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HILBERT'S SIXTEENTH PROBLEM (the second part): ITS PRESENT STATE

SONGLING SHI

RÉSUMÉ. Les développements théoriques récents et les calculs par ordinateur ouvrent de nouvelles perspectives concernant le $16^{\text{ième}}$ problème de Hilbert, deuxième partie.

Dans cet article nous exposons la surprenante histoire et l'état actuel de ce problème. Contrairement à certains énoncés de la littérature récente, nous indiquons que ce problème reste ouvert, même dans sa version la plus simple. Néanmoins, de nouveaux développements permettent d'être optimiste.

ABSTRACT. Recent theoretical developments and the new possibilities brought about by the computer calculations, raise new hopes concerning Hilbert's 16^{th} problem, second part.

In this paper we survey the surprising history of this problem and we give an account of its present state. Contrary to some recent claims in the litterature, we show that this problem is still open, even in the simplest of its cases. Nevertheless, the latest developments in this area justify an increased level of optimism.

In 1900, at the international Congress of Mathematicians in Paris, Hilbert delivered a lecture in which he stated 23 open problems [**1**], which he considered significant for the advancement of science.

Hilbert said the following words:

> *"Permit me in the following, tentatively as it were, to mention particular definite problems, drawn from various branches of mathematics, from the discussion of which an advancement of science may be expected."* (From D. Hilbert's lecture, Paris, 1900)

We observe that in the list of the Encyclopedic Dictionary of Mathematics [**2**] the 16^{th} problem is given as being:

> *(16) To conduct topological studies of algebraic curves and surfaces.*

This is in fact only the first part of Hilbert 16^{th} problem. Hilbert stated the second part of the 16^{th} problem as follows.

> *" ... In connection with this purely algebraic problem, I wish to bring forward a question which, it seems to me, may be attacked by the same method of continuous variation of coefficients, and whose answer is of corresponding value for the topology of families of curves defined by differential equations. This is the question as to the maximum number and position of Poincaré's boundary cycles (cycles limites) for a differential equation of the first order and degree of the form*

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$$\frac{dy}{dx} = \frac{Y}{X}$$

where X and Y are rational integral functions of the nth degree in x and y. Written homogeneously, this is

$$X\left(y\frac{dz}{dt} - z\frac{dy}{dt}\right) + Y\left(z\frac{dx}{dt} - x\frac{dz}{dt}\right) + Z\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right) = 0$$

where X, Y, and Z are rational integral homogeneous functions of the n^{th} degree in x, y, z, and the latter are to be determined as functions of the parameter t."

We see that this problem has also a second part, which deals with differential equations involving polynomials.

This second part of Hilbert's 16^{th} problem proved to be one of the most stubborn problems on Hilbert's list. In the 90 years that passed, in spite of all attempts, this problem is still unsolved even in the simplest of the cases. In 1976, the American Mathematical Society published the Proceedings of the Symposium in pure and applied mathematics which was held at Northern Illinois University in May 1974 [**3**]. In these proceedings all of Hilbert's problems are discussed with two exceptions: the 16^{th} problem and the third problem of Hilbert which was solved shortly after Hilbert stated the problems. In this book there is an article entitled: "Problems of present day mathematics" whose contents were written by 26 authors. The short article on Hilbert's 16^{th} problem was written by V. Arnold. He said:

"In contrast to the recent progress with the algebraic part of the 16^{th} Hilbert's problem (due to Goudkov, Rohlin, Harlamov and others), there is not much progress with the second part, dealing with limit cycles: it is still unknown whether a plane vector field, given by two polynomials of degree 2, can have more than 3 limit cycles."

Let us briefly explain the meaning of this second part of Hilbert's 16^{th} problem. We first consider some examples:

EXAMPLE 1 (FIGURE 1).

$$\frac{dx}{dt} = -y,$$
$$\frac{dy}{dt} = x.$$

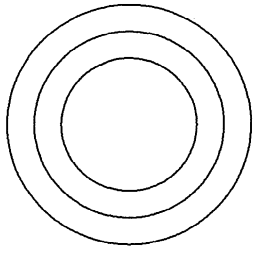

Figure 1.

EXAMPLE 2 (FIGURE 2).

$$\frac{dx}{dt} = -y + x(x^2 + y^2 - 1),$$
$$\frac{dy}{dt} = x + y(x^2 + y^2 - 1).$$

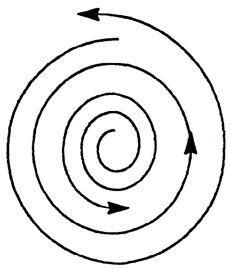

Figure 2.

