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A NOTE ON THE WELL-POSED ULTRA-WEAK CAUCHY PROBLEM

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Résumé

Dans ce travail on considère un problème de Cauchy ultra-faible et bien posé pour l'équation différentielle abstraite $u'(t) - Au(t) = \theta$ dans un espace de Banach, les données initiales appartenant à un sous-ensemble linéaire dense. On associe aux solutions du problème mentionné un semi-groupe d'opérateurs $U(t)$ fortement continu pour $t > 0$. On établit des résultats de commutativité et on démontre que les solutions ultra-faibles dont les données initiales sont dans le domaine $D(A)$ sont des solutions usuelles (fortes).

Abstract

In this work we consider well-posed very weak Cauchy problems for the abstract differential equation $u'(t) - Au(t) = \theta$ in a Banach space, when the initial data belong to a linear and dense subset; an operator semi-group $U(t)$ which is strongly continuous on the interval $]0,\infty[$ is connected with solutions of the Cauchy problem. Then, putting additional conditions on the operator A, some commutativity properties between $U(t)$, $(\lambda-A)^{-1}$ and A are established. Finally, the ultraweak solutions are shown to be strong when the Cauchy data belong to the definition domain $D(A)$.

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Introduction

The Cauchy problem for abstract differential equations — that is, finding vector-valued functions $u(t)$ such that $u'(t) = Au(t)$ and $u(0) = u_o \in D(A)$ — was systematically investigated by E. Hille; in the classical monograph by Hille-Phillips [1] it is examined in Chapter XXIII as "The abstract Cauchy problem". There is a discussion of the uniqueness and relations to semi-group theory are established in the well-posed case. From a somewhat different point of view, the same problems are thoroughly investigated in Krein's monograph [3], and not only for the strong Cauchy problem but also for the so called "weakened" problem.

The concept of ultra-weak solutions and the corresponding Cauchy problem was discussed in Lions [4], [5] and for equations in Banach spaces in the paper [2] by Kato and Tanabe. Further research on the ultra-weak Cauchy problem can be found in our papers [6] and [7]. The investigation of ultra-weak solutions being still in a preliminary stage, many new unsolved problems pose themselves in a natural way. Some of these are worked out in this note: Whereas in [7] the weak Cauchy problem was "sufficiently" well-posed to generate C_o-operator semi-groups, in the present work the kind of continuous dependence on the initial data which is assumed is less restrictive than in [7]. Accordingly, the semi-group associated to the problem will not be in the class C_o, but will only be strongly continuous for $t > 0$. With this main difference, our results now are closely related to some of the statements in [7] and the proofs are, up to a point, quite similar.

1. In the Banach space X we consider a linear (unbounded) operator A, with dense domain $D(A)$ and with dual operator A^* acting from $D(A^*) \subset X^*$ into X^* (the dual space to X). Given any $b > 0$ (or even $b = +\infty$), we consider continuous functions $u(t)$: $[0,b[\to X$, verifying the integral identity

$$(1.1) \qquad \int_0^b \langle \frac{d}{dt} \varphi^*(t) + A^*\varphi^*(t), u(t)\rangle \, dt = 0, \qquad \forall\varphi^* \in K_{A^*}]0,b[$$

(this class of test-functions consists of $\varphi^*(t) \in C^1(]0,b[;X^*)$, $\varphi^* = \theta$ near 0 and b, $\varphi^*(t) \in D(A^*)$ for $t \in]0,b[$ and $A^*\varphi^* \in C(]0,b[;X^*)$.

(Such functions $u(t)$ are the ultra-weak solutions of the equation $u' - Au = \theta$ in the open interval $]0,b[$.) If an additional condition at $t = 0$ is given: $u(0) = u_o$, we have a ultra-weak Cauchy problem. Let now $M \subset X$ be a given linear manifold (subspace of X, not necessarily closed). We say that the ultra-weak Cauchy problem is solvable on $[0,b[$ for the set M if a solution $u(t)$ such that $u(0) = u_o \in M$ exists in the above sense (for all $u_o \in M$).

This problem is uniquely solvable if it is solvable and the solution is unique on $[0,b[$. It is correct (well-posed) or uniformly well-posed if the solution depends continuously on the initial data in M, pointwise in $[0,b[$ or uniformly there.

