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VECTORIAL OPTIMIZATION PROGRAMS WITH MULTIFUNCTIONS AND DUALITY

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Résumé

Nous définissons, dans le présent travail, les notions de sous-différentielle et de conjuguée d'une multifonction définie sur un espace vectoriel réel prenant ses valeurs dans un espace vectoriel ordonné réel. Nous étudions (section 2) les premières propriétés de ces concepts, dans le but d'étendre une dualité pour les problèmes d'optimisation vectorielle au cas où les fonctions à traiter sont des multifonctions (section 3). Nous généralisons ainsi certains résultats de base de Tanino et Sawaragi [8], en particulier pour le cas où les fonctions à traiter prennent leurs valeurs dans un espace localement convexe quasi-complet ordonné par un cône fermé et nucléaire (supernormal).

1. Preliminaries

It is known that, in general, the set of solutions for an optimization problem with point-to-point objective function is characterized by the subdifferential of a function and that the conjugate functions play important roles in the duality theory for convex programs ([1], [6]).

Tanino and Sawaragi ([7], [8]) construct and develop a duality theory for multi-objective optimization problems and Kawasaki ([2], [3]) extends their results, using the concepts of "conjugate" and "subdifferential" for multifunctions from a linear space to R^n_∞ (the union of the n-dimensional Euclidian space R^n and two n-dimensional points consisting of $+\infty$ and $-\infty$, respectively).

In this paper, we define the subdifferential and the conjugate of a multifunction defined on a real linear space and taking values in a real ordered linear space and we investigate their immediate properties (Section 2), with the aim of extending a duality of vectorial optimization problems in which the objective functions are multifunctions, using the concept of vectorial conjugate, Lagrangian and its saddle points (Section 3). Thus, we show some ways and new dual concepts for the study of the vectorial optimization programs with point-to-set objective maps, generalizing basic results of [8], especially for the case when the objective multifunctions take values in quasi-complete locally convex spaces ordered by a closed and nuclear (supernormal) cones.

Let $(Z,\leq)$ be a real ordered vector space, i.e., a real linear space endowed with an order (or preorder) "$\leq$" possibly induced by a cone. We add to Z a smallest element denoted by $-\infty$ and a largest element denoted by $+\infty$, respectively, we consider $\overline{Z} = Z \cup \{-\infty,+\infty\}$ and we extend the addition and the scalar multiplication of Z to $\overline{Z}$ using the following calculation conventions:

$$(-\infty) + x = x + (-\infty) = -\infty, \quad (+\infty) + x = x + (+\infty) = +\infty \quad \text{for every } x \in Z;$$

$$(-\infty) + (-\infty) = -\infty, \quad (+\infty) + (+\infty) = +\infty;$$

$$\lambda \cdot (\pm\infty) = \pm\infty \quad \text{for } \lambda > 0 \quad \text{and} \quad \lambda \cdot (\pm\infty) = \pm\infty \quad \text{for } \lambda < 0.$$

Throughout the paper we denote by Φ the empty set and by $\mathcal{P}(\overline{Z})$ the family of all subsets of $\overline{Z}$. If $A \neq \Phi,\{+\infty\}$ is an arbitrary subset of $\overline{Z}$ and $p \in \overline{Z}$, then

DEFINITION 1.1. We shall say that $p \in \mathrm{INF}(A)$ iff there exists no $a \in A$ such that $a < p$.

DEFINITION 1.2. We shall say that $p \in \mathrm{INF}_1(A)$ iff $p \in \mathrm{INF}(A)$ and for every $p' \in Z$ with $p < p'$, there exists $a \in A$ such that $a < p'$.

It $A = \Phi$ or $A = \{+\infty\}$, we consider $\mathrm{INF}(A) = \mathrm{INF}_1(A) = \{+\infty\}$. In a similar manner we define the sets $\mathrm{SUP}(A)$ and $\mathrm{SUP}_1(A)$ for $A \neq \Phi,\{-\infty\}$.

DEFINITION 1.3. We shall say that $p \in \mathrm{SUP}(A)$ iff there exists no $a \in A$ with $p < a$.

DEFINITION 1.4. We shall say that $p \in SUP_1(A)$ iff $p \in SUP(A)$ and for every $p' \in Z$ with $p' < p$, there exists $a \in A$ such that $p' < a$.

As for INF, if $A = \Phi$ or $A = \{-\infty\}$, we shall consider $SUP(A) = SUP_1(A) = \{-\infty\}$.

