

SURVIVAL OPTIMIZATION FOR A DYNAMIC SYSTEM

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Abstract

We consider a continuous time dynamic system which is subject to random perturbations. The state variable obeys a system of time-invariant linear differential equations, and the random perturbations are taken to be white noise. Our aim is to maximize the time spent by the process in a continuation region, taking into account the quadratic control costs and the termination cost. The main results of this paper are obtained by making use of Wald's identity.

Résumé

On considère un système dynamique en temps continu qui est soumis à des perturbations aléatoires. Notre but est de maximiser le temps que le processus passe dans une certaine région de continuation, en prenant en considération les coûts de commande quadratiques et le coût de terminaison. Les résultats principaux de cet article sont obtenus en utilisant l'identité de Wald.

1. Introduction

Consider a continuous time dynamic system with state variable x (in R^n), control variable u (in R^m) and process equation

$$(1.1) \quad dx/dt = Ax + Bu + \epsilon,$$

where A, B are constant matrices and ϵ is Gaussian white noise of zero mean and covariance rate N . Suppose that the initial value of the process, which we shall also denote by x , belongs to a certain set C and that we wish to choose the control to minimize the expected value of the cost function

$$(1.2) \quad J(x) = \int_0^{\tau} [u'Qu/2 - \lambda] dt + K[x(\tau)],$$

where Q is a symmetric positive definite matrix, λ is a positive parameter, K is a general terminal loss function, and τ is the first moment at which the process $x(t)$ leaves the continuation set C , having started from x . The term $-\lambda$ in the integrand means that one is effectively trying to maximize survival time in C , account being taken of control costs $u'Qu/2$ and terminal cost K .

We assume state observable, and have then the dynamic programming equation

$$(1.3) \quad \min_u [u'Qu/2 - \lambda + (Ax+Bu)'F_x + \text{tr}(NF_{xx})/2] = 0 \quad (C)$$

where $F(x)$ is the minimal expected cost incurred from state value x , and F_x and F_{xx} are respectively the column vector of first derivatives and the matrix of second derivatives of F .

The (C) indicates that the equation holds in C , and we have correspondingly the boundary condition

$$(1.4) \quad F = K \quad (D)$$

where D , the stopping region, is the complement of C in R^n .

The minimizing value of u in (1.3),

$$(1.5) \quad u = -Q^{-1}B'F_x,$$

is the optimal value.

Suppose now that the proportionality relation

$$(1.6) \quad N = \alpha BQ^{-1}B'$$

holds between noise power and control power matrices, α being the scalar proportionality factor. Substituting for u from (1.5) into (1.3) and making the change of variable

$$(1.7) \quad \Phi(x) = \exp[-F(x)/\alpha]$$

we find that equation (1.3) transforms to the linear equation

$$(1.8) \quad \theta\Phi + (Ax)' \Phi_x + \text{tr}(N\Phi_{xx})/2 = 0 \quad (C),$$

where

$$(1.9) \quad \theta = \lambda/\alpha,$$

with terminal condition

$$(1.10) \quad \Phi = \exp[-K/\alpha] \quad (D).$$

From (1.8), (1.10) one would make the identification

$$(1.11) \quad \Phi(x) = E\{\exp[\theta\tau - K(x(\tau))/\alpha] \mid x(0) = x\}$$

where the expectation is over the time τ and coordinate $x(\tau)$ of first passage into D for the *uncontrolled* process

$$dx/dt = Ax + \varepsilon.$$

For this interpretation to be valid it is necessary that termination be certain and the expectation (1.11) be well-defined. For example, suppose we increase λ in the cost function (1.2), i.e., give u greater weight to survival. The effect will be to increase θ in expression (1.11); in most cases it will be true that expression (1.11) diverges at some critical value of θ . This critical value of θ corresponds just to the value of λ for which $E(\tau)$ becomes infinite in the original controlled problem. Note that one could well choose λ , and so θ , negative. The implication would then be that one was trying to encourage early departure from C , rather than delay it.

