hence it is parallel with u_i and has its direction since $\langle x - p(x), u_i\rangle = \langle x, u_i\rangle > 0$ by hypothesis.

Whence $x - p(x) \in \mathbf{K}^0$ and since $P_{\mathbf{K}}(p(x)) \in F \subset \ker u_i$, it follows that

$$\langle x - p(x), P_{\mathbf{K}}(p(x)) - w\rangle = -\langle x - p(x), w\rangle \geq 0,$$

for every $w \in \mathbf{K}$.

In conclusion we have,

$$\langle x - P_F(p(x)), P_F(p(x)) - w\rangle \geq 0, \qquad \forall w \in \mathbf{K},$$

whence using again the characterization (2.6) of the projection, we conclude that the relation (4.4) holds.

(c). Let us consider again that $z \leq y$ and suppose that $y \notin \operatorname{Int}\mathbf{K}$. This condition is equivalent with the existence of some subscript i such that $\langle y, u_i\rangle \geq 0$.

Since $y - z \in \mathbf{K}$ we have $\langle y - z, u_i\rangle \leq 0$ whence we have also $\langle z, u_i\rangle \geq 0$.

If $F = (\ker u_i) \cap \mathbf{K}$ and $p = P_{\operatorname{sp} F}$, then we have by the result proved in (a) (see relation (4.3)), that

$$p(z) \leq_F p(y). \tag{4.5}$$

Use now the fact that both $\langle y, u_i\rangle$ and $\langle z, u_i\rangle$ are nonnegative and the result proved in (b), formula (4.4) to conclude that

$$P_{\mathbf{K}}(y) = P_F(p(y)) \quad \text{and} \quad P_{\mathbf{K}}(z) = P_F(p(z)). \tag{4.6}$$

Since $p(y)$ and $p(z)$ are in $\operatorname{sp} F$ we have according to the induction hypothesis (4.1) via (4.5) that

$$P_F(p(z)) \leq P_F(p(y)).$$

Using now (4.6) we conclude that $P_{\mathbf{K}}(z) \leq_F P_{\mathbf{K}}(y)$, whence $P_{\mathbf{K}}(z) \leq P_{\mathbf{K}}(y)$. (Particularly in this case it follows that both y and z project on the same proper face F).

(d). Suppose now that $y \in \operatorname{Int}\mathbf{K}$. If $z \in \mathbf{K}$, then we have nothing to prove.

If $z \notin \mathbf{K}$, then the line segment $\{y_t \mid t \in (0,1)\}$ with $y_t = tz + (1-t)y$ pierces the boundary of $\mathbf{K}$ at some point y_{t_0}, that is, we have $\langle y_{t_0}, u_i\rangle = 0$ for some subscript i and $\langle y_{t_0}, u_j\rangle \leq 0$ for $j \neq i$.

But $z \leq y_{t_0} \leq y$. From the result established by induction in the point (c) we have

$$P_{\mathbf{K}}(z) \leq P_{\mathbf{K}}(y_{t_0}) = y_{t_0}.$$

Since $y_{t_0} \leq y = P_{\mathbf{K}}(y)$ the last two relations show that $P_{\mathbf{K}}(z) \leq P_{\mathbf{K}}(y)$ also in this case.

Thus the proof of (iii) $\Longrightarrow$ (i) is complete. □

Remark. Putting together the results of sections 3 and 4 we conclude that the assertions (i), (ii) and (iii) of our theorem are equivalent.

Hence we got in turn a new proof of the equivalence of (ii) and (iii) which was given in [2].

5. Proof of the implications (iii) $\Longrightarrow$ (iv) $\Longrightarrow$ (ii). Suppose that (iii) holds. If we consider the normals u_i, $i = 1, \ldots, m$ to the maximal faces of the polyhedral cone $\mathbf{K}$, then $\mathbf{K} = \left(\{u_i\}_{i=1}^m\right)^0$ and using the correctness of $\mathbf{K}$, we have by Lemma 6 that $\langle u_i, u_j \rangle \leq 0$, for $i \neq j$. Thus the implication (iii) $\Longrightarrow$ (iv) was established.

