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EXISTENCE OF ASYMPTOTICALLY ALMOST-PERIODIC AND OF ALMOST-AUTOMORPHIC SOLUTIONS FOR SOME CLASSES OF ABSTRACT DIFFERENTIAL EQUATIONS

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RÉSUMÉ. **Dans ce travail, on étudie le problème de l'existence des solutions asymptotiquement presque-périodiques au sens de Fréchet ou presque-automorphes au sens de Bochner pour équations différentielles dans les espaces de Banach de la forme: $u'(t) - Au(t) = f(t)$, où A est un opérateur linéaire avec certaines propriétés, tandis que f est asymptotiquement presque-périodique ou presque-automorphe.**

ABSTRACT. **In this paper, we consider the problem of the existence of solutions which are asymptotically almost-periodic in the sense of Fréchet (a.a.p.) or almost-automorphic in the sense of Bochner (a.a.) for differential equations in Banach spaces which are of the form: $u'(t) - Au(t) = f(t)$ where A is a linear operator with some special properties while f is a.a.p. or a.a.**

Introduction. The present work is dedicated to non-homogeneous differential equations in a Banach space where the (forcing) term appearing in the right-hand side is asymptotically almost-periodic (a.a.p.) or almost-automorphic (a.a.); one looks for solutions belonging to the same class of functions (respectively). The asymptotically almost-periodic functions were introduced by Fréchet [**5**]; they appear to be quite useful in applications to ordinary differential equations (see [**4**]). If extension to Banach space is done, the theory is quite similar (except for a few changes, here and there) (see [**8**] and [**13**]). There are, as far as we know, some applications to abstract differential equations (as in [**10**], [**12**], [**13**]). Here we add one more result, giving the existence of a (non-necessarily unique) a.a.p. solution of the equation $u' - Au = f$, when A is a simple (diagonal) linear operator with some special properties.

We note that the "solution" so far obtained will only be in the "ultra-weak" sense of Lions [**7**]; this is quite reasonable due to the way the construction is done.

The remaining of the paper is dedicated to almost-automorphic solutions; this class of functions (scalar-valued) was introduced by Bochner ([**1**]); extensions to Banach space-valued functions and various applications are given for example, in [**9**]; the results have similarities to those concerning the almost-periodic solutions and this is true also for the propositions in the present paper (for their almost-periodic counterpart see [**14**], [**15**]).

1. Let X be a Banach space and $f(t), [0,\infty) \to X$ be strongly continuous and satisfying the following property:

$\forall\epsilon > 0,\ \exists\, T(\epsilon) \geq 0$ such that the set of real numbers

$$\left\{\tau \mid \tau \geq 0,\ \sup_{t \geq T(\epsilon)} \|f(t+\tau) - f(t)\| < \epsilon\right\} = \lambda_{+,T}(f,\epsilon) \quad (1.1)$$

is relatively dense on $[0,\infty)$ (i.e. for some $L(\epsilon) > 0$, any interval $[a, a+L]$, $a \geq 0$, contains at least one point of $\lambda_{+,T}(f,\epsilon)$). (This is the definition of the class $\mathcal{F}(R^+; X)$ in [**13**]).

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Furthermore, it is proved in [**13**] that this class is the same as the class of asymptotically almost-periodic functions, $R^+ \to X$, consisting of all continuous functions $g(t), R^+ \to X$ admitting a representation of the form $g(t) = \tilde{g}(t) + w(t)$, where $\tilde{g}(t)$ is almost-periodic, $R \to X$, while $w(t), R^+ \to X$, converges strongly to θ as $t \to \infty$. Thus, if $f(t) \in \mathcal{F}(R^+; X)$ and $z(t)$ is continuous, $R^+ \to X$ and $z(t) \to \theta$ as $t \to +\infty$, it follows that $f(t)+z(t)$ belongs again to $\mathcal{F}(R^+; X)$ ($f = \tilde{f} + w \to f + z = \tilde{f} + w + z, \tilde{f}$ a.p., $w + z \to \theta$ as $t \to +\infty$).

