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NOTIONS OF COMPACTNESS ON THE LATTICE AND ON THE POINT SET IN TERMS OF MEASURES

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RÉSUMÉ. Nous introduisons les concepts de presque compacité et de presque compacité dénombrable sur un treillis à l'aide de mesures et nous relions ces concepts à leurs contreparties ensemblistes. Utilisant ces notions, nous donnons des représentations équivalentes de la pseudocompacité.

ABSTRACT. We introduce the concept of almost compact and the concept of almost countably compact on a lattice in terms of measures and tie them in to their counterparts on the point set. With these notions, we give equivalent representations of pseudocompactness.

1. Introduction. The concept of almost compact has been around for a while [**1**, **2**]. What we wish to do here is define almost compact and almost countably compact on an arbitrary lattice of subsets of an arbitrary space X, in terms of measures. We will see that almost compact on the lattice will reduce to the usual notion of almost compact in the point set framework, and that our definition of almost countably compact will reduce to lightly compact [**3**, **4**] in the point set framework. What makes this note of particular interest will be the measure representations of these concepts we give along with the measure style proofs. We will also give some new results involving these notions of compactness on the lattice. Our notation will be that already standard in the literature [**5**]. All our lattice of subsets from the set X will contain both $\varnothing$ and X.

We can take our lattice $\mathcal{L}$ and extend it to the smallest boolean algebra containing $\mathcal{L}$, $A(\mathcal{L})$. We denote the complemented lattice of $\mathcal{L}$ as $\mathcal{L}'$ and note that $A(\mathcal{L}) = A(\mathcal{L}')$. The set of all zero-one valued measures defined on $A(\mathcal{L})$ we denote $I(\mathcal{L})$. We say a measure is $\mathcal{L}$–regular or just regular if $\mathcal{L}$ is clear, if

$$\mu(A) = \sup_{\substack{L \subseteq A \\ L \in \mathcal{L}}} \mu(L) = \inf_{L' \supseteq A} \mu(L').$$

We denote the set of all regular measures by $I_R(\mathcal{L})$. A measure is σ-smooth on the lattice if for $L_n \in \mathcal{L}$ such that $\bigcap_n L_n = \varnothing$ with L_n decreasing (we denote this by $L_n \downarrow \varnothing$) we have $\lim_n \mu(L_n) = 0$. We denote this subset of $I(\mathcal{L})$ as $I_\sigma(\mathcal{L})$. If the above holds true for all $A_n \in A(\mathcal{L})$ then our measure is σ-smooth on the algebra which we denote $I^\sigma(\mathcal{L})$. We note that if $\mu \in I_R(\mathcal{L})$ then $\mu \in I^\sigma(\mathcal{L})$ iff $\mu \in I_\sigma(\mathcal{L})$. For any measure μ, we define its support by $S(\mu) = \{\bigcap_\alpha L_\alpha \mid \mu(L_\alpha) = 1,\ L_\alpha \in \mathcal{L}\}$.

2. Results on a lattice.

DEFINITION 1. A lattice $\mathcal{L}$ is *almost compact* if for $\mu \in I_R(\mathcal{L}')$, $S_{\mathcal{L}}(\mu) \neq \varnothing$.

DEFINITION 2. A lattice $\mathcal{L}$ is *almost countably compact* if $I_R(\mathcal{L}') \subseteq I_\sigma(\mathcal{L})$.

DEFINITION 3. A lattice $\mathcal{L}$ is *countably compact* if $I(\mathcal{L}) \subseteq I_\sigma(\mathcal{L})$.

DEFINITION 4. A lattice $\mathcal{L}$ is *compact* if $S(\mu) \neq \varnothing$ for $\mu \in I(\mathcal{L})$.

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REMARK 1. It is immediate from the definitions that $\mathcal{L}$ countably compact implies $\mathcal{L}$ almost countably compact.

THEOREM 1. *If the lattice $\mathcal{L}$ is almost compact then $\mathcal{L}$ is almost countably compact.*

PROOF: Let $L_n \in \mathcal{L}$ be such that $\bigcap_n L_n = \varnothing$ with L_n decreasing. If $\mu \in I_R(\mathcal{L}')$ and $\lim \mu(L_n) = 1$, this implies L_n is considered in the support of μ and since $S(\mu) \neq \varnothing$ this contradicts $\bigcap_n L_n = \varnothing$. □

THEOREM 2. *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two lattices such that $\mathcal{L}_1 \subseteq \mathcal{L}_2$. If $\mathcal{L}_2$ is almost countably compact, then $\mathcal{L}_1$ is almost countably compact.*

PROOF: Let $\mu \in I_R(\mathcal{L}_1')$, we can find $\lambda \in I_R(\mathcal{L}_2')$ such that $\mu = \lambda|_{A(\mathcal{L}_1')}$ (see [**6**]). But $\mathcal{L}_2$ a.c.c. gives us $\mu \in I_\sigma(\mathcal{L}_2)$ and since $I_\sigma(\mathcal{L}_2) \subseteq I_\sigma(\mathcal{L}_1)$ our result follows. □

Now we give conditions where almost countably compact implies almost compact. We first recall two definitions [**5**].

