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NOTES ON ABSTRACT DIFFERENTIAL EQUATIONS[1]

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Abstract

In this paper we present a number of (new) results concerning linear differential equations in Banach spaces. The following topics are enclosed: Some regularity properties of weak solutions; mollification of weak solutions and resolvent regularization of them; bounded solutions (a necessary condition for existence and uniqueness); periodic solutions, existence and uniqueness theorems; almost-periodic solutions.

All the equations here considered are of the form: $u'(t) = Au(t) + f(t)$ where $u(t)$, $f(t)$ are functions from a real interval into a Banach space E, and A is a linear (usually unbounded) operator in E with (dense) domain $D(A)$.

Résumé

Dans ce travail on présente certains nouveaux résultats concernant les équations différentielles linéaires en espaces de Banach. On étudie des propriétés de régularité des solutions faibles; ensuite, la régularisation de ces solutions par deux procédés différents; une condition nécessaire pour l'existence et l'unicité des solutions bornées; solutions périodiques, théorèmes d'existence et unicité; solutions presque périodiques.

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Les équations considérées sont de la forme: $u'(t) = Au(t) + f(t)$, u et f étant des fonctions d'un intervalle réel dans un espace de Banach E, tandis que A est un opérateur linéaire dans E de domaine $D(A)$.

Introduction

The present work is dedicated to some properties of differential equations in Banach spaces; precisely, we take equations of the form $u'(t) = Au(t) + f(t)$ where $u(t)$ is a function from a real interval into a Banach space, as well as $f(t)$; the operator A is a linear operator, usually unbounded (that is, discontinuous) and with a domain $D(A)$ (strictly) contained in the Banach space X.

To start with, one defines a certain class of weak solutions (see for ex. [1], [5]). In this context we prove two kinds of results:

i) if the (weak) solution has a (strong) derivative, it belongs to the domain $D(A)$ or to $D(A^{**})$;

ii) if the (weak) solution belongs to $D(A)$, then the strong derivative $u'(t)$ exists.

For another class of solutions, the ultra-weak solutions, similar investigations were done in our paper [6].

Next, we apply the usual mollification process to a weak solution u (that is, the convolution of u with a regular function α); we get a function $u * \alpha$ which has a strong derivative and belongs to $D(A)$ or to $D(A^{**})$.

Finally, if the operator $(\lambda_0 - A)^{-1}$ exists as a bounded linear operator in X (for some complex number λ_0), and if we apply it to a weak solution u, we get a strong solution (it will have a strong derivative). Again, we already proved this kind of results for the ultra-weak solutions of Kato-Tanabe and Lions, in our previous papers [6], [7].

Another section of the paper deals with bounded solutions on the whole real line. A necessary condition for existence and uniqueness of a bounded solution u in correspondence to any (given) bounded function f, was explained in [3] for the case of operators A which are continuous and everywhere defined. It seems, as we shall see, that similar reasonings apply when A is only a linear closed operator which is densely defined.

In the final part of this article we deal with some (simple) results about periodic solutions $u(t)$ when a periodic function $f(t)$ is given, or almost-periodic solutions, in a very simple situation.

If the operator A generates a C_o-semigroup with exponential decay, there exists a unique periodic solution $u(t)$ in the mild or the strong sense.

Next, for a general operator (linear closed but not always generating a semi-group) we indicate how one can look for periodic solutions $u(t)$ when a periodic function $f(t)$ is given, using some simple Fourier series arguments.

Also, in the case where A generates an unitary group of operators in a Hilbert space, we show that the existence of a bounded solution of the equation $u' = Au + f$ implies the existence of a periodic solution with the same period as f (the result is a particular case of [8] but the proof here is somehow simpler to grasp). Finally, we consider the system $u'(t) = Au(t) + f(t)$, where $u(t)$ is a function from R into the product space X^n (X is a Banach space), $f(t)$ is an almost-periodic function from R into X^n, $A = (a_{ij})_{i,j=1}^n$ — a square matrix of complex numbers, and prove, extending a classical result, the almost-periodicity of functions-solutions, $u(t)$ which have relatively compact range in X^n; we also establish an existence and uniqueness theorem in the case where the eigen-values of the matrix A have non-zero real part.

1. Regularity properties of weak solutions

Let X be a B-space, X^* and X^{**} the dual and bidual of X. Let A be a linear, closed, densely defined operator, $D(A) \subset X \to X$. The dual operator

A^* is defined on the set

(1.1) $D(A^*) = \{x^* \in X^* \text{ s.t. } \exists y^* \in X^*, \text{ with } x^*(Ax) = y^*(x) \quad \forall x \in D(A)\}$

by the formula

(1.2) $$A^*x^* = y^*.$$

Thus the relation

(1.3) $$x^*(Ax) = (A^*x^*)(x) \quad \text{holds } \forall x \in D(A), \quad \forall x^* \in D(A^*).$$

Consider a function $f(t)$ which is Bochner integrable, $[0,T] \to X$.

DEFINITION. The strongly continuous function $u(t)$, $[0,T] \to X$ is said to be weak solution of the equation

(1.4) $$\frac{du}{dt} = Au + f \quad \text{on } [0,T]$$

if the following holds:

For all $x^* \in D(A^*)$, the numerical-valued function $x^*(u(t))$ is absolutely continuous on $[0,T]$, and the equality

(1.5) $$\frac{d}{dt} x^*(u(t)) = (A^*x^*)(u(t)) + x^*(f(t))$$

holds *almost-everywhere* on $[0,T]$.

