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ON THE ASYMPTOTIC BEHAVIOUR OF SUMS $\Sigma g(n)\{xn^{-\alpha}\}^m$

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Résumé

Dans ce papier, nous prouvons que l'ordre de grandeur de la différence entre la somme donnée en titre (où $g(t)$ est une fonction à valeurs réelles, positive, non décroissante, m est un entier positif et la sommation se fait sur tous les entiers positifs $n^{\alpha} \le x$) et l'intégrale correspondante est $O(g(x^{1/\alpha})x^{\lambda(\alpha)} \times (\log x)^{\delta_{\alpha,2}})$. De plus, nous montrons que ces différences, pour toutes valeurs de m, sont "pratiquement toutes égales" avec un terme d'erreur égal à $O(g(x^{1/\alpha})x^{\mu(\alpha)})$.

Abstract

In this paper we estimate the difference between the sum given in the title (where $g(t)$ is an arbitrary real-valued, positive, non-decreasing function, m is a positive integer and summation is extended over all positive integers $n^{\alpha} \le x$) and the corresponding integral, obtaining the bound $O(g(x^{1/\alpha})x^{\lambda(\alpha)}(\log x)^{\delta_{\alpha,2}})$. Furthermore, we show that these differences (for given g and varying m) are all "approximatively equal" with an error term of $O(g(x^{1/\alpha})x^{\mu(\alpha)})$.

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1. Introduction

This paper is a continuation of the authors article [5] where the special case $\alpha = 1$ was dealt with. Again $g(t)$ is supposed to be a real-valued, positive, non-decreasing function defined for $t \ge 1$, $m \ge 1$ and $\alpha > 1$ are arbitrary real numbers, x is a large variable and $\{.\}$ denotes the fractional part. We study the asymptotic behaviour of the sums

$$T_m(x) = \sum_{n^\alpha \le x} g(n)\{xn^{-\alpha}\}^m, \tag{1}$$

(for $x \to \infty$, m and α fixed), in particular the difference between $T_m(x)$ and the corresponding integral

$$I_m(x) = \int_1^{x^{1/\alpha}} g(t)\{xt^{-\alpha}\}^m\, dt = \frac{1}{\alpha}\, x^{1/\alpha} \int_1^x g(x^{1/\alpha}u^{-1/\alpha})\{u\}^m u^{-1-1/\alpha}\, du, \tag{2}$$

(some remarks on $I_m(x)$ itself will be formulated at the end of the paper). As in [5], it follows that

$$I_m(x) - T_m(x) = \int_0^1 S_m(x,t)\, dt + O(g(x^{1/\alpha})), \tag{3}$$

where

$$S_m(x,t) = \sum_{n \le x^{1/\alpha}-1} (g(n+t)\{x(n+t)^{-\alpha}\}^m - g(n)\{xn^{-\alpha}\}^m). \tag{4}$$

Our main results reads

THEOREM 1. *Under the suppositions given above, we have*

$$S_m(x,t) = O(g(x^{1/\alpha})x^{\lambda(\alpha)}(\log x)^{\delta_{\alpha,2}}), \tag{5}$$

uniformly in $0 \le t \le 1$, *and therefore, by* (3),

$$T_m(x,t) = I_m(x,t) + O(g(x^{1/\alpha})x^{\lambda(\alpha)}(\log x)^{\delta_{\alpha,2}}), \tag{6}$$

where

$$\lambda(\alpha) = \begin{cases} \frac{2}{3(\alpha+1)}, & \textit{for} \quad 1 < \alpha \le 2; \\ \frac{2}{3+2(\alpha+1)}, & \textit{for} \quad \alpha \ge 2, \end{cases}$$

and δ *is Kronecker's symbol.*

Secondly, we obtain a result which says that, roughly speaking, the differences in question are all "approximately equal" for different values of m.

THEOREM 2. *Under the assumptions of Theorem* 1, *we have*

(7) $$S_m(x,t) - S_1(x,t) = O(g(x^{1/\alpha})x^{\mu(\alpha)}),$$

uniformly for $0 \le t \le 1$, *and therefore, by* (3),

(8) $$T_m(x,t) - I_m(x,t) = T_1(x,t) - I_1(x,t) + O(g(x^{1/\alpha})x^{\mu(\alpha)}),$$

where $\mu(\alpha) = 4(9+5\alpha)^{-1}$ *for* $1 < \alpha < 3$ *and* $\mu(\alpha) < \lambda(\alpha)$ *for any* $\alpha > 1$.