Suppose now that we have (iv) fulfilled.

We shall show first that the vectors u_i, $i \in I$ satisfying this condition are linearly independent.

Since $\mathbf{K}$ is a generating closed cone, in this set, there exist n linearly independent vectors (see the first part of the proof of Lemma 4)

Suppose that $u_1, u_2, \ldots, u_n$ are linearly independent vectors in this set and let us verify the assertion:

(a) *Let $u_1, u_2, \ldots, u_n$ be linearly independent elements in R^n satisfying the conditions $\langle u_i, u_j \rangle \leq 0$, $i \neq j$, $i, j = 1, 2, \ldots, n$. If for some $v \in R^n$ one has $\langle v, u_i \rangle \leq 0$, $i = 1, 2, \ldots, n$, then*

$$v = c_1 u_1 + \cdots + c_n u_n \quad \text{with } c_i \leq 0,\ i = 1, 2, \ldots, n. \tag{5.1}$$

We shall use in the proof a process, which yields an orthogonal basis $w_1, w_2, \ldots, w_n$, every w_i being a linear combination of elements u_j with nonnegative coefficients.

Put $w_1 = u_1$ and suppose that $w_1, w_2, \ldots, w_{k-1}$ were determined $\langle w_i, w_j \rangle = 0$, $i, j \leq k-1$, $i \neq j$ and each of them is a linear combination with nonnegative coefficients of the vectors u_j with $j \leq k-1$.

Let be $w_k = t_1 w_1 + \cdots + t_{k-1} w_{k-1} + u_k$, where the real coefficients $t_1, t_2, \ldots, t_{k-1}$ will be determined.

According to the conditions on $w_1, w_2, \ldots, w_{k-1}$, we have $\langle w_j, u_k \rangle \leq 0$, $j \leq k-1$.

Hence we can determine $t_1, t_2, \ldots, t_{k-1}$ such that $t_j \geq 0$, $j \leq k-1$, from the relation

$$0 = \langle w_k, w_j \rangle = t_j \langle w_j, w_j \rangle + \langle u_k, w_j \rangle$$

This shows that w_k is a linear combination with nonnegative coefficients of $u_1, u_2, \ldots, u_k$ and is orthogonal to w_j, $j \leq k-1$.

We have obviously that $w_1, w_2, \ldots, w_n$ are linearly independent.

Let us consider the representation,

$$v = d_1 w_1 + \cdots + d_n w_n, \quad d_j \in R, \quad j = 1, 2, \ldots, n. \tag{5.2}$$

Since $\langle v, u_i \rangle \leq 0$, $i = 1, 2, \ldots, n$, by hypothesis and since w_k are combinations with nonnegative coefficients of $u_1, u_2, \ldots, u_k$, $k = 1, 2, \ldots, n$, we have $\langle v, w_k \rangle \leq 0$, $k = 1, 2, \ldots, n$.

Assume that we have in (5.2) $d_k > 0$ for some k. Multiplying this relation with w_k we obtain

$$0 \geq \langle v, w_k \rangle = d_k \langle w_k, w_k \rangle > 0$$

The obtained contradiction shows that $d_k \leq 0$, $k = 1, 2, \dots, n$

Let we put in (5.2) the representations of w_k, $k = 1, 2, \dots, n$ as linear combinations of $u_j, j = 1, 2, \dots, n$. Since the coefficients in these representations are nonnegative and d_k, $k = 1, 2, \dots, n$ are nonpositive, we get a representation of v as a linear combination of $u_1, u_2, \dots, u_n$ with nonpositive coefficients. But the resulting coefficients must be the coefficients $c_1, c_2, \dots, c_n$ in (5.1) and the assertion (a) is proved.