The passage from (1.3) to the linear equation (1.8) with interpretation (1.11) is a slight generalization of that previously proved in Whittle and Gait [8]; see also Whittle [7], page 289.

Note that the optimal control (1.5) is, in terms of Φ ,

$$(1.12) \quad u = \alpha Q^{-1} B' \Phi_x / \Phi.$$

The particular problem we shall now consider is that for which C is the interval

$$(1.13) \quad -d < x_1 < d$$

and $K = 0$. That is, one is trying to hold the component x_1 in the interval $(-d, d)$ as long as possible, due account being taken of the control costs thus incurred. For example, x_1 may represent the height of an aircraft with dynamics governed by (1.1): the value $x_1 = -d$ representing ground level, and the value $x_1 = +d$ representing a height at which radar detection is likely. The craft is thus trying to hold height in such a way as to survive these opposing hazards for as long as possible.

Seeing that there will be no overshoot into D , the boundary condition (1.4) may be written $F(\pm d, x_2, \dots, x_n) = 0$, so that equation (1.10) becomes

$$(1.14) \quad \Phi(\pm d, x_2, \dots, x_n) = 1.$$

2. The first-order case

Suppose that x is scalar, so that $x_1 = x$, and that the process equation (1.1) reduces simply to

$$(2.1) \quad dx/dt = Bu + \varepsilon.$$

That is, the intended force Bu and the random force ε affect height "viscously" (i.e., without inertial effects), and height remains constant when these forces are absent.

Equation (1.8) then reduces to

$$(2.2) \quad \theta\phi + N\phi_{xx}/2 = 0$$

and the following theorem is easily established.

THEOREM 1. Consider the process equation

$$dx/dt = Bu + \epsilon,$$

where x is a scalar. Then the function ϕ defined in (1.11) is given by

$$(2.3) \quad \phi(x) = \cos(kx)/\cos(kd) \quad (|x| < d)$$

where

$$(2.4) \quad k = |B|/N [2\lambda/Q]^{1/2}$$

and the corresponding control is

$$(2.5) \quad u = -(\text{sgn}B)[2\lambda/Q]^{1/2} \tan(kx).$$

PROOF. From (2.2) we have

$$(2.6) \quad \phi_{xx} + (2\theta/N)\phi = 0.$$

Equation (2.6) is solved by

$$\phi(x) = a \cos(kx) + b \sin(kx)$$

with

$$k = [2\theta/N]^{1/2},$$

and using the boundary conditions $\phi(\pm d) = 1$ (see (1.14)) we obtain

$$\phi(x) = \cos(kx)/\cos(kd).$$

Now by (1.9) and (1.6) we have

$$k = [2\theta/N]^{1/2} = [2\lambda/N\alpha]^{1/2} = [2\lambda B^2/N^2 Q]^{1/2},$$

i. e.,

$$k = |B|/N [2\lambda/Q]^{1/2}.$$

Finally, relation (1.12) implies that

$$\begin{aligned} u &= -\alpha Q^{-1} B \sin(kx)/\cos(kx) k \\ &= -NQ/B^2 Q^{-1} B \tan(kx) |B|/N [2\lambda/Q]^{1/2} \\ &= -[2\lambda/Q]^{1/2} (\text{sgn}B) \tan(kx). \quad \square \end{aligned}$$

We also have

THEOREM 2. *The critical value of θ is*

$$(2.7) \quad \theta_c = N\pi^2/8d^2.$$

PROOF. Expression (2.3) becomes infinite as λ (or θ) approaches the value such that $kd = \pi/2$; thus

$$(2.8) \quad \lambda_c = Q/8 [\pi N/Bd]^2$$

and using (1.6) we find that

$$\theta_c = \lambda_c/\alpha = B^2/NQ \lambda_c = N\pi^2/8d^2. \quad \square$$

Looking at expression (2.5) we see that, for λ equal to λ_c , the absolute value of the control u becomes infinite as x approaches the boundary values $\pm d$. That is, when the premium given to survival increases to this critical value, then one is willing to use infinite control to ensure it. One ensures survival to the point that $E(\tau) = \infty$; possibly, for $\lambda > \lambda_c$, to the point that termination is uncertain, so that survival is certain. It is difficult to analyse matters for λ greater than or equal to λ_c , however, because one is balancing infinite control costs against infinite survival time.