Consider now a linear closed operator $A, D(A) \subset X \to X$ which is the generator of a Co-semigroup $S(t)$, verifying the exponential decay estimate $\|S(t)\| \leq Me^{\beta t}$, $\beta < 0$ ($\forall t \geq 0$). If $f(t)$ is any continuous function, $R^+ \to X$, the mild solution over $[0, \infty)$ of the abstract differential equation

$$\frac{du}{dt} = Au + f \tag{1.2}$$

with initial data $u_o \in X$, is the function

$$u(t) = S(t)u_o + \int_o^t S(t-\sigma)f(\sigma)\, d\sigma. \tag{1.3}$$

As seen in [**13**], if $f \in \mathcal{F}(R^+; X)$, the function given by the expression

$$\int_o^t S(t-\sigma)\, f(\sigma)\, d\sigma, \text{ belongs to the same class.}$$

As the limit: $S(t)u_o \to \theta$ for $t \to +\infty$ holds we see that the mild solution $u(t)$ is also in $\mathcal{F}(R^+; X)$. Therefore we have that *all mild solutions on* $[0, \infty)$ *of the equation (1.2) are in*
$\mathcal{F}(R^+; X)$ *when* $f \in \mathcal{F}(R^+; X)$.

We note the particular case when A is a (complex) number with $Re\, A < 0$. All solutions of the *ordinary* differential equation

$$u'(t) = Au(t) + f(t), \quad t \geq 0, \quad u(t) : R^+ \to C \tag{1.4}$$

are given by the formula

$$u(t) = e^{At}u_o + \int_o^t e^{A(t-\sigma)} f(\sigma)\, d\sigma. \tag{1.5}$$

In particular, the solution

$$v(t) = \int_o^t e^{A(t-\sigma)} f(\sigma)\, d\sigma$$

with zero Cauchy data is a.a.p. It also verifies the following estimate:

$$|v(t)| \leq \sup_{R+} |f| \int_o^t e^{(Re\, A)t} e^{(-Re\, A)\sigma} d\sigma = e^{(Re\, A)t} \|f\|_\infty \int_o^t e^{(-Re\, A)\sigma} d\sigma$$

$$= e^{(Re\, A)t} \|f\|_\infty \frac{1}{|Re\, A|}(e^{(-Re\, A)t} - 1) = \frac{\|f\|_\infty}{|Re\, A|}(1 - e^{(Re\, A)t}) \leq \frac{\|f\|_\infty}{|Re\, A|}, \forall t \geq 0. \tag{1.6}$$

2. Consider now a separable Hilbert space H with orthonormal basis $(e_j)_1^\infty$ and let $(\lambda_j)_1^\infty$ be a given sequence of complex numbers. Define the linear operator A in H in the following way. The domain $D(A)$ consists of all finite linear combinations of basic vectors e_j

$$D(A) = \left\{ \sum_1^p \alpha_j e_j \, l \, p \in \mathbf{N}, \, \alpha_j \in C \right\} \tag{2.1}$$

and,

$$\forall\, h \in D(A), \;\; h = \sum_1^p \alpha_j e_j, \;\text{ we put }\; Ah = \sum_1^p \alpha_j \lambda_j e_j.$$

Thus A maps each e_j into $\lambda_j e_j$, A is a linear operator ("diagonal" operator), and $D(A)$ is dense in H.

Next, let $f(t)$, $R^+ \to H$ be in $\mathcal{F}(R^+;X)$. It is then obvious that the scalar-valued function $(f(t), e_j)_H$ is in $\mathcal{F}(R^+;\mathbb{C})$ $\forall j = 1,2\ldots$ (this follows directly from the definition of $\mathcal{F}(R^+;X)$ and also from the decomposition of f (as an a.a.p. function). We shall give below a result expressing a sufficient condition in order that the inhomogeneous equation

$$u'(t) = Au(t) + f(t) \tag{2.2}$$

on $[0,\infty)$ admits a *ultra-weak solution* $u(t)$ belonging again to $\mathcal{F}(R^+;X)$.