REMARK 1.1. $INF_1(A) \subseteq INF(A)$, $SUP_1(A) \subseteq SUP(A)$, $SUP(A) = -INF(-A)$ and $SUP_1(A) = -INF_1(-A)$ for every subset A of $\overline{Z}$. Also we have $SUP(B) \supseteq SUP(C)$ and $INF(B) \supseteq INF(C)$ for every non-void subsets B, C of $\overline{Z}$ with $B \subseteq C$ and if we denote by Z_+ the cone which induces the relation "$\leq$", then it is easy to see that $INF_1(A) = INF_1(A+Z_+)$ and $SUP_1(A) = SUP_1(A-Z_+)$.

2. The vectorial subdifferential and conjugate of a multifunction

Let X and Y be two real linear spaces.

DEFINITION 2.1. We call Z-duality between X and Y every bilinear map $(.,.): X \times X \to Z$.

In this case, we shall say that $(X,Y,(.,.))$ is a Z-*dual system*. If, in addition, for every $x \in X\backslash\{0\}$, there exists $y \in Y$ such that $(x,y) \neq 0$ and for every $y \in Y\backslash\{0\}$, there exists $x \in X$ such that $(x,y) \neq 0$, then we call $(X,Y,(.,.))$ a *separate* Z-*dual system*. For example, if we denote by $L(X,Z)$ the real linear space of all linear operators $T: X \to Z$, then we can define the following Z-duality: $(.,.): X \times L(X,Z) \to Z$, $(u,t) = Tu$, $u \in X$ and $T \in L(X,Z)$. It is clear that $(X,L(X,Z),(.,.))$ is a separate Z-dual system if and only if $L(X,Z) \neq \{0\}$ and it separates the points of X, i.e., for every $x,y \in X$ with $x \neq y$, there exists $T \in L(X,Z)$ such that $Tx \neq Ty$.

With respect to this Z-duality we define the *vectorial subdifferential* of a multifunction $f: D(f) \subseteq X \to P(\overline{Z})$ at a point $x_0 \in D(f) = \{x \in X: f(x) \neq \Phi\}$ such that $f(x_0) \neq \{+\infty\}$ by

$$(2.1)\quad \partial_v f(x_0) = \{T \in L(X,Z): \text{there exists } y_0 \in f(x_0) \text{ such that } Tx_0 - Tu \nleq y_0 - y \text{ for every } u \in D(f) \text{ and every } y \in f(u)\}.$$

Throughout the paper we shall consider that $L(X,Z)$ contains only the linear and continuous operators from X to Z whenever X and Z are linear topological spaces. Every element of $\partial_v f(x_o)$ will be called the *vectorial subgradient* of f at x_o and it is clear that the existence of vectorial subgradients is closely related to the (pre)order relation in Z.

REMARK 2.1. This concept of subdifferential generalizes the notions introduced by Definitions 3.1 and 5.1 from [8] and [2] respectively. We also notice that there exists a strong connection between the vectorial subdifferential and the classical subdifferential. Indeed, if X is a real linear topological space and $\varphi: X \to \overline{R}$ is a function, then we can define the multifunction $\tilde{\varphi}: X \to P(\overline{R})$ by $\tilde{\varphi}(x) = \begin{cases} \{t: t \geq \varphi(x)\}, & x \in D(\varphi) \\ \Phi, & x \in X \setminus D(\varphi) \end{cases}$ where $D(\varphi) = \{x \in X: \varphi(x) < +\infty\}$. If we denote the subdifferential of φ at a point $x_o \in D(\varphi)$ by $\partial\varphi(x_o)$, we shall prove that $\partial_v\tilde{\varphi}(x_o) = \partial\varphi(x_o)$.

We have

$$(2.2) \quad \partial_v\tilde{\varphi}(x_o) = \{T \in L(X,R): \text{there exists } y_o \geq \varphi(x_o) \text{ such that } y_o - y \leq Tx_o - Tu \text{ for every } u \in X \text{ and every } y \geq \varphi(u)\}$$

and

$$(2.3) \quad \partial\varphi(x_o) = \{T \in L(X,R): \varphi(x_o) - \varphi(u) \leq Tx_o - Tu \text{ for every } u \in X\}.$$

Let $T \in \partial_v\tilde{\varphi}(x_o)$. Then there exists $y_o \geq \varphi(x_o)$ such that $y_o - y \leq Tx_o - Tu$ for every $u \in X$ and every $y \geq \varphi(u)$. Therefore $\varphi(x_o) - \varphi(u) \leq Tx_o - Tu$ for every $u \in X$, that is, $T \in \partial\varphi(x_o)$.