3. The second-order case

The minimal move towards realism would be to modify the treatment of the last section to include inertial effects, so that it is acceleration rather than

velocity which is proportional to force. This leads to a second-order formulation, in which we identify the components x_1 and x_2 as height and rate of change of height respectively. Seeing that it is variants of this case that we shall consider from now on, it would be a simplification to write (x_1, x_2) as (x, v) . That is, x now denotes height itself rather than the whole state vector, and v is the rate of change of height: vertical velocity.

The plant equation will be

$$(3.1) \quad \begin{cases} dx/dt = v \\ dv/dt = bu + \varepsilon, \end{cases}$$

say, if we suppose that control and process noise act as forces, and that these forces alone contribute to acceleration. The noise term ε is scalar and we shall again denote its power rate by N , although it is properly

$$\begin{bmatrix} 0 \\ \varepsilon \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 0 \\ 0 & N \end{bmatrix}$$

which correspond to the ε and N of the general formulation (1.1).

Equation (1.8) now reduces to

$$(3.2) \quad \theta\phi + v\phi_x + N\phi_{vv}/2 = 0,$$

where now ϕ_x is simply the partial derivative of ϕ with respect to x and ϕ_{vv} is the second partial derivative of ϕ with respect to v . This equation holds in $-d < x < d$.

In the deterministic case $N = 0$ equation (3.2) has solutions

$$\phi = \text{cst. exp}[-\theta x/v].$$

The solution that meets boundary conditions (1.14) at all boundary points at which there is a flux out of C is

$$(3.3) \quad \phi(x, v) = \begin{cases} \exp[+\theta(d-x)/v] & (v > 0) \\ \exp[-\theta(d+x)/v] & (v < 0). \end{cases}$$

At other boundary points there is a discontinuity in Φ .

If $\theta > 0$ then $\Phi(x,0) = +\infty$, reflecting the fact that τ can be made infinite in the deterministic case, and that the critical value of θ in this case is in fact $\theta_c = 0$. The control rule deduced from (1.12), (3.3),

$$(3.4) \quad u(x,v) = \begin{cases} -\lambda b/Qv^2 (d-x) & (v > 0) \\ \lambda b/Qv^2 (d+x) & (v < 0), \end{cases}$$

with its infinite switch in value as v changes sign is also an indication of the form the problem takes when $N = 0$ but $\lambda > 0$; not nonsensical, but ill-posed. That is, in the deterministic case we cannot use the cost criterion defined in (1.2) with λ positive.

The reason for using infinite control near $v = 0$ is, of course, that it can ensure the zero-velocity state, implying infinite survival without further control.

The problem of solving equation (3.2) in the general stochastic case $N > 0$ appears very difficult; surprisingly so, in view of the naturalness of the problem as formulated. The difficulties of evaluating statistics of first-passage times for a second-order process are well-known and the literature on the subject contains few explicit results: in the case of the integrated Wiener process, see McKean Jr. [6], Wong [9] and Goldman [4]; Buckholtz and Wasan [2] have obtained a first-passage time density for a two-dimensional Brownian motion. See also Matkowsky and Schuss [5] and Buckholtz and Wasan [1].

The fact that Φ is discontinuous in v for $N = 0$, but continuous for N positive (except on the boundary), shows that approximations to Φ for small N are not at all evident, as they are not close in any uniform sense to the deterministic solution.

For this reason we shall consider a formulation of the problem in the next section in which velocity is discretized.