We shall give now the following

THEOREM 1. *Let us assume*

1^o) $u(t)$ *has a strong derivative* $u'(t)$ *almost-everywhere on* $[0,T]$;

2^o) $u(t)$ *is weak solution of* $\frac{du}{dt} = Au + f$ *on* $[0,T]$;

3^o) *the domain* $D(A^*)$ *is dense in* X^*.

Let us call J *the canonical imbedding of* X *into* X^{**}. *Then* $(Ju)(t) \in D(A^{**})$ *a.e. in* $[0,T]$ *and the relation*

(1.6) $$\frac{d}{dt}(Ju) = A^{**}(Ju) + Jf$$

holds, a.e. in $[0,T]$.

PROOF. From hypothesis, $\forall x^* \in D(A^*)$, we have the equality

(i) $\frac{d}{dt} x^*(u(t)) = (A^*x^*)(u(t)) + x^*(f(t))$ a.e. on $[0,T]$, i.e., on $[0,T]/\varepsilon_o$, where $m\varepsilon_o = 0$.

(ii) Also, the derivative $u'(t) = \frac{du}{dt}$ exists strongly, $\forall t \in [0,T]/\varepsilon_1$, $m\varepsilon_1 = 0$.

Hence, $\forall t \in [0,T]/(\varepsilon_o \cup \varepsilon_1)$, both (i) - (ii) are true.

As $\frac{d}{dt} x^*(u(t)) = x^*(u'(t))$ on $C(\varepsilon_o \cup \varepsilon_1)$, we get, a.e. on $[0,T]$, the equality

$$x^*(u'(t) - f(t)) = (A^*x^*)(u(t)). \tag{1.7}$$

On the other hand, let us remember that the isometric linear operator J, X into X^{**} is defined by the relation

$$(Jx)(x^*) = x^*(x) \quad \forall x^* \in X^*. \tag{1.8}$$

Therefore we have, $\forall t \in [0,T]$

$$(Ju)(t)(A^*x^*) = (A^*x^*)(u(t)) \tag{1.9}$$

and consequently, a.e. on $[0,T]$, the equality

$$(Ju)(t)(A^*x^*) = x^*(u'(t) - f(t)), \quad \forall x^* \in D(A^*) \tag{1.10}$$

is verified.

It follows that

$$(Ju)(t)(A^*x^*) = (J(u'(t) - f(t)))(x^*), \quad \forall t \in [0,T]/\varepsilon, \quad m\varepsilon = 0.$$

We can write, once t is fixed in $[0,T]/\varepsilon$, $(Ju)(t) = F^{**} \in X^{**}$, $J(u'(t)-f(t)) = G^{**} \in X^{**}$, so that we have the relation

$$F^{**}(A^*x^*) = G^{**}(x^*) \qquad \forall x^* \in D(A^*). \tag{1.11}$$

Accordingly, we derive that

$$F^{**} \in D(A^{**}) \quad \text{and} \quad A^{**}F^{**} = G^{**} \tag{1.12}$$

that is, almost-everywhere on $[0,T]$

$$(Ju)(t) \in D(A^{**}) \quad \text{and} \quad A^{**}(Ju)(t) = G^{**} = J(u'(t) - f(t)).$$

Note also that, when $u'(t)$ exists, we have $Ju'(t) = \frac{d}{dt}(Ju)(t)$; thus we get

$$\frac{d}{dt}(Ju)(t) = A^{**}(Ju)(t) + (Jf)(t), \tag{1.13}$$

a.e. on $[0,T]$. □

Consider now a somewhat different situation, expressed under the statement of

THEOREM 2. *Let us assume that* $u(t) \in C([0,T];X)$ *is a weak solution of the equation* (1.4) *and that, furthermore,* $u(t) \in D(A)$ *a.e. on* $[0,T]$ *and* $Au(t)$ *is Bochner integrable on* $[0,T]$. *Then, the strong derivative* $u'(t)$ *exists a.e. on* $[0,T]$ *and the equality* $u'(t) = Au(t) + f(t)$ *holds, a.e. on* $[0,T]$.

PROOF. We have the equality $x^*(Au(t)) = (A^*x^*)(u(t))$ a.e. on $[0,T]$, $\forall x^* \in D(A^*)$. Thus, from (1.5) we obtain that

$$\frac{d}{dt} x^*(u(t)) = x^*(Au(t) + f(t)), \text{ a.e. on } [0,T], \qquad \forall x^* \in D(A^*). \tag{1.14}$$

From the absolute continuity of the function $x^*(u(t))$ we obtain the equality

$$x^*(u(t)) - x^*(u(0)) = \int_0^t \frac{d}{ds} x^*(u(s))\, ds \tag{1.15}$$

and accordingly the relation

$$x^*[u(t)-u(0)] = \int_0^t x^*(Au(s) + f(s))\, ds = x^*\left(\int_0^t [Au(s) + f(s)]\, ds\right) \quad \forall x^* \in D(A^*). \tag{1.16}$$

In this theorem, the domain $D(A^*)$ is not assumed to be dense in X^*, but is in any case a "total" set in X^* (this means that $x^*(x) = 0 \quad \forall x^* \in D(A^*) \Rightarrow x = 0$).

