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SOLUTIONS TO SOME PROBLEMS IN THE THEORY OF COMBINATORIAL SPECIES

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Résumé

Le présent article résoud trois problèmes récents concernant la théorie combinatoire des espèces de structures. Le premier, posé par G. Labelle, demande d'étendre au cas multisorte, une formule d'intégration combinatoire due à A. Joyal [3]. Les autres, posés par S. Schanuel, concernent les deux équations fonctionnelles $S(2X) = 2S(X)$ et $S(mX,nX) = 0$ (où $m,n \in \mathbb{Z}$) à l'intérieur de la théorie des $\mathbb{K}$-espèces au sens de [13, 14].

Summary

The present paper solves three recent problems concerning the combinatorial theory of species of structures. The first, raised by G. Labelle, asks for an extension to the multisorted case, of a combinatorial integration formula due to A. Joyal [3]. The others, raised by S. Schanuel, concern the two functional equations $S(2X) = 2S(X)$ and $S(mX,nX) = 0$ (where $m,n \in \mathbb{Z}$) within the theory of $\mathbb{K}$-species in the sense of [13, 14].

1. Introduction

This paper gives answers to the following three recent problems concerning the theory of combinatorial species.

PROBLEM 1 (G. Labelle [9]). A. Joyal have shown [3] that every virtual species F has a virtual integral G given by

$$G = E_1 \cdot F - E_2 \cdot F' + E_3 \cdot F'' + \dots + (-1)^{k-1} E_{k+1} \cdot F^{(k)} + \dots$$

where E_k $(= X^k/k!)$ denotes the species of k-points sets. Extend that formula to d-species.

PROBLEM 2 (S. Schanuel [12]). Let S be a K-species. Suppose $S(2X)$ $2S(X)$, can we conclude the following result:

$$S(nX) = nS(X) \quad \text{for all} \quad n \in \mathbb{K}?$$

PROBLEM 3 (S. Schanuel [12]). Let $S = S(X,Y)$ be a given 2-species, such that $S(mX,nX) = 0$ for every $m,n \in \mathbb{Z}$. Must we necessarily have $S = 0$?

Let *Sets* be the category of all finite sets and maps and B be the category of all finite sets and bijections. Recall then ([1], [5], [4], [7] and [13]) that a *species* is a functor $S: \mathcal{B} \to$ Sets and a *morphism* τ from species S to species T is a natural transformation from functor S to functor T. When τ is an isomorphism, we often write $S \approx T$ (and use the notation $S = T$ when we work "up to an isomorphism"). The concept of d-species is defined in an analogous manner by replacing the category $\mathcal{B}$ by $\mathcal{B}^d = \mathcal{B} \times \mathcal{B} \times \dots \times \mathcal{B}$ (d-times). Of course, the concept of 1-species and species coincide. Various operations have been defined for d-species ([1], [8] and [13]); the main ones are the sum "+", product "$\cdot$", cartesian product "$\times$", substitution "$\circ$", and (partial differentiation "$\partial/\partial X$".

A d-species M is called molecule iff $M \neq 0$ (the empty species) and $M = S + T$ implies $S = 0$ or $T = 0$. It is well-known (see [5], [7], [8] and [13] for more detail) that the molecular d-species are of the form

$$X_1^{n_1} \cdot X_2^{n_2} \dots X_d^{n_d}/K$$

where X_i is the d-species of all singletons of sort i and K is a subgroup of $n_1! \times n_2! \times \dots \times n_d!$ (here $n!$ denotes the symmetric group on $\{1,2,\dots,n\}$).

Every d-species is, up to isomorphism, uniquely expressible as a (possibly infinite) sum of molecular d-species.