EXAMPLE 3 (FIGURE 2′).

$$\frac{dx}{dt} = [-y + x(x^2 + y^2 - 1)][(x-1)^2 + y^2],$$
$$\frac{dy}{dt} = [x + y(x^2 + y^2 - 1)][(x-1)^2 + y^2].$$

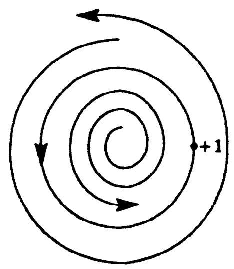

Figure 2′.

For a dynamical system, in particular for all the polynomial vector fields, there are three kinds of orbits: singular points, closed orbits and unclosed orbits, as shown in examples 1–3. A bounded unclosed orbit must tend to a closed orbit which we call limit cycle (Example 2) or to a boundary cycle (Example 3). A limit cycle is an isolated closed solution which is approched by the neighbouring solutions for $t \to +\infty$ or $t \to -\infty$.

Let n be a positive integer. Consider the set $K(n)$ of all polynomial systems of equations:

$$\frac{dx}{dt} = P(x, y),$$
$$\frac{dy}{dt} = Q(x, y)$$

with $n = \max\{\text{degree}(P), \text{degree}(Q)\}$.

Let us denote by $H(n)$ the maximum number of limit cycles which a system in $K(n)$ could have. Then we can state Hilbert's 16^{th} problem, second part, as follows:

Hilbert's 16^{th} problem is to determine what is $H(n)$ and the position of boundary cycles.

We observe that for any given N, we can construct a polynomial system having exactly N limit cycles. Indeed it is easy to show that the following system of degree $n = 2N+1$ has exactly N limit cycles:

$$\frac{dx}{dt} = -y + x(1-r^2)(4-r^2)\dots(N^2-r^2),$$
$$\frac{dy}{dt} = x + y(1-r^2)(4-r^2)\dots(N^2-r^2)$$

where $r^2 = x^2+y^2$. Using polar coordinates $x = r\cos\phi$, $y = r\sin\phi$, we obtain the equation for r:

$$\frac{dr}{dt} = r(1-r^2)(4-r^2)\dots(N^2-r^2) \tag{1}$$

or

$$\frac{dr}{dt} = rf(r)$$

where $f(r) = (1-r^2)(4-r^2)\dots(N^2-r^2)$. This equation is clearly of degree $2N+1$. Using this equation, one easily shows that the system (1) has N limit cycles. Hence we have that $H(2N+1) \geq N$. In general if we consider an equation $\frac{dr}{dt} = rg(r^2)$, where g is a polynomial of degree N, then if g has a positive root, say a^2, we have the circle $r = a$ as a limit cycle for the corresponding system. From this construction we get a feeling that limit cycles of polynomial systems are analogous (more or less) to roots of algebraic equations. This analogy can be made very precise in the case of small amplitude limit cycles via the Weierstrass-Malgrange preparation theorem and the Poincaré return map. However, we don't know whether or not there is a theorem on limit cycles corresponding to the fundamental theorem of algebra. By this we mean that we do not know whether or not Petrovski's conjecture in [**7**, **8**] is true. Petrovski's conjecture in [**7**, **8**] can be stated as follows: the polynomial vector fields of degree n have, generically, the same number of limit cycles. Another artificial example is constructed as follows. We consider an algebraic curve

$$f(x,y) = 0,$$

where f is a polynomial of degree n in x, y. Suppose that this curve has N ovals. Then, the system

$$\frac{dx}{dt} = \frac{\partial f}{\partial y} + f(x,y),$$
$$\frac{dy}{dt} = -\frac{\partial f}{\partial x} \tag{2}$$

has, usually, N limit cycles. For detail and more precision, see Sverdlove's paper [**4**].

The systems (1) and (2) are very simple and for this reason they are useful for quick thinking about the problem of limit cycles. We used these examples in order to check claimed results and they helped us to construct a counterexample to a claimed result of Chin to which I shall come back later.