Accordingly we get the (representation) formula

$$u(t) - u(0) = \int_0^t (Au + f)(s)\,ds. \tag{1.17}$$

As $Au + f$ is Bochner integrable on $[0,T]$ we derive the result.

2. Convolution of weak solutions with scalar regular functions (mollification)

Given any function $u(t) \in C([0,T];X)$, we shall consider the convolution

$$(u * \alpha)(t) = \int_{t-\varepsilon}^{t+\varepsilon} u(s)\alpha(t-s)\,ds \tag{2.1}$$

where $\alpha \in C_0^1(\mathbb{R})$, $\alpha \geq 0$, $\alpha(t) = 0$ for $|t| \geq \varepsilon$. Thus $(u * \alpha)(t)$ is well-defined for $\varepsilon < t < T-\varepsilon$. Furthermore, the strong derivative $(u * \alpha)'(t)$ exists and $= u(t+\varepsilon)\alpha(-\varepsilon) - u(t-\varepsilon)\alpha(\varepsilon) + \int_{t-\varepsilon}^{t+\varepsilon} u(s)\alpha'(t-s)\,ds = \int_{t-\varepsilon}^{t+\varepsilon} u(s)\alpha'(t-s)\,ds$.

Let us assume now (1.5) for $u(t)$; we shall prove a similar relation for $(u * \alpha)(t)$, namely: $\forall x^* \in X^*$

$$\frac{d}{dt} x^*((u*\alpha)(t)) = (A^*x^*)(u * \alpha)(t) + x^*(f * \alpha)(t), \qquad \varepsilon < t < T-\varepsilon\ , \tag{2.2}$$

where

$$(f * \alpha)(t) = \int_{t-\varepsilon}^{t+\varepsilon} f(s)\alpha(t-s)\,ds = \int_0^T f(s)\alpha(t-s)\,ds, \qquad \varepsilon < t < T-\varepsilon. \tag{2.3}$$

(Note that the derivative exists $\forall t \in (\varepsilon, T-\varepsilon)$, as it exists strongly.)

We have the equality

$$\begin{aligned}\frac{d}{dt} x^*((u*\alpha)(t)) &= x^*((u * \alpha)'(t)) = x^*\Big(\int_{t-\varepsilon}^{t+\varepsilon} u(s)\alpha'(t-s)\,ds\Big)\\ &= \int_{t-\varepsilon}^{t+\varepsilon} x^*(u(s))\alpha'(t-s)\,ds.\end{aligned} \tag{2.4}$$

Now, due to absolute continuity of $x^*(u(s))$ we can integrate by parts and get, after use of (1.5), the following relations

$$\int_{t-\varepsilon}^{t+\varepsilon} x^*(u(s))\alpha'(t-s)\,ds = -x^*(u(s))\alpha(t-s)\Big|_{t-\varepsilon}^{t+\varepsilon} + \int_{t-\varepsilon}^{t+\varepsilon} \alpha(t-s)\frac{d}{ds}x^*(u(s))\,ds$$

$$= \int_{t-\varepsilon}^{t+\varepsilon} \alpha(t-s)[(A^*x^*)(u(s)) + x^*(f(s))]\,ds$$

$$(2.5) \qquad = \int_{t-\varepsilon}^{t+\varepsilon} (A^*x^*)(\alpha(t-s)u(s))\,ds + \int_{t-\varepsilon}^{t+\varepsilon} x^*(\alpha(t-s)f(s))\,ds$$

$$= (A^*x^*)(u * \alpha)(t) + x^*(f * \alpha)(t). \square$$

Using Theorem 1 and the above remarks we get the following mollification result:

THEOREM 3. *Assume* $D(A^*)$ *is dense in* X^*; $u(t) \in C([0,T];X)$ *is weak solution of* (1.4). *Then* $J(u * \alpha)(t) \in D(A^{**})$ *and the equality*

$$(2.6) \qquad \frac{d}{dt} J(u * \alpha)(t) = A^{**}J(u * \alpha)(t) + ((Jf) * \alpha)(t)$$

holds, $\forall t \in (\varepsilon, T-\varepsilon)$.

In fact, from Theorem 1 we obtain (2.6) to hold almost-everywhere. However, left and right-hand side of (2.6) are continuous functions, so that the equality is actually true for any $t \in (\varepsilon, T-\varepsilon)$ (if it would be false in t_0 it would be so in $(t_0-\delta, t_0+\delta)$ for some $\delta > 0$ which has positive measure).

(It is not necessary to use Theorem 1 but just imitate its proof and get the result directly, for all $t \in (\varepsilon, T-\varepsilon)$.)

3. Resolvent regularization of weak solutions

Let us assume again that the function $u(t) \in C([0,T];X)$ is weak solution of the equality (1.4) and assume also that the resolvent $(\lambda_0 - A)^{-1}$ exists and $\in L(X)$ for some $\lambda_0 \in \mathbb{C}$.

PROPOSITION 1. *The function* $v(t) = R(\lambda_0;A)u(t)$ *is also weak solution of* (1.4) *with* f *replaced by* $R(\lambda_0;A)f$.