DEFINITION 5. A lattice $\mathcal{L}$ is *prime complete* if for $\mu \in I_\sigma(\mathcal{L})$, $S(\mu) \neq \varnothing$.

DEFINITION 6. A lattice $\mathcal{L}$ is *almost replete* if $S_{\mathcal{L}}(\mu) \neq \varnothing$ for $\mu \in I_R(\mathcal{L}') \bigcap I_\sigma(\mathcal{L})$.

THEOREM 3. *Let $\mathcal{L}$ be almost countably compact and prime complete then $\mathcal{L}$ is almost compact.*

PROOF: Obvious. □

THEOREM 4. *Let $\mathcal{L}$ be almost countably compact and almost replete then $\mathcal{L}$ is almost compact.*

PROOF: If $\mu \in I_R(\mathcal{L}')$ then $\mu \in I_R(\mathcal{L}') \bigcap I_\sigma(\mathcal{L})$ and since $S(\mu) \neq \varnothing$ our result follows. □

3. Applications to topology. Our first application will be to show that if we let our lattice be the lattice of closed sets, i.e. $\mathcal{L} = \mathcal{F}$ then almost compactness on the lattice reduces to almost compactness on the point set.

THEOREM 5. *Let X be a $T_{3\frac{1}{2}}$ topological space. Then for any arbitrary open covering of X there exists a finite subfamily whose closures cover X iff for $\mu \in I_R(\mathcal{O})$ we have $S_{\mathcal{F}}(\mu) \neq \varnothing$. ($\mathcal{O}$ is the lattice of open sets).*

PROOF:

(1) Take $X = \bigcup_\alpha O_\alpha$ and suppose $X \neq \bigcup_{i=1}^n \overline{O}_{\alpha_i}$ then $X \neq \overline{O}_\alpha$ for any α or $\varnothing \neq \overline{O}_\alpha{}'$ for any α. We see that the $\overline{O}_\alpha{}'$ constitute a filter base. We take the filter formed by $\overline{O}_\alpha{}'$ and enlarging it to an ultrafilter we now can construct a $\mu \in I_R(\mathcal{O})$ such that $\mu(\overline{O}_\alpha{}') = \mu(O'_\alpha) = 1$ for all α (see [**6**]). Considering the support of μ we have $\bigcap_\alpha O'_\alpha = \varnothing$ by our assumption which gives $S_{\mathcal{F}}(\mu) = \varnothing$ and our result follows one way.

(2) Now, suppose $S_{\mathcal{F}}(\mu) = \varnothing$ for $\mu \in I_R(\mathcal{O})$. This gives $\mu(F_\alpha) = 1$ and $\bigcap_\alpha F_\alpha = \varnothing$ for all α. We can write $X = \bigcup_\alpha F'_\alpha$ and $\mu(F'_\alpha) = 0$ for all α. Now if $X = \bigcup_{i=1}^n \overline{F'_{\alpha_i}}$ we see that $\mu(\overline{F'_\alpha}) = 0$ and $\mu(X) \leq \sum_{i=1}^n \mu(\overline{F'_{\alpha_i}}) = 0$ but $\mu(X) = 1$ and our result stands. □

Now for almost countably (lightly compact).

THEOREM 6. *Let X be a $T_{3\frac{1}{2}}$ topological space. Then for any countable open cover of X there exists a finite subfamily whose closures cover X iff $I_R(\mathcal{O}) \subseteq I_\sigma(\mathcal{F})$.*

PROOF:

(1) Take $F_n \in \mathcal{F}$ such that $F_n \downarrow \varnothing$ or $X = \bigcup_n F_n'$. Assuming $X = \bigcup_{k=1}^n \overline{F_{n_k}'}$ we can write $\varnothing = \bigcap_{k=1}^n F_{n_k}^0$. Now if $\lim_n \mu(F_n) = 1$ for $\mu \in I_R(\mathcal{O})$ this gives us $\lim_n \mu(F_n^0) = 1$ but this contradicts $\varnothing = \bigcap_{k=1}^n F_{n_k}^0$ and we have $\mu \in I_\sigma(\mathcal{F})$.