Conversely, let $T \in \partial\varphi(x_o)$. Then we have $\varphi(x_o) - \varphi(u) \leq Tx_o - Tu$ for every $u \in X$ which implies that $\varphi(x_o) - y \leq \varphi(x_o) - \varphi(u) \leq Tx_o - Tu$ for every $y \geq \varphi(u)$. Thus we obtained the contrary inclusion $\partial\varphi(x_o) \subseteq \partial_v\tilde{\varphi}(x_o)$ and the equality $\partial_v\tilde{\varphi}(x_o) = \partial\varphi(x_o)$ is proved.

REMARK 2.2. It is easy to see that a multifunction $f: D(f) \subseteq X \to P(\overline{Z})$ is vectorial subdifferentiable at $x_o \in D(f)$ with $f(x_o) \nsubseteq \{+\infty\}$, i.e., $\partial_v f(x_o) \neq \Phi$, if and only if there exist $y_o \in f(x_o)$ and $T \in L(X,Z)$, such that $y_o - Tx_o \in$

$\mathrm{INF}\{y - Tu: u \in D(f)$ and $y \in f(u)\}$ or equivalently, if and only if there exist $y_o \in f(x_o)$ and $T \in L(X,Z)$, such that $y_o - Tx_o \in \mathrm{INF}_1\{y\text{-}Tu: u \in D(f)$ and $y \in f(u)\}$.

REMARK 2.3. If for a nonempty subset M of a real linear space $\widetilde{X}$ we define its *vectorial dual cone* by $M_V = \{T \in L(\widetilde{X},Z): Tx \nleq 0$ for every $x \in M\}$ and for a multifunction $f: D(f) \subseteq X \to P(Z)$ we consider its graph $G_f = \{(x,y): x \in D(f)$ and $y \in f(x)\}$, then $T \in \partial_V f(x_o)$ if and only if $(-T,I) \in (G_f - (x_o,y_o))_V$ for some $y_o \in f(x_o)$, where $I: Z \to Z$ and $I(z) = z$ for every $z \in Z$.

DEFINITION 2.2. If $G \subseteq X$ is a nonempty subset, then an element $g_o \in G$ is said to be a best vectorial approximation of an element $x_o \in X$ by the elements of G with respect to a multifunction $f: D(f) \subseteq X \to P(\overline{Z})$ if there exists $y_o \in f(g_o-x_o)$ such that for every $g \in G$ with $g - x_o \in D(f)$ and every $y \in f(g-x_o)$ it follows that $y \nleq y_o$.

If $x_o = 0$, then g_o is called *minimal element for the vectorial program* $(G,f/_G)$. It is clear that g_o is a best vectorial approximation of x_o by G if and only if $g_o - x_o$ is minimal element for the vectorial program $(G-x_o,f/_{G-x_o})$. Also it is easy to see that $g_o \in G$ is a best vectorial approximation of $x_o \in X$ by the elements of G with respect to a multifunction f if and only if

$$f(g_o-x_o) \cap \mathrm{INF}[\bigcup_{g\in G} f(g-x_o)] = f(g_o-x_o) \cap \mathrm{INF}_1[\bigcup_{g\in G} f(g-x_o)] \neq \Phi.$$

The following theorem establishes the characterization of the set of minimal elements for a vectorial program by the vectorial subdifferential of a multifunction.

THEOREM 2.1. *If* $f: D(f) \subseteq X \to P(Z \cup \{+\infty\})$ *is a multifunction,* $G \subseteq X$ *is a non-void set and we consider the multifunction* $I_{f,G}: D(f) \to P(Z \cup \{+\infty\})$ *defined by*

$$I_{f,G}(x) = \begin{cases} \{0\}, & x \in G \\ \{+\infty\}, & x \in D(f)\setminus G \end{cases}$$

then the following conditions are equivalent:

(i) $g_o \in G$ *is a minimal element for the vectorial program* $(G, f/_G)$;

(ii) $g_o \in G$ *is a minimal element for the vectorial program* $(D(f), f+I_{f,G})$;

(iii) $0 \in \partial_v(f+I_{f,G})(g_o)$.