2. Proof of Theorem 1

In view of Theorem 2, we have only to consider the case $m = 1$. Proceeding exactly as in [5], we can show that

(9) $$S_1(x,t) = O(g(x^{1/\alpha})(m_x+1)),$$

with

$$m_x = \max_{u \le x^{1/\alpha}} |\sigma_u(x)|,$$

$$\sigma_u(x) := \sum_{n \le u} (\{xn^{-\alpha}\} - \{x(n+t)^{-\alpha}\}) = H_0(x,u) - H_t(x,u)$$

$$H_\tau(x,u) := \sum_{n \le u} \{x(n+\tau)^{-\alpha}\} \qquad (\tau = 0 \text{ or } t, \text{ hence } 0 \le \tau \le 1).$$

To evaluate H_τ, we use a lattice point counting argument. Consider the planar domain (in the (p,q)-plane, say) B_τ: $x^{1/(\alpha+1)} < p \le u$, $0 < q \le F_\tau(p)$ where $F_\tau(p) := x(p+\tau)^{-\alpha}$; denote by $L(B_\tau)$ the number of lattice points in B_τ and by

$A(B_\tau)$ the area. Then we obtain

$$(10)\quad L(B_\tau) = \sum_{x^{1/(\alpha+1)}<p\le u} [F_\tau(p)] = \sum_{x^{1/(\alpha+1)}<p\le u} F_\tau(p) - H_\tau(x,u) + S_{1,\tau} + \tfrac{1}{2}[x^{1/(\alpha+1)}],$$

with

$$(11)\qquad S_{1,\tau} := \sum_{1\le p\le x^{1/(\alpha+1)}} \Psi(F_\tau(p)), \qquad \Psi(w) := \{w\} - \tfrac{1}{2}.$$

We evaluate $\sum F_\tau(p)$ by Euler's summation formula and infer from the second mean-value theorem that

$$\int_{x^{1/(\alpha+1)}}^{u} \Psi(p)F_\tau'(p)\,dp = O\left(\max_{x^{1/(\alpha+1)}<p\le u} |F_\tau'|\right) = O(1).$$

Thus we obtain from (10)

$$(12)\qquad L(B_\tau) = A(B_\tau) - \Psi(u)F_\tau(u) + \Psi(x^{1/(\alpha+1)})F_\tau(x^{1/(\alpha+1)}) - H_\tau(x,u) + S_{1,\tau} + \tfrac{1}{2}x^{1/(\alpha+1)} + O(1).$$

Counting the lattice points of B_τ "from the other direction" (and writing $G_\tau = F_\tau^{-1}$, i.e., $G_\tau(q) = (x/q)^{1/\alpha} - \tau$), we get

$$L(B_\tau) = ([u] - [x^{1/(\alpha+1)}])[F_\tau(u)] + \sum_{F(u)<q\le F(x^{1/(\alpha+1)})} [G(q)] - ([F(x^{1/(\alpha+1)})] - [F(u)])[x^{1/(\alpha+1)}].$$

Defining

$$(13)\qquad S_{2,\tau} := \sum_{F(u)<q\le F(x^{1/(\alpha+1)})} \Psi(G_\tau(q)),$$

eliminating the square-brackets by $[w] = w - \Psi(w) - \frac{1}{2}$ and applying again Euler's formula, we arrive at

$$(14)\qquad L(B_\tau) = A(B_\tau) - \tfrac{1}{2}u + \tfrac{1}{2}x^{1/(\alpha+1)} - \Psi(u)F_\tau(u) + \Psi(x^{1/(\alpha+1)})F_\tau(x^{1/(\alpha+1)}) + \int_{F_\tau(u)}^{F_\tau(x^{1/(\alpha+1)})} \Psi(q)G_\tau'(q)\,dq - S_{2,\tau} + O(1).$$

We now compare (12) and (14) to conclude that

$$(15)\qquad H_\tau(x,u) = \tfrac{1}{2}u + S_{1,\tau} + S_{2,\tau} - \int_{F_\tau(u)}^{F_\tau(x^{1/(\alpha+1)})} \Psi(q)G_\tau'(q)\,dq + O(1).$$