(b). Let $u_1, u_2, \dots, u_n$ be linearly independent vectors in the set $\{u_i \mid i \in I\}$ considered in assertion (iv) of the theorem.

We shall show that they are the only nonzero vectors of this set.

Indeed, if v would be another nonzero vector in $\{u_i \mid i \in I\}$, then by the condition in (iv) and by assertion (a) we would obtain the representation (5.1) with $c_i \leq 0$.

But then $-v \in \operatorname{cone}\{u_1, u_2, \dots, u_n\} \subset \operatorname{cone}\{u_i \mid i \in I\}$, that is, v and $-v$ would be both in $\operatorname{cone}\{u_i \mid i \in I\}$ and hence $\mathbf{K}$ would be contained in the hyperplane perpendicular to v, contradicting the hypothesis on $\mathbf{K}$ to be generating. Thus we must have in fact that

$$\mathbf{K} = \{x \in R^n \mid \langle x, u_i \rangle \leq 0, i = 1, 2, \dots, n; u_1, u_2, \dots, u_n \text{ linearly independent}\}, \tag{5.3}$$

relation which together with the theorem of Youdine shows that $\mathbf{K}$ is latticial.

(c). To see that $\mathbf{K}$ is correct we shall prove first that if $F = (\ker u_n) \cap \mathbf{K}$ then $P_{\operatorname{sp} F}(\mathbf{K}) \subset F$.

From representation (5.3) deduced above and from Lemma 3, there exist the linearly independent vectors $e_1, e_2, \dots, e_n$ such that

$$\begin{aligned} \mathbf{K} &= \operatorname{cone}\{e_1, e_2, \dots, e_n\}, \\ \langle e_i, u_j \rangle &= 0 \text{if } i \neq j \text{ and} \\ \langle e_i, u_i \rangle &< 0, \quad i, j = 1, 2, \dots, n. \end{aligned} \tag{5.4}$$

Hence, we have that $\ker u_n = \operatorname{sp} F = \operatorname{sp}\{e_1, e_2, \dots, e_{n-1}\}$

The condition $P_{\operatorname{sp} F}(\mathbf{K}) \subset F$ is then equivalent with $P_{\operatorname{sp} F}(e_n) \in F$, since for an arbitrary $x \in \mathbf{K}$ we have

$$x = c_1 e_1 + \cdots + c_{n-1} e_{n-1} + c_n e_n$$

with $c_j \geq 0$, $j = 1, 2, \dots, n$ and hence

$$P_{\operatorname{sp} F}(x) = c_1 e_1 + \cdots + c_{n-1} e_{n-1} + c_n P_{\operatorname{sp} F}(e_n)$$

by the linearity of $P_{\operatorname{sp} F}$.

We can suppose without loss of generality, that u_n is a unit vector and, then

$$P_{\operatorname{sp} F}(e_n) = e_n - \langle e_n, u_n \rangle u_n.$$

One has further,

$$\langle P_{\operatorname{sp} F}(e_n), u_j \rangle = - \langle e_n, u_n \rangle \langle u_n, u_j \rangle \leq 0,$$

for $j = 1, 2, \dots, n-1$ (since $\langle e_n, u_n \rangle < 0$ and $\langle u_n, u_j \rangle \le 0$ by hypothesis).

Since obviously $\langle P_{\mathrm{sp}\,F}(\epsilon_n), u_n \rangle = 0$, it follows that

$$P_{\mathrm{sp}\,F}(e_n) \in \mathbf{K} \cap \mathrm{sp}\, F = F.$$

(d). Let us see next that F has in $\mathrm{sp}\,F$ the property similar to those of $\mathbf{K}$ in R^n, that is,

$$F = \left\{ x \in \mathrm{sp}\,F \;\middle|\; \begin{array}{l} \langle x, v_j \rangle \le 0,\ j = 1, 2, \dots, n-1 \\ v_1, v_2, \dots, v_{n-1} \text{ linearly independent in } \mathrm{sp}\,F \\ \text{and } \langle v_i, v_j \rangle \le 0,\ i \ne j,\ i, j = 1, 2, \dots, n-1 \end{array} \right\} \tag{5.5}$$