(2) Take $X = \bigcup_n O_n$ with O_n increasing and suppose $X \neq \bigcup_{k=1}^n \overline{O}_{n_k}$. This gives us $\overline{O}_{n_k}{}'$ which can serve as a filter base. Taking that filter and enlarging it to an ultrafilter, we can construct $\mu \in I_R(\mathcal{O})$ such that $\mu(\overline{O_{n_k}}{}') = 1 = \mu(O_{n_k}')$ for all n and k. Now $\varnothing = \bigcap_n O_n'$ and $\lim_k \mu(O_{n_k}') \neq 0$ therefore $\mu \notin I_\sigma(\mathcal{F})$ and the theorem stands. □

REMARK 2. The fact that almost compactness for a topological space implies almost countably compact (lightly compact) follows immediately from Theorem 1.

THEOREM 7. *Let X be a $T_{3\frac{1}{2}}$, almost realcompact, and almost countably compact topological space, then X is almost compact.*

PROOF: Let $\mathcal{L} = \mathcal{F}$ the lattice of closed sets, then almost replete is equivalent to almost realcompact [7]. With Theorem 4 the result follows. □

We have shown [8] that if $I(\mathcal{L}) = I_R(\mathcal{L})$, then the lattice $\mathcal{L}$ is complemented and therefore a boolean algebra. We now see the following:

THEOREM 8. *Let X be $T_{3\frac{1}{2}}$ and almost compact, and let $I(\mathcal{F}) = I_R(\mathcal{F})$, then X is compact.*

PROOF: Take $\mu \in I(\mathcal{F}) = I_R(\mathcal{F})$ which is equivalent to $I(\mathcal{F}) = I_R(\mathcal{O})$, but $S_{\mathcal{O}}(\mu) \neq \varnothing$ since X is almost compact and the result follows. □

THEOREM 9. *Let X be $T_{3\frac{1}{2}}$ almost countably compact and let $I(\mathcal{F}) = I_R(\mathcal{F})$, then X is countably compact.*

PROOF: We see that $\mu \in I(\mathcal{F}) = I_R(\mathcal{O}) \subseteq I_\sigma(\mathcal{F})$. □

REMARK 3. Since $I(\mathcal{F}) = I_R(\mathcal{F})$ implies X is extremely disconnected [8], we see that if X is $T_{3\frac{1}{2}}$ and extremely disconnected then almost compact and compact coincide and almost countably compact and countably compact coincide.

REMARK 4. When we impose the condition that $I(\mathcal{F}) = I_R(\mathcal{F})$ on the space X we can then bring the Theorems about $C(X)$, the space of all real values continuous functions on X, and βX, the Stone-Čech compactification of X, to bear, i.e.: $C(X)$ with the condition, is now a conditionally complete lattice, and βX with the condition is extremely disconnected.

We shall now give some equivalent representations of pseudocompactness in terms of the lattice of closed sets $\mathcal{F}$, the δ-lattice formed from taking countable intersections and finite unions of regular closed sets $\delta(\mathcal{F}_R)$, and the lattice of zero sets $\mathcal{Z}$. We note that $\mathcal{F} \supseteq \delta(\mathcal{F}_R) \supseteq \mathcal{Z}$. We can now show

THEOREM 10. *Let X be a $T_{3\frac{1}{2}}$ topological space then the following are equivalent:*

(1) *X is pseudocompact.*
(2) *The lattice of closed sets $\mathcal{F}$ is almost countably compact (lightly compact).*
(3) *The delta lattice formed from regular closed sets $\delta(\mathcal{F}_R)$ is almost countably compact.*
(4) *The lattice of zero sets $\mathcal{Z}$ is almost countably compact.*

PROOF:

(1) implies (2) we can see in [4].
(2) implies (3) implies (4) follows from Theorem 2.
(4) implies (1). Take $\mu \in I(\mathcal{Z}) = I(\mathcal{Z}')$, there exist $\gamma \in I_R(\mathcal{Z}')$ such that $\mu \leq \gamma$ on $\mathcal{Z}'$ [6], but since $\mathcal{Z}$ is almost countably compact, we have $\mu \leq \gamma \in I_R(\mathcal{Z}') \subseteq I_\sigma(\mathcal{Z})$. This now gives us $\gamma \leq \mu$ on $\mathcal{Z}$ with $\gamma \in I_\sigma(\mathcal{Z})$ but since $\mathcal{Z}$ is normal and countably paracompact this implies that $\mu \in I_\sigma(\mathcal{Z})$ therefore $\mathcal{Z}$ is countably compact which gives us X pseudocompact. □

COMMENTS. The introduction of measures to define topological properties is a valuable tool that can be used to investigate topological spaces. Yet, many questions remain unanswered. Not all topological properties have yielded measure equivalents (metacompactness, paracompactness) and further investigation is needed to determine if they exist. Furthermore, there are measure properties that as of yet have no topological counterpart. This interplay with measures and topologies still has much information to yield.

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