Harnack's theorem claims that the maximum number of ovals of real plane algebraic curves of degree n is $\frac{1}{2}(n-1)(n-2)+1$. So we know from (2) that $H(n) \geq \frac{1}{2}(n-1)(n-2)+1$. This estimate is an improvement on the previous estimate which is : $H(2n+1) \geq n$.

From the above artificial constructions of limit cycles we see that it is difficult to get more limit cycles if we fix the degree n of an equation.

Trivially, there is no limit cycle for linear vector fields. What was the first example with limit cycles for a quadratic system, which appeared in the literature? Many people do not know that the first such example was the following one given by A. Sommerfield in 1929 [**5**: page 458, reference 77]:

$$\frac{dx}{dt} = xy - \mu\left(-\frac{(x-1)(x+2)}{3} + \frac{y^2}{2} + \frac{xy}{3} + \frac{y}{3}\right),$$
$$\frac{dy}{dt} = -\frac{(x-1)(x+2)}{3} + \frac{y^2}{2} + \frac{xy}{3} + \frac{y}{3} + \mu xy$$

where $\mu < 0$ and $|\mu| \ll 1$. This is a system with two limit cycles whose phase portrait is given in Figure 3.

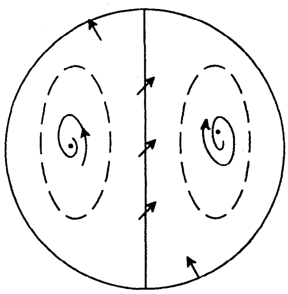

Figure 3.

In his 1985 monograph [**20**] Professor Chin Yuanxun claimed that the first example of a quadratic system with limit cycle is the one given by him in his 1958 paper [**22**]. This example is of a quadratic system with only one limit cycle which is an ellipse.

In 1959 Tung obtained an example of a quadratic system with 3 limit cycles [**6**]. If we compare this example with the one given by Sommerfield, we see that there is a great similarity between these examples. In fact there is just a change of a sign from Sommerfield's example to Tung's example. Tung's example is the following one:

$$\frac{dx}{dt} = xy - \mu\left(-\frac{(x-1)(x+2)}{3} + y^2 + \frac{xy}{3} - \frac{y}{3}\right),$$
$$\frac{dy}{dt} = -\frac{(x-1)(x+2)}{3} + \frac{y^2}{2} + \frac{xy}{3} - \frac{y}{3} + \mu xy.$$

Tung does not mention Sommerfield's example in his paper [**6**] (1959) and neither does he mention it in the bibliography of this paper.

In a paper published in 1955, Petrovski and Landis claimed to have proved that $H(2) = 3$ [**7**]. Later, in a 1957 paper [**8**], they claimed that

$$H(n) \leq \begin{cases} \frac{1}{2}(6n^3 - 7n^2 - 11n + 16), & \text{for } n \text{ odd,} \\ \frac{1}{2}(6n^3 - 7n^2 + n + 4), & \text{for } n \text{ even.} \end{cases}$$

Later on an error in the proof of [7] was found by a study group of mathematicians led by S.P. Novikov. Right after Petrovski and Landis informed the mathematical community of this error and published a note (1967) where they said the following:

> *"In our article (1955) there is an error in the proof of lemma 12 (p. 242, translation p. 213) which was pointed to us by S. P. Novikov. A reference to this lemma is made on p. 153 (translation p. 158) of our (1957) article."*

Although the claim that $H(2) = 3$ was retracted by Petrovski and Landis, the statement $H(2) = 3$ remained a possibility.

I began working on this problem in 1970 and in 1978 I obtained a counterexample to Petrovski's claim that $H(2) = 3$. More precisely, I constructed an example of a quadratic system with four limit cycles. I presented this result at a meeting at Academia Sinica in Bejing on December 28, 1978 which appeared in the chinese edition of the journal Scientia Sinica in November 1979 and in the English version of the same journal in February 1980 ([**10**], see also [**13**]).

My counterexample to Petrovski and Landis' claim that $H(2) = 3$ is the following one (Figure 4):

$$\begin{aligned}\frac{dx}{dt} &= \lambda x - y - 10x^2 + (5+\delta)xy + y^2,\\ \frac{dy}{dt} &= x + x^2 + (-25 + 8\varepsilon - 9\delta)xy\end{aligned}$$

where

$$\begin{aligned}\delta &= -10^{-13},\\ \varepsilon &= -10^{-52},\\ \lambda &= -10^{-200}.\end{aligned}$$

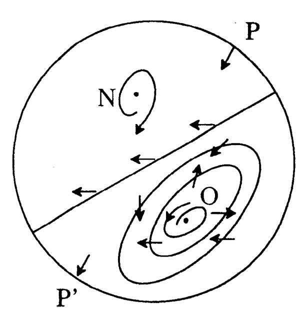

Figure 4.