4. The process of order one-and-a-half

Consider the uncontrolled process, and suppose process noise such that instead of velocity following a diffusion process, as in (3.1), it follows a discrete-state Markov process in which it adopts values v_j , and changes from v_j to v_k with probability intensity λ_{jk} . Here j, k range over some set E of values which enumerates the possible velocity values. Then equation (3.2) becomes

$$(4.1) \quad \theta\Phi(x, v_j) + v_j\Phi_x(x, v_j) + \sum_k \lambda_{jk}[\Phi(x, v_k) - \Phi(x, v_j)] = 0.$$

This equation holds of course for $|x| < d$ and j in E , and is subject to the boundary condition

$$(4.2) \quad \Phi(\pm d, v_j) = 1.$$

In fact, there are again discontinuities at the boundary, and condition (4.2) will be enforced on the solution of (4.1) only at those boundary points at which there is a flux out of C . That is, at $x = d$ for j such that $v_j > 0$, and at $x = -d$ for j such that $v_j < 0$.

We shall term this the "process of order one-and-a-half", intermediate as it is between the processes of order one and two. Of course, in formulating it we have departed from the original motivation for considering the uncontrolled process that from it could be derived the optimal control and the evaluation of F for the controlled process. The process of order one-and-a-half cannot be thus generated. However, the process is of interest in itself, and some version of relation (1.12) will generate a control which is presumably at least plausible.

We shall denote $\Phi(x, v_j)$ by $\phi_j(x)$, or simply by ϕ_j when there is no need to display the argument. If we try a solution

$$(4.3) \quad \phi_j(x) = \Psi_j \exp(\alpha x)$$

of (4.1) then we obtain the equation system

$$(4.4) \quad (\theta + v_j\alpha)\Psi_j + \sum_k \lambda_{jk}(\Psi_k - \Psi_j) = 0.$$

This will have enough solutions for the eigenvalue α and eigenvector $\Psi = (\Psi_j)$ that one can combine the corresponding solutions (4.3) to obtain an evaluation of $\phi_j(x)$ which satisfies the effective boundary conditions. However, even this solution can be made explicit in only a few cases, and in the remaining sections we shall approach this solution from another direction.

We shall henceforth confine ourselves to the process of order one-and-a-half, and shall also uniformly make the assumptions: (i) velocity distribution is symmetric, in that the model is unchanged by a reversal of all velocity values, $v_j \rightarrow -v_j$ for all j ; (ii) velocity changes are local, in that if $v_j \rightarrow v_k$ is a transition for which $\lambda_{jk} > 0$, then there is no v_j intermediate in value between v_j and v_k .

5. Use of Wald's identity

We confine ourselves to the model of order one-and-a-half from now on. More precisely we suppose that velocity is periodic: j belongs to $E = \{0, 1, \dots, 4r+3\}$ and

$$(5.1) \quad v_j = v \sin(2\pi j / (4r+4)).$$

That is, we assume that the airplane can take $2r+3$ different vertical velocities. The periodicity assumption will be used in Section 6 in order to obtain an explicit expression for Ψ_j . We also suppose that

$$(5.2) \quad \lambda_{jk} = \begin{cases} \lambda & \text{if } |j-k| = 1 \\ 0 & \text{otherwise,} \end{cases}$$

with $0 = 4r+4$, so that

$$\sum_{k=0}^{4r+3} \lambda_{jk} = 2\lambda \quad \text{for all } j.$$

Consider the moments of recurrence to $v = 0$, and let Δx and Δt be the change in x and the change in t over a recurrence epoch (so that Δt is just

the random recurrence time to $v = 0$). Define the moment-generating function

$$(5.3) \quad M(\alpha, \theta) = E[\exp(\alpha \Delta x + \theta \Delta t)].$$

Now, the values of $(\Delta x, \Delta t)$ for different recurrence epochs are identically and independently distributed random variables, so that one can appeal to Wald's identity in the form

$$(5.4) \quad E\{\exp[\alpha (x(t_r) - x) + \theta t_r] M(\alpha, \theta)^{-r}\} = 1.$$

Here the expectation is conditional on initial conditions at $t = 0$: $x(0) = x$ and $v(0) = 0$. The quantity t_s is the s^{th} recurrence to $v = 0$, and t_r is the first such recurrence time for which $|x| \geq d$. That is, t_r is the first value of t for which the combined event $\{v(t) = 0 \text{ and } |x(t)| \geq d\}$ occurs.