We note that it is possible to associate to any well-posed ultra-weak Cauchy problem on $[0,\infty[$ an operator semi-group $U(t)$ with usual properties on $[0,\infty[$ as is shown in the following

THEOREM. *Let us assume that* M *is dense in* X, *the weak Cauchy problem is uniquely solvable and correct in* M *on* $[0,\infty[$ *and the solution* $u(t)$ *is contained in* M *when* $t \geq 0$. *Then, there exists an operator semi-group* $U(t)$, $t \in [0,\infty[$, $U(0) = I$, *which is strongly continuous for* $t > 0$ *and such that for all* $x_o \in M$, $U(t)x_o = u(t;x_o)$, *the unique solution on* $[0,\infty[$ *with* $u(0) = x_o$.

PROOF. We follow references [3], [8] where a similar fact is established for more regular solutions, and also our paper [7] for weak solutions and $M = X$.

Let us consider therefore the family of operators $U_M(t)$, mapping M into itself, and defined for all $t \geq 0$ by: $U_M(t)x = u(t;x)$, the unique weak solution $u(t)$ such that $u(0) = x$. We see that $U_M(0)x = x$, $\forall x \in M$, hence $U_M(0) = I_M$ — the identity operator on M. Besides, the relation: $\lim_{t\to 0} U_M(t)x = x$, $\forall x \in M$ holds, as follows from the strong continuity of the solution $u(t)$ on $[0,\infty[$.

Next, it is easy to see[1] that $U_M(t)$ is, for any $t \geq 0$, a linear mapping of M into itself. Also, from the correctness hypothesis, it follows that $U_M(t)$

[1] Using uniqueness of the Cauchy problem.

is, for any $t \geq 0$, a continuous operator $M \to M$, hence a bounded one there (see for instance [8], Lemma 2.1, p. 8).

Let us establish now the semi-group property: $U_M(t_1+t_2) = U_M(t_1) \cdot U_M(t_2)$, for all t_1, t_2 in the interval $[0,\infty[$. Given any element $x_o \in M$ let us define the function $w_a(t) = U_M(t+a)x_o$ where t and a belong to $[0,\infty[$. Thus, we see that $w_a(0) = U_M(a)x_o$. Next, we shall see that $w_a(t)$ *is a weak solution* on $]0,\infty[$, (which is strongly continuous for $t \geq 0$). We have obviously, if $u(t) = U_M(t)x_o$, the relation

$$(1.2) \qquad \int_0^\infty \left\langle \frac{d}{dt}\varphi^*(t) + A^*\varphi^*(t), u(t) \right\rangle dt = 0, \quad \text{for all } \varphi^* \in K_{A^*}]0,\infty[\, .$$

We note also that $w_a(t) = u(t+a)$, and we shall see that the equality

$$(1.3) \qquad \int_0^\infty \left\langle \frac{d}{dt}\psi^*(t) + A^*\psi^*(t), u(t+a) \right\rangle dt = 0, \qquad \forall \psi^* \in K_{A^*}]0,\infty[$$

holds true.

Note first that

$$\int_0^\infty \left\langle \frac{d}{dt}\psi^*(t) + A^*\psi^*(t), u(t+a) \right\rangle dt = \int_a^\infty \left\langle \frac{d}{dz}\psi^*(z-a) + A^*\psi^*(z-a), u(z) \right\rangle dz$$

$$= \int_0^\infty \left\langle \frac{d}{ds}\varphi^*(s) + A^*\varphi^*(s), u(s) \right\rangle ds$$

where the new function φ^* defined by: $\varphi^*(s) = \psi^*(s-a)$, for $s \geq a$ and $= \theta$ for $0 \leq s \leq a$, belongs obviously to the set $K_{A^*}]0,\infty[$. Hence (1.3) follows. On the other hand, let us consider *the weak solution* $z(t) = U_M(t) \cdot U_M(a)x_o$; we see that $z(0) = U_M(a)x_o$ belongs to M, and that $z(0) = w_a(0)$.

As both functions $z(t)$, $w_a(t)$ are weak solutions on $]0,\infty[$, using the uniqueness hypothesis we may now infer the equality

$$w_a(t) = U_M(t+a)x_o = z(t) = U_M(t)U_M(a)x_o, \qquad \forall t \geq 0, \ \forall a \geq 0, \ \forall x_o \in M.$$

Next, using the density of M in X and the boundedness of $U_M(t)$ on M, we extend continuously, in a unique way, the operator $U_M(t)$ from the set M to the whole space X. Let us denote with $U(t)$, $0 \le t < \infty \to L(X)$ the operator-valued function thus obtained. It will be obviously a semi-group of operators (representation of the abelian semigroup of all non-negative real numbers into the Banach algebra $L(X)$). Furthermore, it is clear that $U(t)x = U_M(t)x$ for all x belonging to M. Finally, continuing as, for instance, in [8], pp. 10-12 (replacing the set $D(A)$ there by the dense set M here), we obtain that $U(t)$ is uniformly bounded in any interval $[\delta, 1/\delta]$ where $\delta > 0$, and accordingly $U(t)x$ is a continuous function on the open interval $]0,\infty[$ for all elements $x \in X$.