We shall make use of the following notations:

n	:	$\{1,2,\dots,n\}$
M_d^*	:	the set of all (isomorphism classes of) non-constant molecular d-species
M_d	:	the set of all (isomorphism classes of) molecular d-species
$\mathbb{N}[[M_d]]$	:	the half-ring of all (isomorphism classes of) d-species
$\mathbb{Z}[[M_d]]$	:	the ring of all (isomorphism classes of) virtual d-species = the ring completion of $\mathbb{N}[[M_d]]$ (see [2], [13])
$\binom{T_A}{i}$	=	$T_A(T_A-1)\dots(T_A-i+1)/1\cdot 2\dots i.$
$\mathbb{K}[\binom{T_A}{i}]_{A\in M}$	:	let $\mathbb{L}$ be a $\mathbb{Q}$-algebra containing $\mathbb{K}$, let $(T_A)_{A\in M}$ be a set of formal variables, then $\mathbb{K}[\binom{T_A}{i}]_{A\in M}$, is the sub-half ring of $\mathbb{L}[\binom{T_A}{i}]_{A\in M}$ generated by $\mathbb{K}$ and $\binom{T_A}{i}$ for all $A\in M$, $i\in\mathbb{N}$

We omit the subscript d in the case of 1-species.

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2. Solution to Problem 1

A. Joyal [3] proved that every virtual species F is integrable in $\mathbb{Z}[[M]]$, i.e., there exists a virtual species G such that $G' = F$. Joyal's formula for such a G is

$$G = JF = \sum_{n\geq 1} (-1)^{n-1} E_n \cdot F^{(n-1)} \in \mathbb{Z}[[M]]$$

where $E_n = X^n/n!$ and $F^{(n-1)}$ denotes the (n-1)th derivative of F. The following theorem generalizes this to the case of d-species.

THEOREM 1. *Let* $F_1, F_2, \ldots, F_d$ *be virtual* d-*species such that* $(\partial/\partial X_j)F_i = (\partial/\partial X_i)F_j$ *for* $1 \le i,\ j \le d$. *Then there exists a virtual* d-*species* G *such that* $(\partial/\partial X_i)G = F_i$ *for* $1 \le i \le d$.

PROOF. We prove it first for the case $d = 2$. In advanced calculus: if f_1 and f_2 are continuously differentiable functions of two variables (say in a given open disk centered at origin) such that $(\partial/\partial x_1)f_2 = (\partial/\partial x_2)f_1$, then there exists a function g (defined on the same disk) such that $(\partial/\partial x_1)g = f_1$ and $(\partial/\partial x_2)g = f_2$.

The most "symmetrical" expression for such a g is given by the formula:

$$g(x_1,x_2) = \int_0^{x_1} f_1\, dt_1 + \int_0^{x_2} f_2\, dt_2 - \int_0^{x_1}\int_0^{x_2} (\partial/\partial t_2)f_1\, dt_2\, dt_1.$$

Using Joyal's formula, we just translate

$$(*) \qquad \int_0^{x_i} f\, dt_i \quad \text{by} \quad J_iF = \sum_{n\ge 1} (-1)^{n-1} E_n^i \cdot (D_i^{n-1} F) \ \ldots$$

where $E_n^i = X_i^n/n!$, $D_i^k = (\partial^k/\partial X_i^k)$, and define a virtual 2-species G by

$$G = J_1F_1 + J_2F_2 - J_1J_2D_2F_1.$$

The fact that $D_iG = F_i$ comes from the hypothesis $D_iF_j = D_jF_i$ if $i \ne j$ and from the obvious equalities: $D_iD_j = D_jD_i$, $D_iJ_j = J_jD_i$ if $i \ne j$, and $D_iJ_i =$ identity. Note that $J_iD_i \ne$ identity.