Relation (5.4) will be valid for all (α, θ) such that $M(\alpha, \theta)$ is defined and $|M(\alpha, \theta)| > \rho$, where the probability that $|x| < d$ at each of the first s recurrences to $v = 0$ is of order ρ^s for large s (see Cox and Miller [3]).

LEMMA 1. Let τ be the time when first $|x| \geq d$. Then

$$(5.5) \quad t_{r-1} < \tau \leq t_r.$$

PROOF. By definition, $\tau \leq t_r$ and $\tau \neq t_{r-1}$. If $\tau < t_{r-1}$, then there must be a velocity reversal in $|x| \geq d$ before time t_{r-1} . By assumption (ii) of the last section this will imply $v = 0$, $|x| \geq d$ at some time before t_{r-1} , against hypothesis. \square

Relation (5.5) implies that τ and t_r differ by a term of the order of the recurrence time of $v = 0$. If this recurrence time is not large relative to τ (i.e., if one may expect several velocity reversals before $|x|$ first equals d), then the approximation $\tau \approx t_r$ (where the symbol \approx means is approximately equal to) is effectively a no-overshoot approximation.

THEOREM 3. Let α_0 be a root of smallest modulus of

$$(5.6) \quad M(\alpha, \theta) = 1$$

for given θ . Then, in the no-overshoot approximation

$$(5.7) \quad \Phi(x, 0) \approx [\exp(\alpha_0 x) + \exp(-\alpha_0 x)] / [\exp(\alpha_0 d) + \exp(-\alpha_0 d)].$$

PROOF. If $\alpha = \alpha_0$ is a root of (5.6) then so is $\alpha = -\alpha_0$ by the assumption of symmetric velocity statistics. If we give α the values $\pm\alpha_0$ and make the no-overshoot approximations $\tau \approx t_r$ and $x(t_r) \approx d$ or $x(t_r) \approx -d$ according as to which boundary is crossed on exit from $|x| < d$ we see that relation (5.4) becomes

$$(5.8) \quad \exp(\alpha d)E_+ + \exp(-\alpha d)E_- \approx \exp(\alpha x) \quad (\alpha = \pm\alpha_0).$$

Here

$$E_+ = P(A_+)E[\exp(\theta\tau) | A_+],$$

where A_+ is the event that $|x| < d$ is left first through the boundary $x = +d$; correspondingly for E_- . Evaluating E_+ , E_- and so

$$\Phi(x, 0) = E_+ + E_-$$

from the two relations (5.8), we deduce relation (5.7). \square

Relation (5.6) will in general have an infinite number of roots for α ; we choose those of smallest modulus so as to minimize the effect of the neglect of overshoot in the exponent of (5.4). In the case $\theta < 0$ we know from the convexity of M as a function of α and θ that the roots of smallest modulus are the two real roots of relation (5.6) (Cox and Miller [3]). In the case $\theta > 0$ it turns out that α_0 is purely imaginary.

The root α_0 of (5.6) turns out to be a root of the equation system (4.4).

THEOREM 4. Define

$$(5.9) \quad M_j(\alpha, \theta) = E\{\exp[\alpha x(t) + \theta t] | x(0) = 0, v(0) = v_j\},$$

where t is the time when first $v(t) = 0$; thus $M_0(\alpha, \theta) = 1$. Write $M_j(\alpha, \theta)$

simply as M_j . Then α_0 and the M_j satisfy the equation system

$$(5.10) \quad (\theta + \alpha v_j) M_j + \sum_k \lambda_{jk} (M_k - M_j) = 0$$

with the normalization $M_0 = 1$.

PROOF. Note the distinction between M_0 and M , respectively defined as expectation for the first occurrence and first recurrence of $v = 0$.