Let us first note that the adjoint (hilbertian) operator A^* has a domain containing $D(A)$. (Hence $D(A^*)$ is dense in H and A^* is a closed operator) (see our paper [**15**]). The class of vector-valued functions $K_{A^*}(0,\infty)$ is composed of functions $\varphi(t), (0,\infty) \to D(A^*)$, such that $\varphi(t) \in C_o^1(0,\infty;H)$ and $A^*\varphi \in C(0,\infty;H)$. Then, a continuous function $u(t), (0,\infty) \to H$ is an ultra-weak solution of (2.2) iff the integral relation

$$\int_o^\infty (u(t), \varphi'(t) + (A^*\varphi)(t))_H \, dt = -\int_o^\infty (f(t), \varphi(t))_H \, dt \tag{2.3}$$

holds, for any $\varphi \in K_{A^*}(0,\infty)$.

Note that if the functions $\{u_N(t)\}_1^\infty$, are ultra-weak solutions corresponding to $\{f_N(t)\}_1^\infty$ and if $u_N(t) \to u(t), f_N(t) \to f(t)$ uniformly on each compact of $(0,\infty)$, then $u(t)$ verifies (2.3) with $f(t)$ as right-hand side, (see [**11**]).

Let us state now (and then prove) the following.

THEOREM 1. *Let A be an (unbounded) diagonal operator in the separable Hilbert space H with basis $(e_j)_1^\infty$, such that $Ae_j = \lambda_j e_j \; \forall j = 1,2\ldots$ and $Re\,\lambda_j < 0$ for all $j = 1,2\ldots$*

Let $f(t), [0,\infty) \to H$ be an as. alm. periodic function, such that

$$\sum_{j=1}^\infty \frac{1}{|Re\,\lambda_j|^2} \left(\sup_{t\in R^+} |(f(t), e_j)| \right)^2 < +\infty. \tag{2.4}$$

Then the equation $u'(t) = Au(t) + f(t)$ on $[0,\infty)$ possesses an ultra-weak solution on $(0,\infty)$ which is in $\mathcal{F}(R^+;H)$.

PROOF: Consider the scalar ordinary differential equations

$$\frac{du_j}{dt} = \lambda_j u_j(t) + f_j(t) \tag{2.5}$$

where $f_j(t) = (f(t), e_j)_H$, $j = 1, 2, \dots$

As noted previously, due to the condition $Re\,\lambda_j < 0$, the functions

$$u_j(t) = \int_0^t e^{\lambda_j(t-\sigma)} f_j(\sigma)\, d\sigma \tag{2.6}$$

are all as. alm. periodic solutions of (2.5) on $[0, \infty)$, with the estimation

$$|u_j(t)| \le \frac{1}{|Re\,\lambda_j|} \sup_{R^+} |f_j(\sigma)|, \ \forall\, t \ge 0. \tag{2.7}$$

Define then the function $u(t), [0, \infty) \to H$, given by the series

$$u(t) = \sum_{j=1}^{\infty} u_j(t)\, e_j. \tag{2.8}$$

In view of (2.4) we have the *uniform* convergence on $[0, \infty)$ of the numerical series

$$\sum_{j=1}^{\infty} |u_j(t)|^2 \tag{2.9}$$

and accordingly the *uniform* convergence over R^+, in H-strong sense of the sequence of partial sums

$$\sum_{j=1}^{N} u_j(t)\, e_j = w_N(t). \tag{2.10}$$

Note that the functions $w_N(t)$ are as. alm. per. as mappings, $R^+ \to H$ and thus we see that $u(t)$, which is the uniform limit of $\{w_N(t)\}_1^\infty$, on R^+, is also as. alm. per. (a.a.p. function). We see also that, for all $j = 1, 2 \dots$ the equalities

$$\frac{d}{dt}(u_j(t)\, e_j) = u_j'(t)\, e_j = \lambda_j u_j(t)\, e_j + f_j(t)\, e_j = A(u_j(t)\, e_j) + f_j(t)\, e_j \tag{2.11}$$

hold.