To deal with the remaining integral, we put

$$J_\tau(x,u) := \int_{\frac{1}{2}}^{F_\tau(u)} \Psi(q)G_\tau'(q)\,dq, \qquad C := \alpha^{-1}\int_{\frac{1}{2}}^{\infty} \Psi(q)q^{-1-1/\alpha}\,dq,$$

and infer from the second mean-value theorem that

$$\int_{F_\tau(x^{1/(\alpha+1)})}^{\infty} \Psi(q)G_\tau'(q)\,dq = O(1).$$

Hence

$$\int_{F_\tau'(u)}^{F_\tau(x^{1/(\alpha+1)})} \Psi(q)G_\tau'(q)\,dq = -Cx^{1/\alpha} - J_\tau(x,u) + O(1),$$

and (15) simplifies to

$$(16)\qquad H_\tau(x,u) = \tfrac{1}{2}u + Cx^{1/\alpha} + J_\tau(x,u) + S_{1,\tau} + S_{2,\tau} + O(1).$$

Since

$$J_t(x,u)-J_o(x,u) = -\alpha^{-1}x^{1/\alpha}\int_{F_o(u)}^{F_t(u)} \Psi(q)q^{-1-1/\alpha}\,dq \ll \alpha^{-1}x^{1/\alpha}\int_{F_o(u)}^{F_t(u)} q^{-1-1/\alpha}\,dq = t \ll 1,$$

we get (recalling the definition of $\sigma_u(x)$)

$$(17)\qquad \sigma_u(x) = H_t(x,u) - H_o(x,u) = S_{1,t} + S_{2,t} - S_{1,o} - S_{2,o} + O(1).$$

It remains to estimate $S_{1,\tau}$ and $S_{2,\tau}$. To this end we employ the method of exponent pairs (see H.E. Richert [8], [9], or for a more general survey, the paper [6] of E. Phillips and the recent textbook of A. Ivić [1]), in particular lemma 17 in [9] (noting that our additional constant τ in $S_{1,\tau}$ is easily covered by Richert's proof). We choose the exponent pair $(k,\ell) = (2/7,4/7)$ and obtain after a short computation that

$$S_{1,\tau} = O(x^{2/3(\alpha+1)}(\log x)^{\delta_{\alpha,2}}) \qquad \text{for} \quad 1 < \alpha \le 2$$

and

$$S_{1,\tau} = O(x^{2/(3+2(\alpha+1))}) \qquad \text{for } \alpha > 2.$$

Applying the same procedure to $S_{2,\tau}$ (with $x^{1/\alpha}$ instead of x and $q^{1/\alpha}$ instead of n^α, in the notation of Richert's lemma) we get (since $F_\tau(x^{1/(\alpha+1)}) = O(x^{1/(\alpha+1)})$ and $1/\alpha < 2$)

$$S_{2,\tau} = O(x^{2/3(\alpha+1)}).$$

Entering these estimates into (17) we arrive at

$$\sigma_u(x) = O(x^{\lambda(\alpha)}(\log x)^{\delta_{\alpha,2}}),$$

which, in view of (9), completes the proof of Theorem 1 for $m = 1$.

(REMARK. Properly speaking, the above lattice point counting argument (formula (10) and sequel) applies only to the case that $u > x^{1/(\alpha+1)}$. But for $u \le x^{1/(\alpha+1)}$, it is easy to see that $H_\tau(x,u) - u/2$ can be estimated in the same way as $S_{1,\tau}$ above; this yields the required estimate for $\sigma_u(x)$ for this case also.)

3. Proof of Theorem 2

Writing $f_m(y) = \{y\}^m - \{y\}$ $(m > 1)$ and proceeding as in [5] (formulae (15) f) we can show that

$$|S_1(x,t) - S_m(x,t)| \le g(x^{1/\alpha})(1 + \max_{u \le x^{1/\alpha}} |S_u(x)|), \tag{18}$$

where

$$S_u(x) = \sum_{n \le u} (f_m(xn^{-\alpha}) - f_m(x(n+t)^{-\alpha})) = \sum_{|h|=1}^{\infty} c_h W_h(x,u), \tag{19}$$

$$W_h(x,u) := \sum_{n \le u} (e(hxn^{-\alpha}) - e(hx(n+t)^{-\alpha})),$$

$e(z) := e^{2\pi i z}$ and the Fourier coefficients of $f_m(y)$ satisfy $c_h = O(|h|^{-2})$, $(h \in \mathbb{Z},\ h \neq 0)$. We put $U_h = |hx|^{1/(\alpha+1)}$ and consider the sum