Relation (5.10) is easily established for $j \neq 0$ from the definition (5.9) and the Kolmogorov backward equation. Correspondingly, we have

$$(5.11) \quad M(\alpha, \theta) = \sum_k \int_0^\infty \{ \exp[(\theta - \lambda_0) t] \lambda_{ok} M_k \} dt,$$

or

$$(5.12) \quad (\lambda_0 - \theta) M = \sum_k \lambda_{ok} M_k,$$

where

$$\lambda_0 = \sum_k \lambda_{ok}.$$

The integration in relation (5.11) is over the time t at which velocity changes from $v = 0$. But equation (5.12) with the condition $M = 1 = M_0$ is just the remaining relation of (5.10): that for $j = 0$. \square

It may seem that, since we return to equation (4.4), the problem has not been reduced. However, it has, in that we now seek a particular solution of (4.4), rather than all solutions.

We have now to try to extend the evaluation (5.7) to the case of non-zero initial velocity: an evaluation of $\Phi(x, v_j)$, to whatever degree of approximation is natural.

THEOREM 5. Suppose x distant enough from the boundaries $\pm d$ that there is likely to be a velocity reversal before either boundary is reached. That is, in terms of τ , suppose that $P[\tau > t]$ is closed to 1, where t is the time of first occurrence of $v(t) = 0$. Then

$$(5.13) \quad \Phi(x, v_j) \approx [\Psi_j(\alpha_0) \exp(\alpha_0 x) + \Psi_j(-\alpha_0) \exp(-\alpha_0 x)] / \Psi_0(\alpha_0) \Omega,$$

where

$$\Omega = [\exp(\alpha_0 d) + \exp(-\alpha_0 d)],$$

and where (α_0, Ψ_j) solves (4.4), $\alpha = \alpha_0$ being the solution of minimum modulus.

PROOF. Suppose that an expected reward $\exp(\alpha x + \theta t)$ is incurred if one starts at time t with $x(t) = x$, $v(t) = 0$. The expected reward starting from $x(0) = x$, $v(0) = v_j$ is then

$$\exp(\alpha x) E\{\exp[\alpha(x(t) - x) + \theta t] \mid x(0) = x, v(0) = v_j\} = \exp(\alpha x) M_j(\alpha, \theta),$$

where t is the time of first occurrence of $v(t) = 0$. So, using expression (5.7) we have

$$(5.14) \quad \begin{aligned} \Phi(x, v_j) &\approx E[\exp(\theta t) \Phi(x(t), 0) \mid x(0) = x, v(0) = v_j] \\ &\approx [M_j(\alpha_0, \theta) \exp(\alpha_0 x) + M_j(-\alpha_0, \theta) \exp(-\alpha_0 x)] / \Omega. \end{aligned}$$

Now, the normalization $M_0 = 1$ implies that $M_j = \Psi_j / \Psi_0$, and expression (5.13) thus follows from (5.14). \square

Note that we have assumed the event $v = 0$ to occur before the event $|x| \geq d$ in carrying out the expectation of (5.14); this implies the assumption made in the enunciation of the theorem.

Finally we have

THEOREM 6. *The critical value of θ is that for which*

$$\exp(\alpha_0 d) + \exp(-\alpha_0 d) = 0$$

$$\text{or, if } \alpha_0 = i\beta(\theta),$$

$$(5.15) \quad \beta(\theta) = \pi/2d.$$

PROOF. By relation (5.14), θ_c is such that

$$\exp(\alpha_0 d) + \exp(-\alpha_0 d) = 0.$$

Hence if $\alpha_0 = i\beta(\theta)$ we may write

$$2 \cos[\beta(\theta)d] = 0,$$

from which we deduce relation (5.15). \square

In the next section we shall complete the analysis by estimating α_0 and $\Psi_j(\alpha_0)$.