Hence we get, after finite summation, the equalities

$$w_N'(t) = A(w_N(t)) + \sum_1^N f_j(t)\, e_j \text{ in the strong sense, over } [0, \infty) \tag{2.12}$$

hence also in the (ultra) weak sense (see again [**11**]).

Note also the following

LEMMA 1. *If $f(t) \in \mathcal{F}(R^+; H)$ and $f_j(t) = (f(t), e_j)$, then the relation*

$$f(t) = \lim_{N\to\infty} \left(\sum_{j=1}^{N} f_j(t)\, e_j \right)$$

holds, in strong H, uniformly on R^+.

PROOF: Consider the linear operators P_j from H into $Sp(e_1 \dots e_j)$ given by

$$P_j h = \sum_{k=1}^{j} (h, e_k) e_k.$$

Then $P_j h \to h \ \forall h \in H$ and the convergence is uniform on rel. compact sets in particular on the range of $f(t)$ (see [**2**] and [**6**]). □

Thus $w_N(t) \to u(t)$ uniformly on $[0, \infty)$, while $\sum_1^N f_j(t) e_j \to f(t)$ also uniformly on R^+; we obtain that

$$u'(t) = Au(t) + f(t)$$

in the ultra-weak sense over $(0, \infty)$, that is (2.3) is satisfied. This proves Theorem 1. □

3. Starting from this section we shall consider almost-automorphic solutions of abstract differential equations. Our first result, quite a simple one, is similar to one on almost-periodic solutions which appears in our recent paper [**14**]. Let us state it under the form of

THEOREM 2. *Let $A = (a_{ij})_{ij=1}^n$ be a square-matrix of complex-numbers, such that $Re\lambda_j \neq 0$ for all eigenvalues λ_j. Then given any a.a. function $f(t), R \to Y^n$, there exists a unique a.a. function $y(t), R \to Y^n$, solving the equation $y' = Ay + f$ (here Y is a (complex) Banach space).*

PROOF: The unicity is a particular case of the unicity of all bounded over R solutions (see [**14**]). To prove existence let us remember first the well-known.

LEMMA 2. *$\exists$ a linear invertible operator $B, \mathbb{C}^n \to \mathbb{C}^n$ such that*

$$B^{-1}AB = \begin{pmatrix} \lambda_1 & c_{12} & \dots & c_{1n} \\ O & \lambda_2 & \dots & c_{2n} \\ \dots & \dots & \dots & \dots \\ O & O & \dots & \lambda_n \end{pmatrix}. \tag{3.1}$$

*(see [**3**]).*

LEMMA 3. *If $Re\,\lambda > 0$ and f is a.a., the integral*

$$z(t) = -\int_t^{\infty} e^{\lambda(t-\sigma)} f(\sigma)\, d\sigma$$

is a.a. and is a solution of $z' = \lambda z + f$; if $Re\,\lambda < 0$ and f is a.a., the integral

$$\int_{-\infty}^{t} e^{\lambda(t-\sigma)} f(\sigma)\, d\sigma \text{ has the same properties.}$$

T: he only thing which needs a proof is for example, that $z(t)$ is a.a., say, when $Re\,\lambda < 0$. We have obviously

$$\int_{-\infty}^{t} e^{\lambda(t-\sigma)} f(\sigma)\, d\sigma = -\int_{+\infty}^{0} e^{\lambda u} f(t-u)\, du = \int_{0}^{\infty} e^{\lambda u} f(t-u)\, du = z(t). \tag{3.2}$$

In order to establish that z is a.a., note first that given any real sequence $\{\alpha'_n\}_1^\infty \exists$ a subsequence $\{\alpha_n\}_1^\infty$ such that

$$f(s+\alpha_n) \to g(s) \text{ and } g(s-\alpha_n) \to f(s), \text{ (as } n \to \infty\text{), in the pointwise sense} \tag{3.3}$$

(hence $g(s)$ is strongly measurable and bounded over R).