$$W_h^{(1)} := \sum_{U_h<n\le u} (e(hxn^{-\alpha}) - e(hx(n+t)^{-\alpha}))$$

(if nonempty). Observing that

$$e(hxn^{-\alpha}) - e(hx(n+t)^{-\alpha}) = 2\pi i\alpha hx \int_0^t (n+w)^{-\alpha-1}\, e(hx(n+w)^{-\alpha})\, dw,$$

we conclude that

$$W_h^{(1)} = 2\pi i\alpha hx \int_0^t S(w)\, dw, \tag{20}$$

$$S(w) := \sum_{U_h<n\le u} (n+w)^{-\alpha-1}\, e(hx(n+w)^{-\alpha}).$$

We now split up the interval of summation by a sequence (n_r), defined by $n_0 = U_h$, $n_r = \min(2n_{r-1},u)$. By a classical lemma of Van der Corput (see [11], p. 90), we get for each subinterval

$$\sum_{n_{r-1}<n\le n_r} (n+w)^{-\alpha-1}\, e(hx(n+w)^{-\alpha}) = O(n_r^{-\alpha}\,|hxn_r^{-\alpha-2}|^{\frac12}) + O(n_r^{-\alpha-1}\,|hxn_r^{-\alpha-2}|^{-\frac12})$$

$$= O(n_r^{-1-3\alpha/2}\,|hx|^{\frac12}) + O(n_r^{-\alpha/2}\,|hx|^{-\frac12}).$$

Summation over r yields (since both exponents of n_r are negative)

$$S(w) = O(|hx|^{-(1+2\alpha)/2(\alpha+1)}),$$

thus, by (20),

$$W_h^{(1)} = O(|hx|^{\frac12(\alpha+1)}). \tag{21}$$

We now put $U_h' = \min(U_h,u)$, then it remains to estimate

$$\sum_{|h|=1}^{\infty} c_h \left|\sum_{n\le U_h'} (e(hxn^{-\alpha}) - e(hx(n+t)^{-\alpha}))\right| \ll \sum_{|h|=1}^{\infty} |h|^{-2}(|E_t| + |E_0|)$$

where

$$E_\tau := \sum_{n\le U_h'} e(hx(n+\tau)^{-\alpha}) \qquad (\tau = 0 \text{ or } t).$$

We define $P_h = |hx|^{\beta}$ (β at our disposition) and split up the interval $1 \le n \le U_h'$ by a sequence (n_r), $n_r = \min(2^r,U_h')$. According to Richert [9], p. 76 (see also [8], lemma 6), we have for any exponent pair (k,ℓ)

$$(22)\qquad \sum_{n_{r-1}<n\le n_r} e(hx(n+\tau)^{-\alpha}) \ll |hx|^k n_r^{\ell-(\alpha+1)k} + |hx|^{-\frac{1}{2}} n_r^{1+\alpha/2}.$$

For $n_r > P_h$, we choose the exponent pair $(k,\ell) = (2/7,4/7)$, then in (22) the first exponent of n_r is negative, and we obtain

$$(23)\qquad \sum_{n_r>P_h}\ \sum_{n_{r-1}<n\le n_r} e(hx(n+\tau)^{-\alpha}) \ll |hx|^{2/7+\beta(4/7-2(\alpha+1)/7)} + |hx|^{\frac{1}{2}(\alpha+1)}.$$

For $n_r \le P_h$, however, we choose $(k,\ell) = (1/6,2/3)$ (in the case $\alpha < 3$), then both exponents of n_r in (22) are positive, and we get (with $P_h' = \min(u,P_h)$)

$$(24)\qquad \sum_{n_r\le P_h'}\ \sum_{n_{r-1}<n\le n_r} e(hx(n+\tau)^{-\alpha}) \ll |hx|^{1/6+\beta(2/3-(\alpha+1)/6)} + |hx|^{\frac{1}{2}(\alpha+1)}.$$