PROOF. It is an immediate consequence of the above considerations.

DEFINITION 2.3. For every multifunction $f: D(f) \subseteq X \to P(\overline{Z})$, the point-to-set map $f^*: D(f^*) \subseteq L(X,Z) \to P(\overline{Z})$ defined by

$$f^*(T) = SUP_1\{Tu - y: u \in D(f) \text{ and } y \in f(u)\} \tag{2.4}$$

is called the *vectorial conjugate* of f, where $D(f^*) = \{T \in L(X,Z): SUP_1\{Tu - y: u \in D(f) \text{ and } y \in f(u)\} \neq \Phi\}$. The *vectorial biconjugate* of f, denoted with f^{**} is defined by

$$f^{**}(x) = SUP_1\{Tx - y^*: T \in D(f^*) \text{ and } y^* \in f^*(T)\} \tag{2.5}$$

for every $x \in D(f^{**}) = \{x \in X: SUP_1\{Tx - y^*: T \in D(f^*) \text{ and } y^* \in f^*(T)\} \neq \Phi\}$.

REMARK 2.4. We have $f^*(0) = SUP_1\{-y: u \in D(f) \text{ and } y \in f(u)\} = -INF_1\{y: u \in D(f) \text{ and } y \in f(u)\}$ which justifies the importance of the vectorial conjugate in vectorial optimization problems with the objective maps multifunctions.

REMARK 2.5. If in the Definition 2.3 f is a real multivalued map, we obtain the concept of conjugate map introduced by Definition 2.3 of [8] and if f is a relation from X to R^n_∞, then we obtain the notion of conjugate relation introduced by Definition 3.1 of [2], taking into account the following immediate equality: $f^*(T) = SUP_1\{Tu - y - z_+: u \in D(f), y \in f(u) \text{ and } z_+ \in Z \text{ such that } 0 \leq z_+\}$ for every $T \in D(f^*)$. As in the case of the vectorial subdifferential, there exists a strong connection between the vectorial conjugate and the classical conjugate. Indeed, if X is a real linear topological space and $\varphi: X \to \overline{R}$ is a function, then we can consider the multifunction $\tilde{\varphi}: X \to P(\overline{R})$ defined by $\tilde{\varphi}(x) = \{t: t \geq \varphi(x)\}$ and if we denote the conjugate of φ by φ^*, we shall prove that $\tilde{\varphi}^* = \{\varphi^*\}$.

We have

$$(2.6) \qquad \widetilde{\varphi}^*(T) = SUP_1\{Tx - y: x \in X \text{ and } y \geq \varphi(x)\}$$

and

$$(2.7) \qquad \varphi^*(T) = SUP\{Tx - \varphi(x): x \in X\}$$

for every $T \in L(X,R)$.

Let $p \in \widetilde{\varphi}^*(T)$. Then there exists no $x \in X$ such that $p < Tx - \varphi(x)$, that is, $p \geq Tx - \varphi(x)$ for every $x \in X$ and if $\varepsilon < p$, it follows that there exist $x_\varepsilon \in X$ and $y \geq \varphi(x_\varepsilon)$ such that $\varepsilon < Tx_\varepsilon - y \leq Tx_\varepsilon - \varphi(x_\varepsilon)$. Therefore $\widetilde{\varphi}^*(T) \subseteq \{\varphi^*(T)\}$ for all $T \in L(X,R)$. Conversely, let $p = \varphi^*(T)$. Then there exists no $x \in X$ and $y \geq \varphi(x)$ such that $p < Tx - y$ since, in the contrary case, we have $p < Tx - y \leq Tx - \varphi(x)$ contradiction, and if $\varepsilon < p$, there exists $x_\varepsilon \in X$ such that $\varepsilon < Tx_\varepsilon - \varphi(x_\varepsilon)$. Thus we obtained the contrary inclusion $\{\varphi^*(T)\} \subseteq \widetilde{\varphi}^*(T)$ for every $T \in L(X,R)$ and the equality $\widetilde{\varphi}^* = \{\varphi^*\}$ is proved.