Chen Lan-sun heard my lecture on December 28, 1978 and made no comment at that time. However, briefly afterwards he sent a joint manuscript of his together with Wang Mingshu (on January 6, 1979) for publication in Acta Mathematica Sinica. Their paper was published in November 1979, at the same time with my paper. In their original paper [**11**] Chen and Wang copied my phase portrait (with two foci, unique singular point at infinity and a straight line without contact), see Figure 5, but they omitted one condition, *i.e.* $\delta_2 \ll \delta_1$, see [**11**]. I pointed out this omission in the third Symposium on

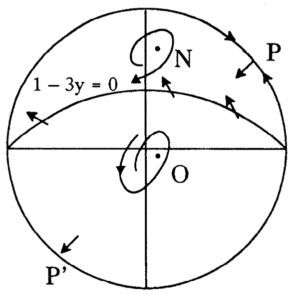

Figure 5.

differential equations, held in Kunming City of China on March, 1980. The authors of the book "Theory of Limit Cycles" [**12**] added this missing condition when they discussed this example in their book without mentioning my name.

To obtain four limit cycles of a quadratic system I used a combination of the method of Sommerfield and the method of Frommer.

In 1930s, Frommer, following a method of Poincaré, obtained a small amplitude limit cycle (in short SALC), the meaning of SALC will be explained later. His example is the following one [**15**]:

$$\frac{dx}{dt} = -y + 2xy - y^2,$$
$$\frac{dy}{dt} = x + (1+\varepsilon)x^2 + 2xy - y^2.$$

Let

$$F(x,y) = \frac{1}{2}(x^2+y^2) + \frac{1+\varepsilon}{3}x^3 - xy^2 - \frac{1}{3}y^3 + \dots$$

we have

$$\frac{dF}{dt} = \frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt}$$
$$= -\frac{\varepsilon}{4}r^4 + \frac{7}{24}r^6 + \dots$$

where $r^2 = x^2 + y^2$. Therefore F decreases for $r < \sqrt{\frac{6}{7}\varepsilon}$ and increases for $r > \sqrt{\frac{6}{7}\varepsilon}$ (Figure 6).

> *"Hence, there must be, for small positive ε, a limit cycle which tends to the origin for $\varepsilon \to 0$,"*

Frommer said.

In this 1–parameter family of polynomial vector field we see that we have a limit cycle in each one of the vector fields for $\varepsilon > 0$ and small. Furthermore these limit cycles are themselves small. This is the meaning of a SALC.

Frommer's method is essentially a degenerate Hopf bifurcation. The main difficulty is to answer how many times one can use Hopf bifurcations for a polynomial vector field.

In 1952, Bautin extended Frommer's result and proved that we could have 3 and at most 3 SALCs in a quadratic system.

In [**17**] I generalized an algorithm Poincaré gave for the center problem [Oeuvre de Henri Poincaré, vol. 1, Gauthier–Villars et Cie, Paris, 1928, p. 95–100] and proved the following:

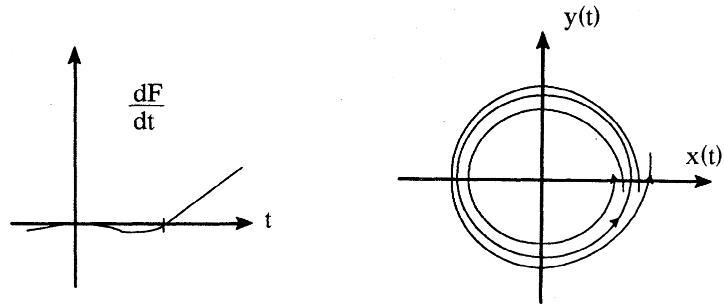

Figure 6.