PROOF. We first need to establish the absolute continuity of the function $x^*(v(t))$ in $[0,T]$, $\forall x^* \in D(A^*)$ (and even for all $x^* \in X^*$). We have in fact, $\forall x^* \in X^*$ the equality

$$x^*(R(\lambda_0;A)u(t)) = ((R(\lambda_0;A))^*x^*)u(t). \tag{3.1}$$

We know also that $\lambda_0 \in \rho(A^*)$ (resolvent set of A^*) and that $R(\lambda_0;A^*) = [R(\lambda_0;A)]^*$. It follows that $(R(\lambda_0;A))^*x^* = R(\lambda_0;A^*)x^* = (\lambda_0-A^*)^{-1}x^*$ which belongs to $D(A^*)$ for all $x^* \in X^*$. Therefore $x^*(v(t)) = y^*(u(t))$ where $y^* = (\lambda_0-A^*)^{-1}x^*$ belongs to $D(A^*)$; hence $x^*(v(t))$ is absolutely continuous on $[0,T]$, $\forall x^* \in X^*$. Next, we have the simple relation

$$\frac{d}{dt}x^*(v(t)) = \frac{d}{dt}y^*(u(t)) = (A^*y^*)(u(t)) + y^*(f(t)) \quad \text{a.e.} \tag{3.2}$$

which is written as

$$\frac{d}{dt}x^*(v(t)) = (A^*(\lambda_0-A^*)^{-1}x^*)(u(t)) + ((\lambda_0-A^*)^{-1}x^*)(f(t)) \quad \text{a.e.} \tag{3.3}$$

Note now that, for $x^* \in D(A^*)$, the commutativity property $A^*(\lambda_0-A^*)^{-1}x^* = (\lambda_0-A^*)^{-1}A^*x^*$, holds. Hence, we derive from (3.3), the equality

$$\begin{aligned} \frac{d}{dt}x^*(v(t)) &= ((\lambda_0-A^*)^{-1}A^*x^*)(u(t)) + ([(\lambda_0-A)^{-1}]^*x^*)(f(t)) \\ &= ([(\lambda_0-A)^{-1}]^*A^*x^*)(u(t)) + ([(\lambda_0-A)^{-1}]^*x^*)(f(t)) \\ &= (A^*x^*)((\lambda_0-A)^{-1}u(t)) + x^*((\lambda_0-A)^{-1}f(t)) \\ &= (A^*x^*)(v(t)) + x^*(g(t)), \quad g(t) = R(\lambda_0;A)f(t). \end{aligned} \tag{3.4}$$

This proves the proposition.

Note also that $v(t) \in D(A)$ $\forall t \in [0,T]$ and that $Av(t) = -u(t) + \lambda_0 R(\lambda_0;A)u(t)$ belongs to $C([0,T];X)$.

From Theorem 2 we infer that $v'(t)$ *exists strongly, a.e. on* $[0,T]$, *and that*

$$v'(t) = Av(t) + R(\lambda_0;A)f(t), \tag{3.5}$$

a.e. on $[0,T]$.

If $f(t)$ is continuous on $[0,T]$ we get that $v'(t)$ is also continuous and consequently $v'(t) = Av(t) + R(\lambda_0;A)f(t) \quad \forall t \in [0,T]$.

4. A necessary condition for existence and uniqueness of bounded solutions

In the B-space X consider a linear closed operator A, with dense domain $D(A)$. We prove the following

THEOREM 4. *Let us assume that for any continuous function* $f(t)$, $\mathbb{R} \to X$ *such that* $\sup_{t\in\mathbb{R}} \|f(t)\| < \infty$ *there exists one and only one solution* $u(t)$ *of the equation*

$$(4.1) \qquad u'(t) = Au(t) + f(t)$$

over the real line, such that

$$(4.2) \qquad \sup_{t\in\mathbb{R}} \|u(t)\| < \infty.$$

Then, any number on the imaginary line $\{i\tau\}_{\tau\in\mathbb{R}}$ *belongs to the resolvent set of* A.

LEMMA 1. *The operator* $(i\tau-A)^{-1}$ *exists* $\forall \tau \in \mathbb{R}$.

If not, there exists $\tau_0 \in \mathbb{R}$ and $x_0 \in D(A)$, $x_0 \neq \theta$ such that $i\tau_0 x_0 = Ax_0$.

Consider the vector-function $u_0(t) = e^{i\tau_0 t} x_0$. We see that $\|u_0(t)\| = \|x_0\|$, so that $u_0(t)$ is bounded over $\mathbb{R}$. Furthermore, we have $u_0'(t) = i\tau_0 u_0(t) = Au_0(t)$, $\forall t \in \mathbb{R}$.

Hence, the homogeneous equation $u'(t) = Au(t)$ has a non-trivial bounded solution over $\mathbb{R}$, a contradiction.

LEMMA 2. *The operator* $(i\tau-A)$ *maps* $D(A)$ *onto* X, $\forall \tau \in \mathbb{R}$.