We shall now see that $z(t+\alpha_n) \to w(t) = \int_0^\infty e^{\lambda u} g(t-u)\,du$ (this is a Bochner integral) and also that $w(t-\alpha_n) \to z(t)$ as $n \to \infty$, pointwise (this will prove that z is alm. aut.).

We have:

$$z(t+\alpha_n) = \int_0^\infty e^{\lambda u} f(t-u+\alpha_n)\,du. \tag{3.4}$$

Now: $f(t-u+\alpha_n) \to g(t-u) \forall u \in R$ (assume t fixed). Furthermore we have the estimate

$$\|e^{\lambda u} f(t-u+\alpha_n)\|_X \le e^{(Re\,\lambda)u} \|f\|_\infty \in L^1(0,\infty) \text{ (as } Re\,\lambda < 0). \tag{3.5}$$

Applying the dominated convergence theorem we get (strongly in X)

$$z(t+\alpha_n) \to \int_0^\infty e^{\lambda u} g(t-u)\,du = w(t) \ (\forall t \in R). \tag{3.6}$$

Next we have

$$w(t-\alpha_n) = \int_0^\infty e^{\lambda u} g(t-u-\alpha_n)\,du.$$

We note that

$$g(t-u-\alpha_n) \to f(t-u) \ \forall u \in R \text{ (once } t \text{ is fixed in } R).$$

also

$$\|g(t-u-\alpha_n)e^{\lambda u}\|_X \le e^{(Re\,\lambda)u} \|f\|_\infty \text{ (as } \|g\|_\infty \le \|f\|_\infty). \tag{3.7}$$

Thus we find again, for same reason as above, that

$$w(t-\alpha_n) \to \int_0^\infty e^{\lambda u} f(t-u)\,du = z(t), \text{ (for all } t \in R).$$

This proves Lemma 3. □

Proof of Theorem 2(continuation): Consider the following system of ordinary diff. equations

$$\begin{aligned}
\frac{dz_1}{dt} &= \lambda_1 z_1(t) + c_{12} z_2(t) + \ldots c_{1n} z_n(t) + g_1(t) \\
\frac{dz_2}{dt} &= \lambda_2 z_2(t) + \ldots c_{2n} z_n(t) + g_2(t) \\
&\vdots \\
\frac{dz_{n-1}}{dt} &= \lambda_{n-1} z_{n-1}(t) + c_{n-1,n} z_n(t) + g_{n-1}(t) \\
\frac{dz_n}{dt} &= \lambda_n z_n(t) + g_n(t)
\end{aligned} \tag{3.8}$$

where the λ_i are the eigenvalues of A, the c_{ij} are those appearing in Lemma 2 while the $g_i(t)$ are defined as follows:

We know that any linear invertible operator B, $\mathbb{C}^n \to \mathbb{C}^n$ is represented by a matrix (non-singular) $B = ((b_{ij}))$, where $b_{ij} \in \mathbb{C}\ \forall i,j = 1,2,\dots n$. We can associate to this matrix a linear continuous operator $Y^n \to Y^n$ in the "usual" way: $\forall y = (y_1 \dots y_n) \in Y^n$,

$$By = \begin{pmatrix} b_{11} & \dots & b_{1n} \\ b_{21} & \dots & b_{2n} \\ \dots & \dots & \dots \\ b_{n1} & \dots & b_{nn} \end{pmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} b_{11} & y_1 & + & \dots & b_{1n} & y_n \\ b_{21} & y_1 & + & \dots & b_{2n} & y_n \\ \dots & & & & & \dots \\ b_{n1} & y_1 & + & \dots & b_{nn} & y_n \end{pmatrix} \tag{3.9}$$

(note that B is also the name of the operator $Y^n \to Y^n$ thus defined).