We now choose (for $\alpha < 3$), $\beta = 5(4+5(\alpha+1))^{-1}$, and infer from (23) and (24), by a short computation, that

$$E_\tau = O(|hx|^{4/(9+5\alpha)}).$$

Together with (21, (19) and (18), this completes the proof of Theorem 2 for the case $\alpha < 3$. In order to show that, for any $\alpha > 1$, we can always obtain an exponent $\mu(\alpha) < \lambda(\alpha)$ in Theorem 2 ($\lambda(\alpha)$ being defined in Theorem 1), we let the estimate (23) unchanged and replace the exponent pair $(1/6,2/3)$ (used to derive (24)) by some $(k,\ell) \neq (0,1)$ with k sufficiently small for given α and $(1-\ell)/k > 3/2$. (By theorem 3 in [6], there exists an infinite sequence of exponent pairs $(k,\ell) \neq (0,1)$ such that $k \to 0$ and $(1-\ell)/k \to \infty$.) For this (k,ℓ), $\ell - (\alpha+1)k > 0$, and we obtain an exponent $k + \beta(\ell - (\alpha+1)k)$ in (24) which, for $\beta = \beta_o := 2(3+2(\alpha+1))^{-1}$ is less than β_o (as a short computation shows). Therefore, choosing β slightly greater than β_o, we obtain both in (24) and in (23) exponents less than β_o, thereby completing the proof of Theorem 2 for the case $\alpha \ge 3$ also.

REMARK. As in most applications of the method of exponent pairs, our results are capable of slight improvements by a more elaborate choice of the exponent pairs employed (depending on the value α). To give an example, we consider the special case $\alpha = 2$: Again by lemma 17 from [9], we get

$$\max(|S_{1,\tau}|,\ |S_{2,\tau}|) = O(x^{k/(k+1)} \log x)$$

for any exponent pair (k,ℓ) with $\ell = 2k$. Appealing to the work of R.A. Rankin [7], we may conclude that, for any $\varepsilon' > 0$, $\left(\frac{\theta+1}{2(\theta+2)} + \varepsilon',\ \frac{\theta+1}{\theta+2} + 2\varepsilon'\right)$ is such an exponent pair, $\theta = 0{,}3290213568\ldots$ being Ranking's constant (cf. [7], formula (3)). Therefore, for any $\varepsilon > 0$,

$$\max(|S_{1,\tau}|,\ |S_{2,\tau}|) = O(x^{\gamma+\varepsilon}),$$

where $\gamma = (\theta+1)/(3\theta+5) = 0{,}22198215\ldots$. Since $\mu(2) < \gamma$, the estimates of Theorem 1 may thus be refined, for $\alpha = 2$, to

$$S_m(x,t) = O(g(x^{\frac{1}{2}})x^{\gamma+\varepsilon}) \tag{5'}$$

and

$$T_m(x) - I_m(x) = O(g(x^{\frac{1}{2}})x^{\gamma+\varepsilon}). \tag{6'}$$

4. <u>Remarks on</u> $I_m(x)$

As enunciated in the introduction, we have to justify that the integral $I_m(x)$ actually dominates the error term we have estimated by our theorems, at least for a reasonably large class of functions $g(t)$. In fact, if $g(t)$ is regularly varying (see E. Seneta [11] for an enlightening study of this concept), i.e., that

$$\lim_{t\to\infty} \frac{g(ut)}{g(t)} = h(u)$$

exists for every $u > 0$ (and equals $h(u) = u^{\rho}$ for some $\rho > 0$, according to [10], p. 9), we can show by the same argument as in [5], that

$$I_m(x) \sim A_{\alpha,\rho}^{(m)}\ x^{1/\alpha}\ g(x^{1/\alpha})$$

where

$$A_{\alpha,\rho}^{(m)} := \frac{1}{\alpha}\int_1^{\infty} \{u\}^m\ u^{-1-(1+\rho)/\alpha}\ du.$$

In the special case $g(t) = t^{\rho}$, this asymptotic relation can be refined to

$$I_m(x) = A_{\alpha,\rho}^{(m)} x^{(1+\rho)/\alpha} + O(1),$$

which, together with our theorems, gives rather precise asymptotic formulas for the corresponding sums $S_m(x)$. Furthermore, like in [5], we can give explicit evaluations for $A_{\alpha,\rho}^{(m)}$ in terms of the Riemann zeta-function and Euler's constant γ, provided that m is and integer and $m \le (1+\rho)/\alpha$: Writing $r = (1+\rho)/\alpha$, for short we have

$$A_{\alpha,\rho}^{(m)} = \frac{1}{r-m} - \sum_{j=1}^{m} \frac{m!\zeta(r-m+j)}{r(r-1)\dots(r-m+j)j!} \qquad \text{for } m < r,$$

$$A_{\alpha,\rho}^{(m)} = 1 - \gamma - \sum_{2\le j\le m} \frac{1}{j}(\zeta(j)-1) \qquad \text{for } m = r.$$

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