Indeed, let us take

$$v_j = u_j - \langle u_j, u_n \rangle\, u_n, \quad j = 1, 2, \dots, n-1. \tag{5.6}$$

Then the vector v_j are obviously linearly independent and

$$\langle v_i, v_j \rangle = \langle u_i - \langle u_i, u_n \rangle u_n, u_j - \langle u_j, u_n \rangle u_n \rangle = \langle u_i, u_j \rangle - \langle u_j, u_n \rangle \langle u_i, u_n \rangle,$$

since $\|u_n\| = 1$.

Because $\langle u_i, u_j \rangle$, $\langle u_i, u_n \rangle$ and $\langle u_j, u_n \rangle$ are all nonpositive if $i \ne j$, we conclude that in this case $\langle v_i, v_j \rangle \le 0$.

We have the representation

$$F = \{x \in R^n \mid \langle x, u_n \rangle = 0 \text{and} \langle x, u_j \rangle \le 0,\ j = 1, 2, \dots, n-1\}$$

and hence taking into account the representations (5.6) of v_j and the relations proved above, we arrive to (5.5).

(e). Denote by G the face

$$G = \mathbf{K} \cap (\ker u_n) \cap (\ker u_{n-1}).$$

Then G is a face of F of codimension one in $\mathrm{sp}\,F$ and since (d) we can apply the assertion proved in (c) for $\mathbf{K}$ replaced by F and R^n replaced by $\mathrm{sp}\,F$.

Denote $p = P_{\mathrm{sp}\,F}$ and $q = P_{\mathrm{sp}\,G}$. With these notations we have

$$P_{\mathrm{sp}\,G}|_{\mathrm{sp}\,F}(F) = q|_{\mathrm{sp}\,F}(F) \subset G.$$

Let us show now that

$$q = q|_{\mathrm{sp}\,F} \circ p.$$

To verify this, consider an arbitrary element $x \in R^n$ and put it in the form $x = u + v$ with $u \in \mathrm{sp}\,F$ and $v \in (\mathrm{sp}\,F)^0$.

Assume further that $u = w + z$ with $w \in \mathrm{sp}\,G$ and $z \in (\mathrm{sp}\,G)^0 \cap \mathrm{sp}\,F$.

Then $x = w + z + v$. Since $\mathrm{sp}\,G \subset \mathrm{sp}\,F$, it follows that $(\mathrm{sp}\,F)^0 \subset (\mathrm{sp}\,G)^0$ and thus $z + v \in (\mathrm{sp}\,G)^0$, whence $q(x) = w$, $p(x) = w + z$ and $q|_{\mathrm{sp}\,F}(w + z) = w$, that is $q(x) = q|_{\mathrm{sp}\,F}(p(x))$.

If we apply twice (c) and use the above conclusions, it follows that $q(\mathbf{K}) = (q|_{\mathrm{sp}\,F} \circ p)(\mathbf{K}) \subset q|_{\mathrm{sp}\,F}(F) \subset G$, that is $P_{\mathrm{sp}\,G}(\mathbf{K}) \subset G$.

(f). If H is an arbitrary face of $\mathbf{K}$ then we can include it in a chain

$$H \subset H_1 \subset H_2 \subset \cdots \subset H_k$$

such that $H_1, H_2, \ldots, H_k$ have the property in their spans similar to those of $\mathbf{K}$ in R^n stated at (iv) of our theorem and so that H is a face of codimension one of H_1, with respect to $\operatorname{sp} H_1$, H_i is a face of codimension one of H_{i+1} with respect to $\operatorname{sp} H_{i+1}$ if $i \leq k-1$ and H_k is a face of codimension one of $\mathbf{K}$.