Now, the inverse matrix $B^{-1} = (b'_{ij})$ is well defined and it also defines a linear continuous operator $Y^n \to Y^n$ in a similar way (it is the inverse of the above considered operator $B, Y^n \to Y^n$, and is denoted by B^{-1}). Then *we put* $g(t) = B^{-1}f(t)$, and $g(t) = (g_1(t), \dots, g_n(t)) \in Y^n, \forall t \in R$. Note that $B^{-1} \in \mathcal{L}(Y^n, Y^n)$, and this implies that g is a.a. $(R \to Y^n)$ so that all $g_i(t)$ are a.a., $R \to Y$.

Apply now Lemma 3 to the last equation in (3.8), to obtain the (unique) a.a. solution $z_n(t), (R \to Y)$. Then again, in the next to last equation, using a.a. of $c_{n-1,n}z_n + g_{n-1}(R \to Y)$, we obtain the (unique) a.a. solution z_{n-1}. Continuing this way we get the unique a.a. solution $(z_1(t), \dots, z_n(t)), R \to Y^n$ of (3.8). Consider now the function $y(t) = Bz(t), R \to Y^n$. We see that $y(t)$ is a.a. Next, we have

$$\frac{dy}{dt} = B\frac{dz}{dt}. \tag{3.10}$$

Actually, the system (3.8) is nothing else than the equation (in Y^n)

$$\frac{dz}{dt} = (B^{-1}AB)z(t) + B^{-1}f(t). \tag{3.11}$$

Therefore, we get the equality

$$\frac{dy}{dt} = B(B^{-1}AB)z(t) + f(t) = ABz(t) + f(t) = Ay(t) + f(t). \quad \square$$

4. Our next result about existence of a.a. solutions refers to the equation $u' = Au + f$ in the Banach space X, when A is the infinitesimal generator of a C_0-semigroup *with exponential decay*, while f is a.., $R \to X$. The function $u(t), R \to X$ is a mild solution of $u' = Au + f$ if it is continuous and if the relation

$$u(t) = S(t-a)u(a) + \int_a^t S(t-\sigma)f(\sigma)\,d\sigma \tag{4.1}$$

holds, $\forall a \in R, \forall t \geq a$.

The *unicity* of a.a. mild solutions follows from the unicity of any *bounded* mild solution. The existence of the a.a. solutions is now quite standard[1]. Precisely, we shall see that the expression

$$v(t) = \int_{-\infty}^t S(t-\sigma)f(\sigma)\,d\sigma \tag{4.2}$$

[1]See, for a special case, Th. 2 in [9].

(an absolutely convergent integral for bounded f) is a.a. whenever f is so).

a) We have

$$v(t) = \int_o^\infty S(s)\ f(t-s)\,ds.$$

If $\{\alpha'_n\}_1^\infty$ is any real sequence, $\exists\{\alpha_n\}_1^\infty$ contained in $\{\alpha'_n\}_1^\infty$ such that $f(\sigma+\alpha_n) \to g(\sigma)$ and $g(\sigma-\alpha_n) \to f(\sigma), \forall\sigma \in R$, in strong X-topology.

Then:

$$v(t+\alpha_n) = \int_o^\infty S(s)f(t+\alpha_n-s)\,ds \to \int_o^\infty S(s)g(t-s)\,ds = w(t) \tag{4.3}$$

(by the dominated convergence theorem; g is now a strongly measurable bounded function, $R \to X$, the integral is a Bochner integral).

Next we have that

$$w(t-\alpha_n) = \int_o^\infty S(s)g(t-\alpha_n-s)\,ds \text{ converges to } \int_o^\infty S(s)f(t-s)\,ds = v(t)\ \forall t \in R.$$

b) $v(t)$ is *continuous*, $R \to X$. In fact, let $t_n \to t_o \in R$.

We have

$$v(t_n) = \int_o^\infty S(s)f(t_n-s)\,ds. \tag{4.4}$$

Then $S(s)f(t_n-s) \to S(s)f(t_0-s)\forall s \in R^+$ (by continuity of f and boundedness of $S(s)$).