In the general case, we use a similar argument and the following fact: Let $f_1, f_2, \ldots, f_d$ be continuously differentiable of d variable functions such that $(\partial/\partial x_i)f_j = (\partial/\partial x_j)f_i$ for $1 \le i,\ j \le d$, then the function

$$g = \sum_{1\le k\le d} (-1)^{k-1} h_k$$

where

$$h_k = \sum_{1\le i_1<i_2<\ldots<i_k\le d} \int_0^{x_{i_1}}\int_0^{x_{i_2}} \cdots \int_0^{x_{i_k}} (\partial/\partial x_{i_2}\partial x_{i_3}\ldots\partial x_{i_k})f_{i_1})\, dx_{i_k}\ldots dx_{i_2}\, dx_{i_1}$$

satisfies $(\partial/\partial x_i)g = f_i$ for $1 \le i \le d$. In this case, the final expression for the virtual species G takes the form

$$G = \sum_{1\le k\le d} (-1)^{k-1} \sum_{1\le i_1<i_2<\ldots<i_k\le d} J_{i_1} J_{i_2} \ldots J_{i_k} D_{i_2} D_{i_3} \ldots D_{i_k} F_{i_1} . \square$$

2. Solution to Problem 2

DEFINITION 1 ([13]). A half-ring $\mathbb{K}$ is called a *binomial half-ring* if

(a) there exists a $\mathbb{Q}$-algebra $\mathbb{L}$ containing $\mathbb{K}$, and

(b) for every $a \in \mathbb{K}$ and $i \in \mathbb{N}$, $\binom{a}{i} \in \mathbb{K}$.

For example $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Q}[i]$ and $\mathbb{N} + \mathbb{Q}\varepsilon$ $(\varepsilon^2 = 0)$ are all binomial half-rings, but $\mathbb{F}_p$ and $\mathbb{Z}[i]$ are not binomial half-rings.

REMARK. When $\mathbb{K}$ is a binomial half-ring, we say that S is a $\mathbb{K}$-species if $S \in \mathbb{K}[[M]]$. The concept of species (resp. virtual species) and $\mathbb{N}$-species (resp. $\mathbb{Z}$-species) coincide.

THEOREM 2 ([13]). *Let* S *be a* $\mathbb{K}$-*species and* $n_A \in \mathbb{K}$ *for all* $A \in M$. *Then*

$$S \circ (\sum_{A\in M^*} n_A A) = \sum_{B\in M} f_B((n_A)_{A\in M})B$$

where $f_B((T_A)_{A\in M}) \in \mathbb{N}[\binom{T_A}{i}_{A\in M}]$ *for all* $B \in M$.

From the above theorem, we can easily prove

THEOREM 3. *Let* S *be a* $\mathbb{K}$-*species, then the following two statements are equivalent:*

(a) $S(2X) = 2S(X)$;

(b) $S(nX) = nS(X)$ *for all* $n \in \mathbb{K}$.

PROOF. (b) $\Rightarrow$ (a) is trivial; we only prove (a) $\Rightarrow$ (b). From Theorem 2, we have

$$S(nX) - nS(X) = \sum_{A\in M} f_A(n) \cdot A$$

for some $f_A(T) \in \mathbb{N}[\binom{T}{i}]$ where $A \in M$.

If $T = 2^k$ for all $k \in \mathbb{N}$, then $f_A(T) = 0$. Therefore $f_A(T) = 0$ for all

$T \in \mathbb{K}$, i.e., $S(nX) = nS(X)$ for all $n \in \mathbb{K}$. □

REMARK. The constant 2 in condition (a) can be replaced by any constant except 0, 1, and -1.

3. Counterexample to Problem 3

The answer to Problem 3 is negative! This is another instance of the fundamental difference between d-species and d-variable power series.

Let the 2-species $S = S(X,Y)$ be defined by the following:

$$2(X^2/2!)Y + XY^2 - 2X(Y^2/2!) - X^2Y.$$

Thus $S(mX,nX) = 0$ for all $m,n \in \mathbb{N}$ since $(X^2/2!) \circ (nX) = \binom{n}{2}X^2 + \binom{n}{1}X^2/2!$, but $S(X,Y) \neq 0$. □

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