Repeating step by step the process just described in (c), (d) and (e) we conclude that $P_{\operatorname{sp} H}(\mathbf{K}) \subset H$, that is $\mathbf{K}$ is correct.

The proof of the implications (iv) $\Longrightarrow$ (ii) is hence completed.

6. Proof of the implications (i) and (ii) $\Longrightarrow$ (v) $\Longrightarrow$ (iv). If (ii) holds, then $\mathbf{K}$ is latticial.

Since $x \leq x^+$ with $x^+ = x \vee 0$, from (i) it follows that $P_{\mathbf{K}}(x) \leq P_{\mathbf{K}}(x^+) = x^+$ and we have (v).

We shall verify the implication (v) $\Longrightarrow$ (iv) by contradiction. That is, we assume that, $\mathbf{K}$ is latticial, that is, it can be represented in the form:

$$\mathbf{K} = \{x \mid \langle x, u_i \rangle \leq 0, i = 1, 2, \ldots, n, u_1, \ldots u_n \text{ linearly independent}\}$$

and that for each $x \in R^n$ we have that $P_{\mathbf{K}}(x) \leq x^+$, but there are some vectors, say u_1 and u_2 in the above representation such that $\langle u_1, u_2 \rangle > 0$.

We shall suppose in what follows that $u_1, u_2, \ldots, u_n$ are unit vectors.

(a). If $n = 2$, then we consider an element $x \in \mathbf{K}$ with $\langle x, u_1 \rangle = 0$, $\langle x, u_2 \rangle < 0$. Since $-x \leq 0$ we must have by (v) that $P_{\mathbf{K}}(-x) \leq (-x)^+ = 0$, that is $P_{\mathbf{K}}(-x) = 0$. Consider now the vector $z = -x + \langle x, u_2 \rangle u_2$. Then $\langle z, u_2 \rangle = 0$ and

$$\langle z, u_1 \rangle = \langle -x + \langle x, u_2 \rangle u_2, u_1 \rangle = \langle x, u_2 \rangle \langle u_2, u_1 \rangle < 0. \tag{6.1}$$

Thus $z \in \mathbf{K}$. We have further,

$$\begin{aligned} \langle -x - z, z - w \rangle &= \langle -x - (-x + \langle x, u_2 \rangle u_2), (-x + \langle x, u_2 \rangle u_2) - w \rangle \\ &= \langle -\langle x, u_2 \rangle u_2, \langle x, u_2 \rangle u_2 - (x + w) \rangle \\ &= -\langle x, u_2 \rangle^2 + \langle x, u_2 \rangle \langle u_2, x + w \rangle \\ &= \langle x, u_2 \rangle \langle u_2, w \rangle \geq 0, \end{aligned}$$

$\forall w \in \mathbf{K}$, since $\langle x, u_2 \rangle < 0$ and $\langle u_2, w \rangle \leq 0$, $\forall w \in \mathbf{K}$.

By the characterization (2.6) of the projection we have then that $P_{\mathbf{K}}(-x) = z$. But by (6.1) it must be $z \neq 0$.

The obtained contradiction shows that in this case we cannot have $\langle u_1, u_2 \rangle > 0$.

(b). Suppose that $n \geq 3$. Let us show first that under the above hypothesis there exists an element w in R^n such that

$$\langle w, u_2 \rangle = 0, \quad \langle w, u_j \rangle < 0, \quad j \geq 3 \quad \text{and} \quad P_{\mathbf{K}}(w) \in \operatorname{rint} F, \tag{6.2}$$

where $F = \mathbf{K} \cap (\ker u_1)$. Consider the cone

$$\mathbf{K}_1 = \{x \in R^n \mid \langle x, u_1 \rangle \geq 0, \langle x, u_j \rangle \leq 0, j = 2, \ldots, n\}.$$

By Lemma 1 there exist some elements y and z in $\mathbf{K}$ such that

$$\langle y, u_1 \rangle > 0, \quad \langle y, u_2 \rangle = 0, \quad \langle y, u_j \rangle < 0, \quad j = 3, \ldots, n,$$

and

$$\langle z, u_1 \rangle = \langle z, u_2 \rangle = 0, \quad \langle z, u_j \rangle < 0, \quad j = 3, \ldots, n.$$

Take $w_t = ty + (1-t)z$ with $t \in (0, 1]$.