MAIN LEMMA. *For a polynomial vector field*

$$\frac{dx}{dt} = \lambda x - y + P(x,y),$$
$$\frac{dy}{dt} = y + \lambda x + Q(x,y) \qquad (E_n)$$

where P and Q are polynomials of degree n, there exists a formal series

$$F = F_2 + F_3 + \dots$$

where

$$F_2 = \frac{1}{2}(x^2 + y^2),$$
$$F_k = \sum_{i=1}^{k} f_i x^i y^{k-i}, \quad k \geq 2$$

such that

$$\frac{dF}{dt} = \lambda F_2 + V_1 F_2^2 + V_2 F_2^3 + V_3 F_2^4 + \dots$$

where V_i are homogenious polynomial of degree $2i$ in the coefficients of P and Q.

In the proof we find that the functions F_i are not uniquely determined and the same thing holds for the constants V_i. For each i, there is an infinite number of constants V_i. The set of all such V_i is a coset modulo the ideal generated by $V_1, \dots, V_{i-1}$. We may call this coset the i^{th} Poincaré-Lyapunov constant [**18**].

THEOREM(POINCARÉ-FROMMER-BAUTIN-SHI). *The maximum number of SALCs of (E_n) is the number $M(n)$ of algebraically independent Poincaré-Lyapunov constants [**17**, **18**] and $M(n)$ is finite.*

Bautin (1952) proved that $M(2) = 3$. So far we do not know the value of $M(n)$ for $n \geq 3$.

In my example with 4 limit cycles, I synthesized Sommerfield's and Bautin's methods.

Sommerfield's method is via a bifurcation of a limit cycle from a boundary cycle at infinity. It can be realized by rotating a specific vector field.

If the singular point A is stable and if O is unstable (Figure 7), then we rotate the vector field by a small negative angle such that the singular points A and O don't change their

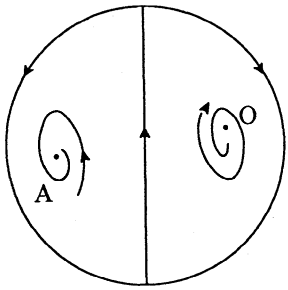

Figure 7.

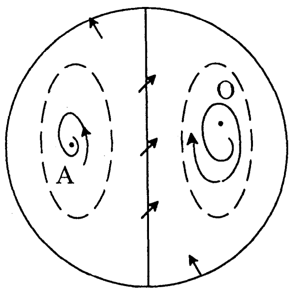

Figure 8.

stability and we obtain two limit cycles (Figure 8). Similarly, if the points A and O have the same stabilities, then we get only one limit cycle by the rotation of the vector field. For more detail, see [**13**].

It remains to combine those two methods and see which one is realizable and which one is not.

For a system

$$\frac{dx}{dt} = \lambda x - y + lx^2 + mxy + ny^2,$$
$$\frac{dx}{dt} = x + ax^2 + bxy$$

there are two foci if and only if

$$\lambda b^2 + 2\lambda(ab - mb + 2am) + (a+m)^2 - 4l(n+b) < 0$$

with $n \neq 0$, $|\lambda| < 2$, $(\lambda n + m)^2 + 4n(b+n) < 0$. There exists a unique singular point at infinity if and only if

$$[9an - m(b-l)]^2 - 4[(b-l)^2 + 3am][m^2 + 3n(b-l)] > 0.$$

The relative position of the straight line without contact and the saddle point at infinity is decided by the sign of this quantity: $a^2n - abm + lb^2$. The origin is a weak focus of order 3 if and only if

$$\lambda = 0,$$
$$an \neq 0,$$
$$m = 5a,$$
$$b = 3l + 5n$$

and

$$V_3 = a(2a^2 + 2n^2 + ln)[(b + 2l)(l + n)(b + n) - am(2l + b + n)] \neq 0.$$

The origin is stable if $V_3 < 0$, and unstable if $V_3 > 0$. The focus $A(0, \frac{1}{n})$ is stable if $\lambda + \frac{m}{n} < 0$ and unstable if $\lambda + \frac{m}{n} > 0$. The conditions

$$\begin{gathered} 3a^2 - l(l + 2n) < 0, \\ 25a^2 + 12n(l + 2n) < 0, \\ l(3l + 5n)^2 - 5a^2(3l + 5n) + na^2 < 0, \\ a^2(5l + 8n) - [(2l + 5n)^2 + 15a^2][25a^2 + 3n(2l + 5n)] > 0 \end{gathered}$$

and $V_3 > 0$ are realizable. Hence, we perturb the system

$$\begin{aligned} \frac{dx}{dt} &= -y + lx^2 + 5axy + ny^2, \\ \frac{dy}{dt} &= x + ax^2 + (3l + 5n)xy \end{aligned}$$

such that we get a new system which has at least 4 limit cycles.