REMARK 2.6. The concept of vectorial biconjugate for multifunctions extends also the classical concept of biconjugate for extended real-valued functions. Indeed, from the above considerations it follows that for every function $\varphi: X \to \overline{R}$ and every $x \in X$ we have $\widetilde{\varphi}^{**}(x) = SUP_1\{Tx - y^*: T \in D(\widetilde{\varphi}^*) \text{ and } y^* \in \widetilde{\varphi}^*(T)\} = SUP_1\{Tx - y^*: T \in D(\varphi^*) \text{ and } y^* \in \{\varphi^*(T)\}\} = \varphi^{**}(x)$.

Strong relationships between the vectorial subdifferential, the vectorial conjugate map and the vectorial biconjugate map are established in the following theorems.

THEOREM 2.2. *A multifunction* $f: D(f) \subseteq X \to P(\overline{Z})$ *is vectorial subdifferentiable at* $x_0 \in D(f)$ *with* $f(x_0) \neq \{+\infty\}$ *if and only if there exist* $y_0 \in f(x_0)$ *and* $T \in L(X,Z)$ *such that* $Tx_0 - y_0 \in f^*(T)$.

PROOF. If $f: D(f) \subseteq X \to (\overline{Z})$ is a multifunction vectorial subdifferentiable at $x_0 \in D(f)$, then there exist $y_0 \in f(x_0)$ and $T \in L(X,Z)$ such that

$Tx_o - y_o \in f^*(T)$, because, in the contrary case, there exist $u \in X$ and $y \in f(u)$, such that $Tx_o - y_o < Tu - y$, i.e., $Tx_o - Tu < y_o - y$ in contradiction with $T \in \partial_v f(x_o)$. Conversely, let $x_o \in D(f)$, $y_o \in f(x_o)$ and $T \in L(X,Z)$ such that $Tx_o - y_o \in f^*(T)$. If $T \notin \partial_v f(x_o)$, then there exist $u \in D(f)$ and $y \in f(u)$ with $Tx_o - Tu < y_o - y$ i.e., $Tx_o - y_o < Tu - y$, in contradiction with $Tx_o - y_o \in f^*(T)$ and the theorem is proved.

THEOREM 2.3. *If* $f: D(f) \subseteq X \to P(\overline{Z})$ *is a multifunction, then*

(i) *for every* $x_o \in D(f)$ *with* $f(x_o) \neq \{+\infty\}$ *and every* $T \in L(X,Z)$ *for which there exist* $y_o \in f(x_o)$ *and* $y^* \in f^*(T)$ *such that* $y_o + y^* = Tx_o$ *we have* $T \in \partial_v f(x_o)$;

(ii) *for every* $x_o \in D(f)$ *with* $\partial_v f(x_o) \neq \Phi$ *and every* $T \in \partial_v f(x_o)$, *there exists* $y_o \in f(x_o)$ *such that* $Tx_o \not< y_o + y^*$ *and* $y_o + y^* \not< Tx_o$ *for every* $y^* \in f^*(T)$.

PROOF. (i) follows directly from Theorem 2.2.

(ii) It is clear that for every $x_o \in D(f)$, $y_o \in f(x_o)$ and $T \in \partial_v f(x_o)$ we have $y_o + y^* \not< Tx_o$ for all $y^* \in f^*(T)$. Assume now that for every $y_o \in f(x_o)$, there exists $y^* \in f^*(T)$ such that $Tx_o < y_o + y^*$, i.e., $Tx_o - y_o < y^*$. Since $y^* \in f^*(T)$, it follows that there exist $u \in D(f)$ and $y \in f(u)$, such that $Tx_o - y_o < Tu - y$, or equivalently, $Tx_o - Tu < y_o - y$, in contradiction with $T \in \partial_v f(x_o)$. This completes the proof.

THEOREM 2.4. *Every multifunction* $f: D(f) \subseteq X \to P(\overline{Z})$ *has the following properties:*

(i) *if* f *is vectorial subdifferentiable at* $x_o \in D(f)$, *then*

$$f(x_o) \cap f^{**}(x_o) \cap \{Tx_o - y^*: T \in L(X,Z) \text{ and } y^* \in f^*(T)\} \neq \Phi;$$

(ii) *if* $f(x_o) \cap f^{**}(x_o) \cap \{Tx_o - y^*: T \in L(X,Z)$ *and* $y^* \in f^*(T)\} \neq \Phi$, *then there exist* $y_o \in f(x_o)$ *and* $T \in L(X,Z)$ *such that* $Tx_o - y_o \in f^*(T)$;

(iii) *for every* $x_o \in D(f) \cap D(f^{**})$, $y_o \in f(x_o)$ *and* $y_o^{**} \in f^{**}(x_o)$ *we have* $y_o \not< y_o^{**}$.