Then $w_t \in \mathbf{K}_1$ and since $\|u_1\| = 1$ we have

$$P_{\mathrm{sp}\,F}(w_t) = w_t - \langle w_t, u_1 \rangle u_1,$$

with $\mathrm{sp}\,F = \ker u_1$. Let us see that for a sufficiently small t we have $P_{\mathrm{sp}\,F}(w) \in \mathrm{rint}\,F$. We have for $j \geq 3$ that

$$\langle w_t, u_j \rangle = t \langle y, u_j \rangle + (1-t) \langle z, u_j \rangle \leq \max\{\langle y, u_j \rangle, \langle z, u_j \rangle\}$$

Put $\delta = \max\{\langle y, u_j \rangle, \langle z, u_j \rangle, j \geq 3\}$.

Then $\delta > 0$ and we can take t so small in $(0, 1]$ to have $0 < \langle w_t, u_1 \rangle < \delta$.

Then for $j \geq 3$ one has

$$\begin{aligned} \langle P_{\mathrm{sp}\,F}(w_t), u_j \rangle &= \langle w_t, u_j \rangle - \langle w_t, u_1 \rangle \langle u_1, u_j \rangle \\ &\leq \langle w_t, u_j \rangle + |\langle w_t, u_1 \rangle \langle u_1, u_j \rangle| \leq -\delta + \langle w_t, u_j \rangle < -\delta + \delta = 0 \end{aligned}$$

and

$$\langle P_{\mathrm{sp}\,F}(w_t), u_2 \rangle = \langle w_t, u_2 \rangle - \langle w_t, u_1 \rangle \langle u_1, u_2 \rangle = -\langle w_t, u_1 \rangle \langle u_1, u_2 \rangle > 0,$$

since $\langle w_t, u_2 \rangle = 0$, $\langle u_1, u_2 \rangle > 0$ and $\langle w_t, u_1 \rangle > 0$.

Since obviously, $\langle P_{\mathrm{sp}\,F}(w_t), u_1 \rangle = 0$, the obtained relation shows that for a such t we have $P_{\mathrm{sp}\,F}(w_t) \in \mathrm{rint}\,F$, whence it follows implicitly that $P_{\mathrm{sp}\,F}(w_t) = P_{\mathbf{K}}(w_t)$.

Take $w = w_t$ and observe that is satisfies the requirements in (6.2).

(c). We shall see next that w^+ is contained in the face $F_{1,2}$ of $\mathbf{K}$ given

$$F_{1,2} = \{x \in \mathbf{K} \mid \langle x, u_1 \rangle = \langle x, u_2 \rangle = 0\}.$$

Since $\langle w, u_2 \rangle = 0$ we have by Lemma 2 that $\langle w^+, u_2 \rangle = 0$. Assuming that

$$\langle w^+, u_1 \rangle < 0 \tag{6.3}$$

consider the element $v_t = tw^+ + (1-t)w$. For any t in (0, 1) one has

$$w < v_t < w^+, \quad \text{(where } x < y \text{ means } x \leq y \text{ and } x \neq y). \tag{6.4}$$

This follows from conditions (6.2) which imply that $w < w^+$.

