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IMPROVING REGULARITY OF WEAK SOLUTIONS OF ABSTRACT DIFFERENTIAL EQUATIONS[1]

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Résumé

Dans ce travail, on indique comment obtenir des solutions régulières (fortes) de l'équation différentielle opérationnelle $v' - Av = \theta$ dans l'intervalle $(a,b) \subset \mathbb{R}$ (A étant un opérateur non borné dans l'espace de Banach X), en partant des solutions faibles continues $u(t)$ de la même équation, moyennant la formule $v(t) = (\lambda_o - A)^{-1} u(t)$, où l'opérateur $(\lambda_o - A)^{-1} \in L(x)$ existe pour un $\lambda_o \in \mathbb{C}$.

Introduction

In this note we continue previous investigations on the weak solutions of differential equations with unbounded operators in Banach spaces (see [3], [4], [5], [6]). The result which will be explained here consists in the following: if $u(t)$ is a continuous weak solution of an equation of the form $u'(t) - A\,u(t) = \theta$ on the interval $(a,b) \subset \mathbb{R}$, A being a linear densely defined operator in the Banach space X, and if $v(t) = R(\lambda_o, A)u(t)$ where λ_o is a regular point of the operator A, then $v(t)$ is a regular (strong) solution of the same equation: $v'(t) - A\,v(t) = \theta$ on (a,b). Precise statements and the proof are given below.

1. Let X be a Banach space and A be a linear operator with dense domain $D(A) \subset X$ and with range in X too. Consider the dual (or adjoint)

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operator A^* acting on $D(A^*) \subset X^*$ and with range in X^*- the dual space to X. If (a,b) is an interval of the real line, we define $K_{A^*}(a,b)$ to be the class of all functions $\phi^*(t) \in C^1[(a,b);X^*]$, which are θ near a and b, such that $\phi^*(t) \in D(A^*)$ for all $t \in (a,b)$ and $A^*\phi^* \in C^0[(a,b);X^*]$; the elements of $K_{A^*}(a,b)$ are vector-valued test-functions.

A strongly continuous function $u(t)$, $(a,b) \to X$ is called weak solution of the differential equation: $u'(t) - A\,u(t) = \theta$, if the integral identity

$$\int_a^b \langle \frac{d}{dt}\phi^*(t) + A^*\phi^*(t), u(t)\rangle\, dt = 0 \tag{1.1}$$

is satisfied, for all $\phi^* \in K_{A^*}(a,b)$ (here $\langle\ ,\ \rangle$ means duality between X and X^*). Our aim is to establish the following:

THEOREM. *Let* $u(t)$ *be a continuous weak solution on the interval* $(a,b) \subset \mathbb{R}$ *of the differential equation* $u'(t) - A\,u(t) = \theta$, *and assume that the operator* A *has a least one regular point* λ_o . *Then the function* $v(t) = (\lambda_o - A)^{-1}u(t)$ *belongs to* $C^1[(a,b);X]$ *and verifies the differential equation:* $v'(t) - A\,v(t) = \theta$ *on* (a,b) *in the strong sense.*

2. The proof

LEMMA 1. *The above defined function* $v(t)$ *is again a continuous weak solution on* (a,b) *of the same differential equation:* $v'(t) - A\,v(t) = \theta$.

Consider in fact any test-function $\phi^*(t) \in K_{A^*}(a,b)$. It readily follows that

$$\int_a^b \langle \frac{d}{dt}\phi^* + A^*\phi^*, v\rangle\, dt = \int_a^b \langle \frac{d}{dt}\phi^* + A^*\phi^*, R(\lambda_o;A)u\rangle\, dt. \tag{2.1}$$

Use now a well-known result (see for instance [1], p. 14, Lemma 4.6) to derive that $\lambda_o \in \rho(A^*)$ (resolvent set of A^*) and the equality $R(\lambda_o;A^*) = (R(\lambda_o;A))^*$. Thus we get the relation

$$\int_a^b \langle \frac{d}{dt}\phi^* + A^*\phi^*, v\rangle\, dt = \int_a^b \langle R(\lambda_o;A^*)(\frac{d}{dt}\phi^* + A^*\phi^*), u\rangle\, dt$$

$$(2.2) \qquad = \int_a^b \langle \frac{d}{dt} R(\lambda_o;A^*)\phi^* + A^*R(\lambda_o;A^*)\phi^*, u\rangle\, dt$$

$$= \int_a^b \langle \frac{d}{dt}\psi^*(t) + A^*\psi^*(t), u(t)\rangle\, dt$$

where $\psi^*(t) = R(\lambda_o;A^*)\phi^*(t)$. It is quite obvious that the new function $\psi^*(t)$ belongs also to our test-functions space $K_{A^*}(a,b)$ (for instance, one sees that $A^*\psi^* = (A^*-\lambda_o I+\lambda_o I)R(\lambda_o;A^*)\phi^* = -\phi^*(t) + \lambda_o R(\lambda_o;A^*)\phi^*(t)$ which belongs to $C[(a,b);X^*]$). Therefore, the last integral in (2.2) vanishes, as desired. Next, we prove the simple

LEMMA 2. *The function* $v(t)$ *belongs to* $D(A)$ *for all* $t \in (a,b)$ *and* $A\, v(t)$ *is a continuous function from* (a,b) *into* X.