REMARK 1. If x is an arbitrary element of X, the continuous function on $]0,\infty[$, $U(t)x$ is a solution (on $]0,\infty[$) of the equation (1.1); this is because $U(t)x$ is the uniform limit on each interval of the form $[\delta, 1/\delta]$ of the ultra-weak solutions $U(t)x_n$ where x_n is any sequence in M converging to x.

Note also the following

COROLLARY 1. *Under the hypothesis of the Theorem, and assuming also that* $D(A) \subset M$ *and* $(\lambda_o - A)^{-1} \in L(X)$ *for some complex number* λ_o, *it follows that, for any* $T > 0$, *there is* $C_T > 0$, *with the property that*

$$\|U(t)(\lambda_o - A)^{-1}\| \le C_T \quad \textit{for} \quad 0 \le t \le T.$$

In fact, it is obvious that

$$\lim_{t\to 0} U(t)(\lambda_o - A)^{-1}x = (\lambda_o - A)^{-1}x, \qquad \forall x \in X.$$

This shows boundedness of the expression

$$\|U(t)(\lambda_o - A)^{-1}x\|, \qquad 0 \le t \le T, \quad \text{for any} \quad x \in X.$$

We state now the following

COROLLARY 2. *Under the same assumption as in Corollary* 1, *we have the commutativity property:*

$$U(t)(\lambda_o-A)^{-1} = (\lambda_o-A)^{-1}U(t), \quad \forall t \geq 0.$$

PROOF. Let be $x_o \in D(A) \subset M$; it follows that $U(t)x_o$ is an ultra-weak solution on $]0,\infty[$ with x_o as initial data. Then, we can use Lemma 1 in [9] and obtain that the function $v(t) = (\lambda_o-A)^{-1}U(t)x_o$ is also an ultra-weak solution on $]0,\infty[$ (the one such that $v(0) = (\lambda_o-A)^{-1}x_o$!).

Now, we see that $(\lambda_o-A)^{-1}x_o \in D(A^2) \subset M$, hence $U(t)(\lambda_o-A)^{-1}x_o$ represents the weak solution on $]0,\infty[$ with initial data $(\lambda_o-A)^{-1}x_o$.

Thus, from uniqueness, we obtain the equality:

$$(\lambda_o-A)^{-1}U(t)x_o = U(t)(\lambda_o-A)^{-1}x_o, \quad \forall x_o \in D(A).$$

Moreover, $D(A)$ is dense in X and both operators $(\lambda_o-A)^{-1}U(t)$ and $U(t)(\lambda_o-A)^{-1}$ belong to $L(X)$.

We are also ready for the

COROLLARY 3. *Under the assumptions of Corollary* 1, *the operator* $U(t)$ *maps* $D(A)$ *into itself for all* $t \geq 0$, *and the relation* $U(t)Ax = AU(t)x$, $\forall x \in D(A)$, $\forall t \geq 0$, *holds true.*

PROOF. If $x \in D(A)$ it follows that $x = (\lambda_o-A)^{-1}(\lambda_o-A)x$. Hence we obtain $U(t)x = U(t)(\lambda_o-A)^{-1}(\lambda_o-A)x$ which equals (by Corollary 2) the expression $(\lambda_o-A)^{-1}U(t)(\lambda_o-A)x$ — obviously an element of $D(A)$. Furthermore, we can write the equalities:

$$AU(t)x = AU(t)(\lambda_o-A)^{-1}(\lambda_o-A)x = A(\lambda_o-A)^{-1}U(t)(\lambda_o-A)x$$
$$= (A-\lambda_oI+\lambda_oI)(\lambda_o-A)^{-1}U(t)(\lambda_o-A)x = U(t)Ax$$

(here x is in $D(A)$).

Our last result is now the

COROLLARY 4. *Under the assumptions of Corollary* 1, *for any* $x \in D(A)$ *the ultra-weak solution* $U(t)x$ *is also a strong, regular solution on the open* *interval* $]0,\infty[$.

In fact, we only note that

$$U(t)x = U(t)(\lambda_0-A)^{-1}(\lambda_0-A)x = (\lambda_0-A)^{-1}U(t)(\lambda_0-A)x.$$

From the above Remark 1, we infer that $U(t)(\lambda_0-A)x$ is a solution (in $]0,\infty[$) of (1.1). Hence, using the main result in [9] we find that $U(t)x$ is a strong (regular) solution on $]0,\infty[$.

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