Since $w^+ - w \in \mathbf{K}$, we have $\langle w - w^+, u_j \rangle \leq 0$, that is, $\langle w^+, u_j \rangle \leq \langle w, u_j \rangle$ whence $\langle w^+, u_j \rangle \leq 0$; for $j \geq 2$ by the conditions (6.2). Hence,

$$\langle v_t, u_j \rangle = t \langle w^+, u_j \rangle + (1-t) \langle w, u_j \rangle \leq 0, \quad j \geq 2, \text{ for any } t \in (0, 1). \tag{6.5}$$

From the hypothesis (6.3), taking into account that $\langle v_t, u_1 \rangle = t \langle w^+, u_1 \rangle + (1 - t) \langle w, u_1 \rangle$ it follows that for t sufficiently close to 1 in $(0, 1)$ we have also $\langle v_t, u_1 \rangle \leq 0$.

But this relation together with (6.5) show that $v_t \in \mathbf{K}$, that is $v_t \geq 0$. Hence $w^+ = w \vee 0 \leq v_t$ and we have got a contradiction with (6.4).

Thus the assertion (c) is proved.

(d). Since $F_{1,2}$ is a face of $\mathbf{K}$, the relation $P_{\mathbf{K}}(w) \leq w^+$ would imply that $P_{\mathbf{K}}(w) \in F_{1,2}$, in contradiction with (6.2).

The obtained contradiction shows that the inequality $\langle u_1, u_2 \rangle > 0$ cannot hold, that is, $\langle u_1, u_j \rangle \leq 0$ for $i \neq j$, i, $j = 1, 2, \ldots, n$. That is, we have the condition (iv) fulfilled. □

Résumé subsantiel en français. Soit $\mathbf{K}$ un cône convexe fermé dans l'espace euclidien R^n. On note par $P_{\mathbf{K}}$ la projection sur $\mathbf{K}$. Le cône $\mathbf{K}$ est*avec projection isotone* si, pour tous x, $y \in R^n$, la relation $y - x$ implique $P_{\mathbf{K}}(x) - P_{\mathbf{K}}(y) \in \mathbf{K}$.

Nous étudions dans ce papier la caractérisation des cônes avec projection isotone dans les espaces euclidiens.

On note par «$\leq$» l'ordre défini par le cône $\mathbf{K}$ et par A^0 le polaire d'un ensemble $A \subseteq R^n$. On dit qu'un sous-ensenble $F \subseteq \mathbf{K}$ est une *face* de $\mathbf{K}$ si :

(i) F est un sous-cône;
(ii) $x \in F$, $y \in \mathbf{K}$ et $y \leq x$ impliquent que $y \in F$.

Le cône $\mathbf{K}$ est *correct* si, pour chaque face $F \subseteq \mathbf{K}$, on a que $P_{\mathrm{sp}\, F}(\mathbf{K}) \subseteq F$, où $\mathrm{sp}\, F$ est le sous-espace vectoriel engendré par F.

Le but de ce papier est de démontrer le résultat suivant :

Théorème. *Soit* $\mathbf{K}$ *un cône convexe qui engendre l'espace* R^n. *Les affirmations suivantes sont équivalentes* :

(i) $\mathbf{K}$ *est un cône avec projection isotone*;
(ii) $\mathbf{K}$ *est correct et* R^n *est un treillis*;
(iii) $\mathbf{K}$ *est polyédrique et correct*;
(iv) *il existe un ensemble de vecteurs* $\{u_i \mid i \in I\}$ *avec la propriété que* $\langle u_i, u_j \rangle \leq 0$ $\forall i,\ j \in I,\ i \neq j$ *et* $\mathbf{K} = \left(\{u_i \mid i \in I\}\right)^0$;
(v) R^n *est un treillis et* $P_{\mathbf{K}}(x) \leq x^+$ *pour chaque* $x \in R^n$, *où* $x^+ = x \vee 0$.

Les cônes avec projection isotone sont importants pour les méthodes numériques de type projection en optimisation et pour l'étude de la complémentarité.

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G. Isac
Collège militaire royal
Saint-Jean (Québec) CANADA J0J 1R0

A. B. Németh
Institutul de Matematică
Str. Republicii, C. P. 68
3400 Cluj-Napoca, ROUMANIE