Consider the function $f(t) = e^{i\tau t}f_o$, where f_o is any fixed element of X. Hence $f(t)$ is bounded over $\mathbb{R}$. Let $x(t)$ be the unique bounded solution of the equation

$$x'(t) = Ax(t) + f(t) \tag{4.3}$$

and put $y(t) = e^{-i\tau t}x(t)$. Then $y(t)$ is bounded over R and

$$y'(t) = (A-i\tau)y(t) + f_o. \tag{4.4}$$

Actually, the equation (4.4) has at most one bounded solution (otherwise, if y_1, y_2 are two bounded solutions of it, $x_1(t) = e^{i\tau t}y_1(t)$, $x_2(t) = e^{i\tau t}y_2(t)$ would be two bounded solutions of (4.3)). It follows that the translated function $y(t+a)$ (a being any real number), which is again a bounded solution of (4.4), must coincide with $y(t)$. Thus, we get: $y(t+a) = y(t)$, $\forall t \in \mathbb{R}$; hence $y(a) = y(0)$. But a is any real number, hence $y(t)$ is constant. From (4.4) we derive $\theta = (A-i\tau)y(0) + f_o$, $f_o = (A-i\tau)(-y(0))$. This proves Lemma 2.

We now end the proof of the theorem. We obtained that, $\forall\tau \in \mathbb{R}$, the operator $(i\tau-A)^{-1}$ exists and is everywhere defined. It is also closed, like $i\tau-A$, hence it is bounded by the closed graph theorem.

5. Periodic solutions

We consider again non-homogeneous differential equations in Banach spaces: $u'(t) = Au(t) + f(t)$, where A is a certain linear unbounded operator while $f(t)$, $\mathbb{R} \to X$ (the B-space) is periodic with period ω ($f(t+\omega) = f(t)$ $\forall t \in \mathbb{R}$).

First, we prove existence and uniqueness of a periodic strong solution $u(t)$ with the same period, under the hypothesis that A is the infinitesimal generator of a C_o-semigroup with exponential decay as $t \to \infty$.

Let $S(t)$, $t \in \mathbb{R}^+ \to L(X)$ be a C_o-operator semigroup, verifying an estimate $\|S(t)\| \le Me^{\beta t}$, $\forall t \ge 0$, where $M > 0$, $\beta < 0$, and let $A = \lim_{\eta\downarrow 0} \frac{S(\eta)-I}{\eta}$ be its infinitesimal generator.

We have

THEOREM 5. *Given* $f(t) \in C^1(\mathbb{R};X)$, *periodic of period* ω, *there exists one and only one (strong) solution over* $\mathbb{R}$ *of*

$$u'(t) = Au(t) + f(t) \tag{5.1}$$

which is periodic with the same period ω.

PROOF. Uniqueness:

Any periodic continuous function is bounded over $\mathbb{R}$. If u_1, u_2 are periodic solutions with period ω, their difference $u(t)$ is a periodic (hence a bounded) solution of $u' = Au$, over the whole $\mathbb{R}$.

This implies $u(t) \equiv \theta$ by Th. 1.1, Ch. V in [9].

Existence:

As is quite easy to see, for any real number A, the integral $\int_A^t S(t-\sigma)f(\sigma)\,d\sigma$ exists (in Riemann's sense), because one can establish easily the continuity of $\sigma \to S(t-\sigma)f(\sigma)$ for $A < \sigma < t$.

Next, we have the estimate

$$\|S(t-\sigma)f(\sigma)\| \leq Me^{\beta(t-\sigma)} \sup_{\sigma\in\mathbb{R}} \|f(\sigma)\| = Ce^{\beta t}e^{|\beta|\sigma}$$

Also $\int_{-\infty}^t e^{|\beta|\sigma}\,d\sigma = \frac{1}{|\beta|}e^{|\beta|t} = \frac{1}{|\beta|}e^{-\beta t}$ is convergent. Thus the integral $\int_{-\infty}^t S(t-\sigma)f(\sigma)\,d\sigma$ is absolutely convergent and the estimate

$$\left\|\int_{-\infty}^t S(t-\sigma)f(\sigma)\,d\sigma\right\| \leq \frac{M}{|\beta|}\|f(.)\|_\infty \tag{5.2}$$

holds.

Next, we prove that the function

$$u(t) = \int_{-\infty}^t S(t-\sigma)f(\sigma)\,d\sigma \tag{5.3}$$

is periodic, with the same period ω. In fact, we have

$$u(t+\omega) = \int_{-\infty}^{t+\omega} S(t+\omega-\sigma)f(\sigma)\, d\sigma = \quad (\sigma-\omega = s)$$

(5.4)

$$\int_{-\infty}^{t} S(t-s)f(s+\omega)\, ds = \int_{-\infty}^{t} S(t-s)f(s)\, ds = u(t), \qquad \forall t \in \mathbb{R}.$$

Next, one has to prove that $u(t)$ is strongly continuous and also a strong solution over $\mathbb{R}$ of $u' = Au + f$. This is done in [9], in the more general case where f is almost-periodic over $\mathbb{R}$. (Note that $f \in C^1(\mathbb{R};X)$ and periodic implies that f' is also continuous periodic, hence bounded over $\mathbb{R}$.) □

Somewhat simpler is the study of periodic *mild* solutions over $\mathbb{R}$ of $u' = Au + f$.

DEFINITION. Given the continuous function $f(t)$; $\mathbb{R} \to X$, the continuous function $u(t)$, $\mathbb{R} \to X$ is said to be a mild solution of the equation

$$u' = Au + f$$

if the functional relation

$$u(t) = S(t-a)u(a) + \int_a^t S(t-\sigma)f(\sigma)\, d\sigma \tag{5.5}$$

holds, $\forall a \in \mathbb{R}$ and $\forall t \geq a$.