In fact, we have:

$$A\, v(t) = (A-\lambda_o I+\lambda_o I)(\lambda_o - A)^{-1}u(t) = -u(t) + \lambda_o R(\lambda_o;A)u(t)$$

which is strongly continuous on (a,b).

We are now ready for the final part of the proof of the Theorem. Using Lemma 1 and the equality: $\langle\phi^*(t), A\, v(t)\rangle = \langle A^*\phi^*(t), v(t)\rangle$ we obtain the relation

$$(2.3) \qquad \int_a^b \langle \frac{d}{dt}\phi^*, v\rangle\, dt = -\int_a^b \langle\phi^*(t), A\, v(t)\rangle\, dt, \qquad \forall\phi^* \in K_{A^*}(a,b).$$

We shall use this equality for a special sequence of functions in $K_{A^*}(a,b)$; precisely, let us take a sequence of (scalar-valued) functions $\{\alpha_m(t)\}_1^\infty \subset C_o^1(\mathbb{R})$, such that $\alpha_m(t) = 0$ for $|t| \geq 1/m$, $\alpha_m(t) \geq 0$, $\int \alpha_m(\sigma)\, d\sigma = 1$. Next, let us fix any point t_o in (a,b) and then consider the X^*-valued function $\phi_m^*(\tau) = \alpha_m(t_o-\tau)x^*$ where x^* is an arbitrary element in $D(A^*)$. It is obvious that for m sufficiently large (depending on t_o), the above function ϕ_m^* is a test-function-it belongs to $K_{A^*}(a,b)$. At this stage we can infer from the above formula (2.3) the new identity

$$(2.4)\qquad \int_a^b \alpha_m'(t_o-\tau)\ \langle x^*,v(\tau)\rangle\ d\tau = \int_a^b \alpha_m(t_o-\tau)\ \langle x^*,A\ v(\tau)\rangle\ d\tau$$

for all $x^* \in D(A^*)$ and $m \geq m_o(t_o)$ and therefore also the equality

$$(2.5)\qquad \langle x^*,\int_a^b \alpha_m'(t_o-\tau)v(\tau)\ d\tau\rangle = \langle x^*,\int_a^b \alpha_m(t_o-\tau)(Av)(\tau)\ d\tau\rangle$$

again for all $x^* \in D(A^*)$ and $m \geq m_o$. Use now the fact that A has a regular point; it follows that it is a closed linear operator with dense domain, and accordingly, the domain of its adjoint, $D(A^*)$ is a total set in X^* (see [2] for definition and result on total sets). We may derive therefore the equality in X

$$(2.6)\qquad \int_a^b \alpha_m'(t_o-\tau)v(\tau)\ d\tau = \int_a^b \alpha_m(t_o-\tau)(Av)(\tau)\ d\tau, \qquad m \geq m_o\ .$$

Consider now the convolution:

$$(v*\alpha_m)(t) = \int_a^b \alpha_m(t-\tau)v(\tau)\ d\tau$$

which has a strong derivative

$$(v*\alpha_m)'(t) = \int_a^b \alpha_m'(t-\tau)v(\tau)\ d\tau.$$

Accordingly, the relation (2.6) can be written as

$$(2.7)\qquad (v*\alpha_m)'(t) = ((Av)*\alpha_m)(t), \qquad \forall t \in (a,b) \text{ and } m \geq m_o(t).$$

At this point we take again a fixed t_o in (a,b), and consider $\delta > 0$ in such a way that $(t_o-\delta,t_o+\delta) \subset (a,b)$. It is now obvious that the above (2.7) will hold for all t in $(t_o-\delta,t_o+\delta)$ as soon as m is greater than some m_o depending on t_o and $\delta > 0$ only. (For in this case all functions $\alpha_m(t-\tau)$ belong to $C_o^1(a,b)$ as necessary.) We shall now integrate (2.7) between a fixed $\bar{t} \in (t_o-\delta,t_o+\delta)$ and an arbitrary t chosen in the same interval and shall derive

$$(2.8)\qquad (v*\alpha_m)(t) = (v*\alpha_m)(\bar{t}) + \int_{\bar{t}}^t ((Av)*\alpha_m)(\sigma)\ d\sigma.$$

When $m \to \infty$, using continuity and uniform continuity of v and Av, as well as the

δ-function properties of the sequence $\{\alpha_m\}_1^\infty$ we deduce the equality

$$(2.9) \qquad v(t) = v(\bar{t}) + \int_{\bar{t}}^{t} (Av)(\sigma)\, d\sigma, \quad \text{for all } t \text{ in } (t_o-\delta, t_o+\delta).$$

Using strong continuity of the function $(Av)(\sigma)$ one may derive from (2.9) the strong derivability of $v(t)$ in $(t_o-\delta,t_o+\delta)$-hence in all (a,b)- , and the equality

$$(2.10) \qquad v'(t) = A\, v(t)$$

in this same interval. Finally from continuity of Av we deduce that $v(t) \in C^1[(a,b);X]$ and the theorem is proved completely.

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