6. Determination of $\alpha_0(\theta)$ and $\Psi_j(\alpha_0)$

Theorem 4 tells us that α_0 and the M_j satisfy the equation system

$$(\theta + \alpha v_j)M_j + \sum_k \lambda_{jk}(M_k - M_j) = 0$$

with the normalization $M_0 = 1$. This normalization implies that $M_j = \Psi_j/\Psi_0$, and it follows that

$$(6.1) \quad (\theta + \alpha_0 v_j)\Psi_j + \sum_k \lambda_{jk}(\Psi_k - \Psi_j) = 0 \quad \text{for all } j \text{ in } E.$$

For our choice of λ_{jk} and v_j (see (5.2), (5.1)) the system (6.1) takes the form, writing $\alpha_0 = \alpha$,

$$(6.2) \quad (\theta + \alpha v \sin[2\pi j/(4r+4)])\Psi_j + \lambda(\Psi_{j+1} + \Psi_{j-1} - 2\Psi_j) = 0 \quad \text{for } j \text{ in } E.$$

To solve (6.2), write

$$(6.3) \quad \Psi_j = \sum_{k=0}^{m-1} c_k z^{jk},$$

where $z = \exp(2\pi i/m)$ and $m = 4r+4$. Next, notice that

$$v_j = v \sin(2\pi j/m) = v[(z^j - z^{-j})/2i].$$

So if we suppose that $\alpha = i\beta$, $\beta \in \mathbb{R}$, (6.2) becomes

$$[\theta + \beta v(z^j - z^{-j})/2] \sum_{k=0}^{m-1} c_k z^{jk} + \lambda \sum_{k=0}^{m-1} c_k z^{jk} (z^{k-2} + z^{-k}) = 0,$$

which implies that

$$(6.4) \quad \theta c_k + \beta v/2 (c_{k-1} - c_{k+1}) + \lambda (z^{k/2} - z^{-k/2})^2 c_k = 0$$

for $k = 0, \dots, m-1$.

The usual method employed to obtain an approximate solution of (6.4) is to assume that $c_k = 0$ for $|k| > p$, with $c_{-k} = c_{m-k}$. The approximate solutions obtained will be valid if θ is small. For example, if p equals 1 then (6.4) is simply

$$(1) \quad \theta c_{-1} - \beta v/2 c_0 + \lambda (z^{-\frac{1}{2}} - z^{\frac{1}{2}})^2 c_{-1} = 0$$

$$(2) \quad \theta c_0 + \beta v/2 (c_{-1} - c_{+1}) = 0$$

$$(3) \quad \theta c_{+1} + \beta v/2 c_0 + \lambda (z^{\frac{1}{2}} - z^{-\frac{1}{2}})^2 c_{+1} = 0.$$

From (1) and (3) we obtain

$$(6.5) \quad c_{-1} = -c_{+1} = \beta v/2 [\theta + \lambda (z^{\frac{1}{2}} - z^{-\frac{1}{2}})^2]^{-1} c_0$$

and substituting into (2) we find that ($c_0 \neq 0$)

$$(6.6) \quad \beta^2 = 2\theta/v^2 [2\lambda(1 - \cos(2\pi/m)) - \theta].$$

Furthermore if we set c_0 equal to 1, then we deduce from relation (6.5) that

$$(6.7) \quad \Psi_j = 1 + i\beta v [2\lambda(1 - \cos(2\pi/m)) - \theta]^{-1} \sin(2\pi j/m) \quad j = 0, \dots, m-1.$$

If p is equal to 2 we obtain

$$(6.8) \quad \beta^2 = (4\theta z_1 z_2)/v^2 [4\lambda(1 - \cos(4\pi/m)) - 3\theta],$$

where

$$z_j = 2\lambda(1 - \cos(2\pi j/m)) - \theta,$$

$$(6.9) \quad \Psi_j = 1 + 2\beta^2 v^2 / D_1 \cos(4\pi j/m) + i[4\beta v z_2 / D_1 \sin(2\pi j/m)],$$

where

$$D_1 = \beta^2 v^2 + 4z_1 z_2.$$

If $p = 3$, we have

$$(6.10) \quad \beta^2 v^2 = -Z + \{Z^2 + 8\theta z_1 z_2 z_3\}^{\frac{1}{2}},$$

where

$$Z = 2z_2 z_3 - \theta(z_1 + z_3),$$

$$(6.11) \quad \Psi_j = 1 + 2\beta^2 v^2 z_3 / D_2 \cos(4\pi j/m) \\ + i[\beta v(\beta^2 v^2 + 4z_2 z_3) \sin(2\pi j/m) + \beta^3 \sin(6\pi j/m)] / D_2,$$

