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A DIRECT PROOF OF POISSON'S THEOREM FOR NON-IDENTICAL BERNOULLI RANDOM VARIABLES

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Abstract

The purpose of this note is to prove directly, by an elementary method, that the distribution of the sum of independent but not necessarily identically distributed Bernoulli random variables converges to a Poisson distribution.

Résumé

Le but du présent travail est de démontrer directement, en utilisant une méthode élémentaire, que la distribution de la somme de variables aléatoires indépendantes mais pas nécessairement identiquement distribuées, de type Bernoulli converge vers une distribution de Poisson.

Introduction

Let $\{X_{n1}, X_{n2}, \ldots, X_{nn}\}$ be $n \geq 1$ independent integer-valued random variables and define the sum $S_n = \sum_{i=1}^{n} X_{ni}$. It is well known that if the X_{ni} are identically distributed Bernoulli random variables with $E(X_{ni}) = p_n$, and $np_n \to \lambda > 0$ as $n \to \infty$, then

$$\lim_{n\to\infty} P(S_n = k) = e^{-\lambda}\lambda^k/k! \tag{1}$$

for all $k = 0,1,2,\ldots$. This result is known in the literature as Poisson's theorem.

Von Mises [3] pointed out in 1921 that (1) also holds if the X_{ni} are not identically distributed, but satisfy, for $E(X_{ni}) = p_{ni}$, $\sum_{i=1}^{n} p_{ni} = \lambda > 0$ and $p_{ni} = O(1/n)$. Later, Koopman [2] independently obtained the same result. We state below Von Mises' version of Poisson's theorem in the form in which it appears in the literature today, which is more general than the original one.

THEOREM (Von Mises). *Let* $X_{n1}, X_{n2}, \ldots, X_{nn}$ *be* n *independent Bernoulli random variables with* $P(X_{ni} = 1) = p_{ni} = 1 - P(X_{ni} = 0)$, *then* (1) *holds if* $\lambda_n = \sum_{i=1}^{n} p_{ni} \to \lambda > 0$ *and* $p_n = \max_{1 \le i \le n} p_{ni} \to 0$ *as* $n \to \infty$.

The original proof of Von Mises ([3]; see also [4], pp. 107-112), as well as the proofs of Koopman [2] and Feller ([1], p. 264) use generating functions and the continuity theorem. In this note we shall prove the theorem by directly proving that (1) holds for each k. This approach is used in most textbooks for the special case when the p_{ni} do not depend on i.

PROOF of the Theorem. We first state a lemma. It can be proved by using the expansion $-\ell n(1-x) = x + x^2/2 + x^3/3 + \ldots$ and its proof shall be omitted.

LEMMA. *For each* $n = 1,2,\ldots$, *let* $\{\alpha_{n1}, \alpha_{n2}, \ldots, \alpha_{nn}\}$ *be* n *non-negative real numbers. If* $\sum_{i=1}^{n} \alpha_{ni} \to \alpha < \infty$ *and* $\max_{1 \le i \le n} \alpha_{ni} \to 0$, *as* $n \to \infty$, *then* $\prod_{i=1}^{n} (1-\alpha_{ni}) \to e^{-\alpha}$ *and* $\sum_{i=1}^{n} \alpha_{ni}^{r} \to 0$, *for all* $r > 1$.

We now proceed to prove the theorem.

For $k = 0$, it follows from the lemma that

$$P(S_n = 0) = \prod_{i=1}^{n} (1-p_{ni}) \to e^{-\lambda}, \qquad \text{as } n \to \infty. \tag{2}$$

For $k \ge 1$, denote $N_n = \{1,2,\ldots,n\}$ and, for $n \ge k$, define the cross product $N_n^k = N_n \times N_n \times \ldots \times N_n$ of N_n with itself k times.

Let A_n^k be the subset of N_n^k defined by

$$A_n^k = \{(i_1,\ldots,i_k) \in N_n^k\colon\ i_j \neq i_\ell \text{ for all } j \neq \ell\} \tag{3}$$

and denote $B_n^k = N_n^k \backslash A_n^k$.

Because the probability $P(S_n = k)$ is invariant under permutations of $(X_1,\ldots,X_n)$, we can write

(4) $$P(S_n = k) = P(S_n = 0)\Big(\sum{}^{\ddagger} \prod_{j=1}^{k} (p_{ni_j}/(1-p_{ni_j}))\Big),$$

where the summation $\sum^{\ddagger}$ is taken over the set

$$\{(i_1,\ldots,i_k) \in N_n^k\colon\ 1 \le i_1 < \ldots < i_k \le n\}$$

which is a subset of A_n^k.

By assumption $p_n \to 0$ as $n \to \infty$, so we can choose an n_o such that if $n \ge n_o$, we have $p_{ni} \le p_n < 1$ for all i. For such n, we can write

(5) $$P(S_n = 0)\Big(\sum{}^{\ddagger} \prod_{j=1}^{k} p_{ni_j}\Big) \le P(S_n = k) \le P(S_n = 0)\Big(\sum{}^{\ddagger} \prod_{j=1}^{k} p_{ni_j}\Big)(1 - p_n)^{-k}.$$

Since for any set of numbers $a_1,\ldots,a_n$, we have

$$\left(\sum_{i=1}^{n} a_i\right)^k = k!\sum{}^{\ddagger} \prod_{j=1}^{k} a_{i_j} + \sum_{B_n^k} \prod_{j=1}^{k} a_{i_j}\ ,$$

so that

(6) $$\sum{}^{\ddagger} \prod_{j=1}^{k} p_{ni_j} = \left(\left(\sum_{i=1}^{n} p_{ni}\right)^k - \sum_{B_n^k} \prod_{j=1}^{k} p_{ni_j}\right)/k!\ .$$

In the last expression, the first term tends to $\lambda^k/k!$ by assumption. The second term is defined to be 0 if $k = 1$, because $B_n^1 = \phi$, and for $k \ge 2$, it is upper-bounded by

(7) $$\sum_{j=2}^{k} \binom{k}{j}\left(\sum_{i=1}^{n} p_{ni}^j \lambda_n^{k-j}\right)/k!\ .$$

By the lemma, this bound tends to 0 as $n \to \infty$, because the term $\sum_{i=1}^{n} p_{ni}^j$ tends to 0 for all $j \ge 2$.

Since $(1 - p_n)^{-k}$ tends to 1 as $n \to \infty$ for all fixed k, the theorem is proved by combining (2) and (5).

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