At the beginning I thought that since from Sommerfield's method one could obtain 2 limit cycles and from Bautin's method one could obtain 3 limit cycles by combining the two methods we could obtain five limit cycles for a quadratic systems and hence $H(2) \geq 5$.

This meant that the conditions

$$\begin{gathered} 3a^2 - l(l + 2n) < 0, \\ 25a^2 + 12n(l + 2n) < 0, \\ l(3l + 5n)^2 - 5a^2(3l + 5n) + na^2 < 0, \\ a^2(5l + 8n)^2 - [(2l + 5n)^2 + 15a^2][25a^2 + 3n(2l + 5n)] > 0, \\ V_3 < 0 \end{gathered} \tag{$*$}$$

were all realizable. I gave a lecture on this on December 18, 1978 at Institute of Applied Mathematics, Academia Sinica in Beijing. From the arguments I gave at that time it followed that the $3 + 1$, the $2 + 1$ and the $1 + 1$ configurations can also all be realized (Figure 9).

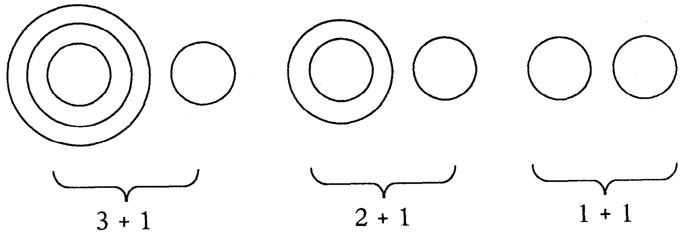

Figure 9.

Upon checking my preprint on $H(2) \geq 5$, I discovered that the calculation of Bautin for the Poincaré-Lyapunov constants for the quadratic systems are not all correct. Indeed, [**13**] found that there is a mistake in the calculation of V_3 of Bautin's paper. This quantity appears there with the wrong sign. This error influences the existence of one of the limit cycles. In my first preprint, in connection with Bautin's wrong calculation, I obtained 5 limit cycles. After I corrected this mistake, I could only obtain 4 limit cycles. The reason is that the above conditions $(*)$ are not realizable (!).

K.S. Sibirsky proved that V_i are algebraic invariants under the rotation group of plane. Other conditions are all represented by algebraic invariants. I believe that the algebraic invariants would control the number of limit cycles. Now, we may state the conjecture.

Conjecture: H(2)=4. In 1983, in a symposium held in Beijing, Chin Yuanshun purported that $H(2) = 4$. He used complex variable arguments. But as I pointed out in that meeting, Chin's arguments contain assertions which are not proved and some are incorrect. However, Chin's paper was published in the Springer Lecture Notes in Mathematics, Vol. 1151 [**19**], and its review in the Mathematical Reviews did not mention the gaps and the errors. In 1987, I gave a counterexample to two of Chin's main assertions. This paper recently appeared in the Bulletin of the London Mathematical Society [**21**].

Let us describe Chin's arguments. Chin begins with a study in the complex domain. The system under consideration is a natural extension of the real system

$$\frac{du}{dt} = U(u, x), \qquad \frac{dx}{dt} = X(u, x) \tag{E}$$

where U, X are real polynomials, to the complex system

$$\frac{dw}{dT} = W(w, z), \qquad \frac{dz}{dt} = Z(w, z) \tag{E^*}$$

where $w = u + iv$, $z = x + iy$, $T = t + i\tau$ and $W(w, z) = U(w, z)$, $Z(w, z) = X(w, z)$.

The geometrical figures of a general solution for (E^*): $F(w, z) =$ constant, are 2–dimensional manifolds in the real 4–dimensional (u, v, x, y) space. We shall call them solution surfaces. The real curves defined by the system (E) are just the intersections of the surfaces of (E^*) with the plane $v = y = 0$.

DEFINITION. A solution surface F is called an isolated limit surface (ILS) if there exists another solution surface F_1 such that $\overline{F}_1 \supset F$ but $F_1 \subsetneq \overline{F}$.

The limit cycle in the real domain corresponds to the limit surface in the complex domain.

The mainstay of Chin's argument is the following one: The number of ILSs of the complex system (E^*) controls the number of limit cycles of the real system (E). But Chin did not study the ILS. In particular he did not give an answer to anyone of the following questions:

(1) Is there a limit cycle on the isolated limit solution surface?
(2) How many limit cycles are there on an isolated limit solution surface?