We have now

THEOREM 6. *Given* $f \in C(\mathbb{R};X)$, *periodic of period* ω, *there exists one and only one mild solution over* $\mathbb{R}$ *of* $u' = Au + f$, *which is periodic with period* ω.

PROOF. <u>Uniqueness:</u>

If u_1, u_2 are two periodic mild solutions with period ω, $u = u_1 - u_2$ verifies

$$u(t) = S(t-a)u(a), \qquad \forall t \geq a, \quad \forall a \in \mathbb{R} \tag{5.6}$$

and is periodic, of period ω, hence bounded over $\mathbb{R}$. Thus

$$\|u(t)\| \leq Me^{\beta(t-a)} \sup_{\mathbb{R}} \|u(.)\| \to 0 \quad \text{as} \quad a \to -\infty.$$

Existence:

Consider the periodic function $u(t) = \int_{-\infty}^{t} S(t-\sigma)f(\sigma)\,d\sigma$ which was previously defined.

It is continuous, due to uniform continuity of f over $\mathbb{R}$.

It is a mild solution: In fact the right-hand side in (5.5) becomes:

$$S(t-a)\int_{-\infty}^{a} S(a-\sigma)f(\sigma)\,d\sigma + \int_{a}^{t} S(t-\sigma)f(\sigma)\,d\sigma$$

$$= \int_{-\infty}^{a} S(t-a+a-\sigma)f(\sigma)\,d\sigma + \int_{a}^{t} S(t-\sigma)f(\sigma)\,d\sigma \tag{5.7}$$

$$= \int_{-\infty}^{t} S(t-\sigma)f(\sigma)\,d\sigma = u(t). \ \square$$

Consider now a linear closed operator A with domain $D(A)$ in the Banach space X and then a continuous periodic function (period T), $f(t)$, from $\mathbb{R}$ into X. Let us define the Fourier coefficients f_k of f, by the usual formula

$$f_k = \frac{1}{T}\int_{0}^{T} f(t)e^{-ikt\frac{2\pi}{T}}\,dt, \qquad k \in \mathbb{Z}. \tag{5.8}$$

Next, let us assume that $u(t)$, $\mathbb{R} \to X$ is a solution of the equation $u'(t) = Au(t) + f(t)$, $t \in \mathbb{R}$, which is also periodic with the same period T as f. Thus, its Fourier coefficients u_k are given by

$$u_k = \frac{1}{T}\int_{0}^{T} u(t)e^{-ikt\frac{2\pi}{T}}\,dt, \qquad k \in \mathbb{Z}. \tag{5.9}$$

We are now looking for some connection between u_k and f_k:

From the equality: $u' = Au + f$ we derive that

$$e^{-ikt\frac{2\pi}{T}}u'(t) = e^{-ikt\frac{2\pi}{T}}Au(t) + e^{-ikt\frac{2\pi}{T}}f(t), \qquad t \in \mathbb{R}, \quad k \in \mathbb{Z}. \tag{5.10}$$

It follows, integrating from 0 to T and using closedness of A

$$\frac{1}{T}\int_{0}^{T} e^{-ikt\frac{2\pi}{T}}u'(t)\,dt = Au_k + f_k. \tag{5.11}$$

The left-hand side integral is transformed, using partial integration and $u(T) = u(0)$, into $\frac{2k\pi i}{T} u_k$. Therefore we get

$$(5.12) \qquad (\frac{2k\pi i}{T} - A)u_k = f_k, \qquad \forall k \in \mathbb{Z}$$

which is the connection we were looking for.

Let us assume now that

$$(5.13) \qquad (\frac{2k\pi i}{T} - A)^{-1} \in L(X) \qquad \forall k \in \mathbb{Z}.$$

It follows that

$$(5.14) \qquad u_k = (\frac{2k\pi i}{T} - A)^{-1} f_k.$$

Assume now that

$$(5.15) \qquad \|(i\tau - A)^{-1}\| \le \frac{C}{|\tau|^2} \quad \text{for large real } \tau.$$

It follows that

$$(5.16) \qquad \|u_k\| \le \frac{C}{(\frac{2k\pi}{T})^2} = \frac{CT^2}{4\pi^2 k^2}$$

and accordingly the Fourier series of u: $\sum_{k\in\mathbb{Z}} u_k e^{\frac{2\pi}{T} ikt}$ is absolutely and uniformly convergent.

Another question to be considered is the following:

In the Banach space X consider a family $A(t)$ of linear operators with domain $D(A(t)) \subset X$, where $t \in \mathbb{R}$ and $A(t+\omega) = A(t)$ $\forall t \in \mathbb{R}$ and some $\omega > 0$.

Assume that $u(t)$, $0 \le t \le \omega \to D(A(t))$ is a solution of the equation $u'(t) = A(t)u(t)$ on $[0,\omega]$ (with right-derivative for $t = 0$ and left-derivative for $t = \omega$). Then, *if* $u(0) = u(\omega)$, *there exists a solution* $v(t)$, $t \in \mathbb{R} \to D(A(t))$, *of the equation* $v'(t) = A(t)v(t)$, *such that* $v(t+\omega) = v(t)$ $\forall t \in \mathbb{R}$.