Next:

$$\|S(s)f(t_n-s)\| \leq Me^{\beta s}\|f\|_\infty \in L^1(0,\infty)\ (\beta \text{ is } < 0)$$

Hence we can apply the dominated convergence theorem. Thus, v is a.a. Finally, v is a mild solution of $v' = Av + f$. In fact (as in our paper [**14**]), we note that

$$S(t-a)v(a) = S(t-a)\int_{-\infty}^a S(a-\sigma)f(\sigma)\,d\sigma = \int_{-\infty}^a S(t-\sigma)f(\sigma)\,d\sigma \tag{4.5}$$

If we add

$$\int_a^t S(t-\sigma)f(\sigma)\,d\sigma \text{ we find } \int_{-\infty}^t S(t-\sigma)f(\sigma)\,d\sigma = v(t). \quad \square$$

5. In this (final) section of our paper we prove existence and uniqueness of a.a. solutions for the equation $u' = Au + f, f$ being a.a. in a Hilbert space, in a situation similar to that of the article [**15**]. As in the previous section 2, we consider a separable Hilbert space H with orthonormal basis $(e_j)_1^\infty$ and a diagonal (unbounded) operator A defined on the linear span of $(e_j)_1^\infty$ by: $Ah = \sum \alpha_j\lambda_j e_j \forall h = \sum_{\text{finite}} \alpha_j e_j, (\lambda_j)_1^\infty$ being a given sequence of complex numbers.

We shall prove the following

THEOREM 3. *Assume that* $Re\,\lambda_j \neq 0\ \forall j = 1\ ,2 \ldots.$ *Let* $f, R \to H$ *be a.a. such that*

$$\sum_{j=1}^{\infty} \frac{1}{|Re\,\lambda_j|^2} \cdot (\sup_R |(f(t), e_j)|)^2 < +\infty. \tag{5.1}$$

Then the equation $u' = Au + f$ *on the (whole) real line admits an (unique) ultra-weak solution which is almost-automorphic.*

PROOF: The unicity of the almost-automorphic ultra-weak solution is a consequence of its *boundedness* over R (as in the final part of the paper [**15**]) and of the sole assumption that $Re\,\lambda_j \neq 0, \forall j = 1, 2 \ldots.$ The existence is proved as in [**15**], in the following way. Note first that the scalar-product $(f(t), e_j)_H = f_j(t)$ is a scalar-valued a.a. function, $\forall j = 1, 2 \ldots$ and then define the a.a. (scalar) function $u_j(t)$ which is

$$\int_{-\infty}^{t} e^{\lambda_j (t-s)} f_j(s)\, ds \text{ for } Re\,\lambda_j < 0 \text{ or } -\int_{t}^{\infty} e^{\lambda_j (t-s)} f_j(s)\, ds \text{ if } Re\,\lambda_j > 0.$$

We have the obvious estimate

$$|u_j(t)| \leq \frac{1}{|Re\,\lambda_j|} \|f_j\|_\infty \tag{5.2}$$

and accordingly the series of numerical-valued functions

$$\sum_{j=1}^{\infty} |u_j(t)|^2 \text{ is uniformly convergent over } R.$$

Hence, the series of vector-valued functions

$$\sum_{j=1}^{\infty} u_j(t)\, e_j \tag{5.3}$$

is uniformly convergent on R, in H-norm (this is due to the equality)

$$\left\| \sum_{N}^{N+p} u_j(t)\, e_j \right\|^2 = \sum_{N}^{N+p} |u_j(t)|^2. \tag{5.4}$$

As all the finite sums

$$\sum_{j=1}^{N} u_j(t)\, e_j$$

are H-valued a.a. functions, it follows (by a known result about uniformly convergent sequences of a.a. functions, [**13**]) that

$$u(t) = \sum_{1}^{\infty} u_j(t)\, e_j$$

is a.a., $R \to H$ as well.

Finally, in order to establish that u is (ultra)-weak solution of $u' = Au + f$ we can follow the proof in [**15**]; we only have to note that, if $f_j(t) = (f(t), e_j)$, then

$$f(t) = \lim_{N \to \infty} \sum_{j=1}^{N} f_j(t)\, e_j$$

uniformly on R; this is due to the relative compactness of the range of f by [**2**]. □

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