The idea of my counterexample to Chin's statements regarding Hilbert's 16^{th} problem is very simple. Any two circles which do not have the same center must have points of intersection in the complex plane. Hence we have two limit cycles as two circles which are on the same ILS. This is contrary to the main argument of Chin. So, Chin did not prove the conjecture $H(2)=4$.

Since the bounds claimed by Petrovski and Landis were obtained by using a lemma which had an error, we have no proof that these are really bounds for $H(n)$. This prompts the question: Are there any bounds for $H(n)$ to be found in the literature?

In a paper published in 1957 N.N. Molčanov [**25**] claimed the result that

$$H(n) < (2\sqrt{2}-2)n^2$$

for large n. S. A. Gal'pern pointed out in USSR Mathematical Review, 6 (1959), No. 5813 that the assertions in this paper are either introduced without proof or depend on other assertions which are not proved. I have a counterexample to Molčanov's conclusion which I shall include in a forthcoming preprint.

In spite of the negative result concerning Hilbert's 16^{th} problem, there is hope due to the new developments in this subject. Thus we have the general and powerful results on the maximum number of small amplitude limit cycles and the following result:

THE FINITENESS THEOREM OF DULAC (1923), IL'YASHENKO, ECALLE, MARTINET, MOUSSU AND RAMIS. *Any polynomial system*

$$\frac{dx}{dt} = P(x,y),$$
$$\frac{dy}{dt} = Q(x,y)$$

has a finite number of limit cycles.

This theorem was first claimed by Dulac in a 1923 paper [**26**]. In the 1980s Il'yashenko [**27**] noticed an error in one of the lemmas used by Dulac and proved the finiteness theorem for polynomial vector fields with nondegenerate singular points [**28**]. Ecalle, Martinet, Moussu, Ramis wrote a preprint in 1987 [**29**] in which they announced this result and we heard that Ecalle is in the process of writing its complete proof.

Although we know that each quadratic system has a *finite* number of limit cycles no upper bound is known for $H(2)$.

Also the problem of determining $H(3)$ or at least giving an upper bound for it is a very difficult one. Before solving it one would have to study many more cubic systems than this has been done up to now.

Hilbert's 16^{th} problem, second part, is not only an open problem; it is also a research field. The aspects involved in this problem are the following ones:

Analytical Aspects. In the proof of finiteness theorem [**27**, **28**, **29**] Il'yashenko and Ecalle *et al.* used modern developments in singularity theory and the theory of normal forms. Also, bifurcation theory methods.

Algebraic Aspects. For example, K.S. Sibirsky developped the theory of algebraic invariants of polynomial vector fields and he applied this theory to obtain specific results in the theory of quadratic systems.

Geometric Aspects. In [**30**] J. Guckenheimer introduced the 5–dimensional orbit space of quadratic systems under the action of the affine group and positive time rescaling. Dana Schlomiuk [**31**] got the first result about a subspace of this orbit space, namely she obtained the bifurcation diagram of the subspace of all quadratic systems possessing a center. She proved in a forthcoming preprint that this bifurcation diagram is completely controlled by the invariant algebraic curves of degree at most three and that the algebraic conditions for the center can be stated in geometrical terms by using these invariant algebraic curves.

Computer Aspects. With the help of the computer techniques, N. Lloyd *et al.* recently announced with a special cubic system [**24**] that $M(3) \geq 6$. Several papers, using computer methods, announced that $H(3) \geq 9$. Due to computer algebra results are produced faster nowadays than it was the case before.

In conclusion, since 1900, there has been no answer to Hilbert's 16^{th} problem, second part, not even for the case of quadratic systems. We still do not know the value of $H(2)$. Up to now only isolated and special questions (even if highly interesting in themselves) have been handled successfully. We need a more comprehensive point of view because of the importance and variety of the questions which face us. We do not deal here with only one problem, but with a whole variety of problems [**27**, **30**, **31**]. As the history of the subject has shown, in this area of study one has to proceed with special rigor. At the present time, there is a lot of activity in this area of study. There is renewed hope that a combination of the more powerful newly discovered theoretical methods, such as those developed by Ecalle *et al* with the computer techniques will bring about a solution to Hilbert's 16^{th} problem, at least for the case of quadratic systems.

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