Let us define in fact a function $v(t)$ by means of the relation $v(t) = u(t-n\omega)$ for $n\omega \le t < (n+1)\omega$ (thus $0 \le t-n\omega < \omega$), $\forall n \in \mathbb{Z}$. Assume $t = n\omega + \alpha$, $0 \le \alpha < \omega$; then $v(t) = u(\alpha)$ and $v(t+\omega) = v((n+1)\omega + \alpha) = u(\alpha) = v(t)$. Hence v is periodic, with period ω. We must also prove that $v'(t)$ exists strongly $\forall t \in \mathbb{R}$. This is obvious for $n\omega < t < (n+1)\omega$; in this case $v'(t) = u'(t-n\omega) = A(t-n\omega)u(t-n\omega) = A(t)v(t)$. Let now $t = n\omega$ (some $n \in \mathbb{Z}$). We have for $h > 0$, $\frac{1}{h}[v(t+h) - v(t)] = \frac{1}{h}[u(h) - u(0)] \to u_+'(0) = A(0)u(0)$; for $h < 0$ we have

$$\frac{1}{h}[v(t+h) - v(t)] = \frac{1}{h}[u(\omega+h) - u(0)] = \frac{1}{h}[u(\omega+h) - u(\omega)] \to u_-'(\omega)$$

$$= A(\omega)u(\omega) = A(0)u(0).$$

Our present discussion ends with a result where existence of a bounded solution implies existence of a periodic solution. In the Hilbert space H consider a unitary group $U(t)$ of linear transformations: that is $U^*(t) = [U(t)]^{-1} = U(-t)$, $\forall t \in \mathbb{R}$, with infinitesimal generator A. Given a continuous periodic function $f(t)$, $\mathbb{R} \to H$ (period p), we define mild solutions of the equation $u'(t) = Au(t) + f(t)$ as continuous functions $u(t)$, $\mathbb{R} \to H$, admitting the representation formula

$$u(t) = U(t)u(0) + \int_0^t U(t-\sigma)f(\sigma)\, d\sigma, \qquad \forall t \in \mathbb{R}. \tag{5.17}$$

Let us assume existence of a mild solution $u(t)$ which is bounded over the real line: $\sup_{t\in\mathbb{R}} \|u(t)\| < \infty$. Then, using Theorem 4.1 in [10], we infer *existence and unicity* of a bounded mild solution w, such that $\sup_{t\in\mathbb{R}} \|w(t)\| \le \sup_{t\in\mathbb{R}} \|v(t)\|$ for all bounded mild solutions of (5.17) (*minimal* bounded mild solution).

We shall now see that *this minimal bounded solution is periodic with the same period as* f.

Note first that from the relation

$$w(t) = U(t)w(0) + \int_0^t U(t-\sigma)f(\sigma)\, d\sigma \tag{5.18}$$

we infer

(5.19)
$$\begin{aligned} w(t+p) &= U(t+p)w(0) + \int_0^{t+p} U(t+p-\sigma)f(\sigma)\,d\sigma \\ &= U(t)[U(p)w(0) + \int_0^{t+p} U(p-\sigma)f(\sigma)\,d\sigma] \\ &= U(t)[U(p)w(0) + \int_0^{p} U(p-\sigma)f(\sigma)\,d\sigma + \int_p^{t+p} U(p-\sigma)f(\sigma)\,d\sigma] \\ &= U(t)w(p) + U(t) \int_p^{t+p} U(p-\sigma)f(\sigma)\,d\sigma. \end{aligned}$$

If $\sigma = s+p$, we have

(5.20)
$$\int_p^{t+p} U(p-\sigma)f(\sigma)\,d\sigma = \int_0^t U(-s)f(s)\,ds$$
$$w(t+p) = U(t)w(p) + \int_0^t U(t-s)f(s)\,ds, \qquad t \in \mathbb{R}.$$

This shows that the translated function: $t \to w(t+p)$ is also a mild solution of (5.17).

Furthermore, $\sup_{t\in\mathbb{R}} \|w(t+p)\| = \sup_{t\in\mathbb{R}} \|w(t)\|$, indicates that w is a *minimal* mild solution, hence, by unicity, $w(t+p) = w(t)$ $\forall t \in \mathbb{R}$. □

(Remark: A similar result for semigroups instead of groups appears in our paper [8].)

6. Almost-periodic solutions

Let $\mathbb{R}$ be the real line, Y a Banach space over $\mathbb{C}$; $A = (a_{ij})_{i,j=1}^n$ a square-matrix of complex numbers, $n \times n$; Y^n being the product Banach space with norm: $\|y\|_{Y^n} = (\sum_1^n \|y_i\|_Y^2)^{\frac{1}{2}}$ $\forall y = (y_1, y_2, \ldots, y_n) \in Y^n$. Our first result is

THEOREM 7. *Let* $f(t)$; $\mathbb{R} \to Y^n$ *be an almost-periodic function while* $y(t)$, $\mathbb{R} \to Y^n$ *is a solution of the equation*

(6.1)
$$\frac{dy}{dt} = Ay(t) + f(t), \qquad t \in \mathbb{R}.$$

Then, if $y(t)$ *has relatively compact range in* Y^n, *it is almost-periodic.*

We shall use the following

LEMMA. *There exists a linear operator* B, $\mathbb{C}^n \to \mathbb{C}^n$, *which has an inverse, such that*

$$B^{-1}AB = \begin{pmatrix} \lambda_1 & C_{12} & \dots & C_{1n} \\ 0 & \lambda_2 & \dots & C_{2n} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix},$$

where the λ_i's *are the eigenvalues of* A (see [2], pp. 91-92).