where

$$D_2 = \beta^2 v^2 (z_1 + z_3) + 4z_1 z_2 z_3.$$

Looking at the three expressions that we have for β^2 , we see that θ must not be greater than $2\lambda(1 - \cos(2\pi/m))$ since we want β to be real. Actually λ depends on r , so that $2\lambda(1 - \cos(2\pi/m))$ tends to a positive limit as r increases.

PROPOSITION 1. Suppose, as in Theorem 5, that x is distant enough from the boundaries $\pm d$ that there is likely to be a velocity reversal before either boundary is reached. I.e., $P[\tau > t] \approx 1$, where t is the time of first occurrence of $v(t) = 0$. Then

$$(6.12) \quad \Phi(x, v_j) \approx [\Psi_j \exp(\alpha_0 x) + \Psi_{m-j} \exp(-\alpha_0 x)] / [\Psi_0 \Omega],$$

where

$$\Omega = \exp(\alpha_0 d) + \exp(-\alpha_0 d),$$

(α_0, Ψ_j) solves (4.4), α_0 being the solution of minimum modulus, and $\alpha_0 = \alpha = i\beta$ is given for small θ by either (6.6), (6.8) or (6.10) and the corresponding Ψ_j by formula (6.7), (6.9) or (6.11) respectively.

PROOF. This proposition is a direct consequence of Theorem 5 since, under the symmetry hypothesis, $\Psi_j(-\alpha_0) = \Psi_{-j}(\alpha_0) = \Psi_{m-j}(\alpha_0)$. \square

For example if we use formulae (6.6) and (6.7), that is if we take p equal to 1, we obtain

$$(6.13) \quad \Phi(x, v_j) = [\cos(\xi x) - (2\theta/z_1)^{\frac{1}{2}} \sin(2\pi j/m) \sin(\xi x)]/\cos(\xi d),$$

where

$$\xi = (2\theta z_1)^{\frac{1}{2}}/v.$$

Also, note that our approximation fails when $\beta d = \pi/2$ (as indicated in Theorem 6) and this gives us θ_c . In the case $p = 1$, again, we have

$$2\theta z_1/v^2 = \pi^2/4d^2$$

and we deduce that

$$(6.14) \quad \theta_c = [\lambda_r - (\lambda_r^2 - \pi^2 v^2/2d^2)^{\frac{1}{2}}]/2,$$

where

$$\lambda_r = 2\lambda(1 - \cos(2\pi/m)).$$

Finally, from (1.12) and (3.1) we deduce that the control in the second-order case that we considered is proportional to Φ_v/Φ . We define, correspondingly,

$$(6.15) \quad u(x, v_j) = \text{cst.} [(\Phi_j - \Phi_{j-1})/(v_j - v_{j-1}) + (\Phi_{j+1} - \Phi_j)/(v_{j+1} - v_j)]$$

for all j in E . The control function u should be equal to zero at $x = d$ for j such that v_j is positive; that is, for $j = 1, \dots, 2r+1$. Looking at formula

(6.15) we see that this is the case, except for $j = 1$ and $j = 2r+1$. We could

modify (6.15) to

$$u(x, v_j) = 2 \text{cst.} [(\Phi_{j+1} - \Phi_j)/(v_{j+1} - v_j)]$$

for $j = 1$ in order to have $u(d, v_1) = 0$, whenever r is greater than zero. However, we are interested in the case of large r and therefore this modification and the corresponding one for $j = 2r+1$ (and $j = 2r+5, 4r+3$ to obtain $u(-d, v_j) = 0$, $v_j < 0$) are unnecessary since $\Phi(d, 0)$ ought to be near 1, and so $u(d, v_1)$ near 0, when r is large.

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