PROOF of Theorem. Let $Z(t) = B^{-1}y(t)$, a function from $\mathbb{R}$ into Y^n (here B^{-1} is the operator generated by the matrix B^{-1}, acting linearly from Y^n into itself). As B^{-1} is also continuous from Y^n into itself, we see that $Z(t)$ has also a relatively compact range in Y^n. Similarly, the function $g(t) = B^{-1}f(t)$, $\mathbb{R} \to Y^n$, is almost-periodic. We obtain now

$$\frac{dZ}{dt} = B^{-1}\frac{dy}{dt} = B^{-1}Ay(t) + B^{-1}f(t) = B^{-1}AB\,Z(t) + g(t). \tag{6.2}$$

Using the above lemma, we derive from (6.2) the following system:

$$\begin{aligned} \frac{dZ_1}{dt} &= \lambda_1 Z_1(t) + C_{12}Z_2(t) + \dots + C_{1n}Z_n(t) + g_1(t) \\ \frac{dZ_2}{dt} &= \lambda_2 Z_2(t) + \dots + C_{2n}Z_n(t) + g_2(t) \\ &\dots\dots\dots\dots\dots\dots\dots\dots \\ \frac{dZ_{n-1}}{dt} &= \lambda_{n-1}Z_{n-1}(t) + C_{n-1,n}Z_n(t) + g_{n-1}(t) \\ \frac{dZ_n}{dt} &= \lambda_n Z_n(t) + g_n(t), \end{aligned} \tag{6.3}$$

where $Z(t) = (Z_1(t),\dots,Z_n(t))$, $g(t) = (g_1(t),\dots,g_n(t))$.

Now, if P_j, $Y^n \to Y$ is the projection $Z \to P_jZ = Z_j$ (for $Z = (Z_1,Z_2,\dots,Z_n)$), we see that P_j is a linear continuous mapping; therefore, each function $Z_i(t)$, $\mathbb{R} \to Y$ has relatively compact range, and each function $g_i(t)$ is almost-periodic, $\mathbb{R} \to Y$. We shall apply a result by Kopec [4], several times, starting with the last equation in (6.3), and obtain that each function $Z_i(t)$ is almost-periodic, $\mathbb{R} \to Y$.

Our last result is the

THEOREM 8. *Let* $A = (a_{ij})_{i,j=1}^{n}$ *a square-matrix of complex numbers, such that, for each eigenvalue* λ_j, $\mathbb{R}e\lambda_j \neq 0$ *holds. Then, given any almost-periodic function* $f(t)$, $\mathbb{R} \to Y^n$, *there exists one and only one almost-periodic function* $y(t)$, $\mathbb{R} \to Y^n$, *solving the equation*

$$\frac{dy}{dt} = Ay + f. \tag{6.4}$$

PROOF. Uniqueness:

Let $u(t)$ be a bounded over $\mathbb{R}$ solution of $\frac{du}{dt} = Au$. Then $Z(t) = B^{-1}u(t)$ is a bounded solution, $\mathbb{R} \to Y^n$ of $Z'(t) = B^{-1}AB\, Z(t)$. Hence we get

$$\begin{aligned} \frac{dZ_1}{dt} &= \lambda_1 Z_1(t) + C_{12}Z_2(t) + \dots + C_{1n}Z_n(t) \\ &\dots\dots\dots\dots\dots\dots\dots\dots \\ \frac{dZ_{n-1}}{dt} &= \qquad\qquad \lambda_{n-1}Z_{n-1}(t) + C_{n-1,n}Z_n(t) \\ \frac{dZ_n}{dt} &= \qquad\qquad\qquad\qquad \lambda_n Z_n(t). \end{aligned} \tag{6.5}$$

The last equation gives (as for scalar-valued functions) that $Z_n(t) = e^{\lambda_n t} Z_n(0)$. As $Z_n(t)$ is bounded over $\mathbb{R}$ and $\mathbb{R}e\lambda_n \neq 0$ we get $Z_n(t) \equiv 0$. Next, $\frac{dZ_{n-1}}{dt} = \lambda_{n-1}Z_{n-1}(t)$ and again $Z_{n-1}(t) \equiv 0$, and so on.

Existence:

We solve first the system (6.3). From the last equation we get an almost-periodic $Z_n(t)$ which is $\int_{-\infty}^{t} e^{\lambda_n(t-\sigma)} g_n(\sigma)\, d\sigma$ for $\mathbb{R}e\lambda_n < 0$ or $-\int_{t}^{\infty} e^{\lambda_n(t-\sigma)} g_n(\sigma)\, d\sigma$ for $\mathbb{R}e\lambda_n > 0$.

Then we find an almost-periodic Z_{n-1} and inductively, almost-periodic $Z_1, Z_2, \ldots, Z_{n-2}$. Next, put $y(t) = B\,Z(t)$. □

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