PROOF. (i) By virtue of Theorem 2.2, if $f: D(f) \subseteq X \to P(\overline{Z})$ is vectorial subdifferentiable at $x_0 \in D(f)$, then there exist $T \in L(X,Z)$ and $y_0 \in f(x_0)$ such that $Tx_0 - y_0 \in f^*(T)$, that is, $y_0 \in f(x_0) \cap \{Tx_0 - y^*: T \in L(X,Z)$ and $y^* \in f^*(T)\}$.

Let us suppose that $y_0 \notin f^{**}(x_0)$. Since $y_0 \in \{Tx_0 - y^*: T \in L(X,Z)$ and $y^* \in f^*(T)\}$, it follows that there exist $T' \in L(X,Z)$ and $y' \in f^*(T')$ with $y_0 < T'x_0 - y'$, that is, $y' < T'x_0 - y_0$, in contradiction with $y' \in f^*(T')$. Therefore $y_0 \in f(x_0) \cap f^{**}(x_0) \cap \{Tx_0 - y^*: T \in L(X,Z)$ and $y^* \in f^*(T)\}$ and (i) is proved.

(ii) Suppose that there exists $y_0 \in f(x_0) \cap f^{**}(x_0) \cap \{Tx_0 - y^*: T \in L(X,Z)$ and $y^* \in f^*(T)\}$ and f is not vectorial subdifferentiable at $x_0 \in D(f)$, that is, for every $T \in L(X,Z)$, there exist $u \in D(f)$ and $y \in f(u)$, such that $Tx_0 - Tu < y_0 - y$, or equivalently, $Tx_0 - y_0 < Tu - y$ which implies that $Tx_0 - y_0 \notin f^*(T)$, i.e., $y_0 \notin Tx_0 - f^*(T)$ for all $T \in L(X,Z)$, contradiction, and taking into account Theorem 2.2(ii) follows.

(iii) Assume that there exist $x_0 \in D(f) \cap D(f^{**})$, $y_0 \in f(x_0)$ and $y_0^{**} \in f^{**}(x_0)$ such that $y_0 < y_0^{**}$. Then there exist $T \in L(X,Z)$ and $y_0^* \in f^*(T)$ with $y_0 < Tx_0 - y_0^*$ that is, such that $y_0^* < Tx_0 - y_0$, contradiction, and theorem is proved.

COROLLARY. *A multifunction* $f: D(f) \subseteq X \to P(\overline{Z})$ *is vectorial subdifferentiable at* $x_0 \in D(f)$ *with* $f(x_0) \neq \{+\infty\}$ *if and only if*

(2.8) $$f(x_0) \cap f^{**}(x_0) \cap \{Tx_0 - y^*: T \in L(X,Z) \text{ and } y^* \in f^*(T)\} \neq \Phi.$$

THEOREM 2.5. *For every multifunction* $f: D(f) \subseteq X \to P(\overline{Z})$ *we have*

(2.9) $$\mathrm{SUP}_1\{Tx - z: T \in L(X,Z),\ z \in Z \text{ and } z \nless Tx - y \text{ for every } x \in D(f) \text{ and every } y \in f(x)\} \cap \{Tx - y^*: T \in L(X,Z) \text{ and } y^* \in f^*(T)\} \subseteq f^{**}(x) \cap \{Tx - z: T \in L(X,Z),\ z \in Z \text{ and } z \nless Tx - y \text{ for every } x \in D(f) \text{ and every } y \in f(x)\}$$

and

(2.10) $f^{**}(x) = \mathrm{SUP}_1\{Sx - z: S \in L(X,Z)$ *and there exists* $y^* \in f^*(S)$ *such that* $y^* \leq z\}$

$x \in X$.

PROOF. (2.9) and (2.10) are immediate consequences of Definition 1.4 and of relation (2.5).

REMARK 2.7. Inclusion (2.9) offers a sufficient condition for the existence of f^{**} and relation (2.10) generalizes Theorem 6.3.7 ([5], p. 343), proving that for a multifunction $f: D(f) \subseteq X \to \mathcal{P}(\overline{Z})$ we have $f = f^{**}$ if and only if $f(x) = \mathrm{SUP}_1\{Sx - z: S \in L(X,Z)$ and there exists $y^* \in f^*(S)$ such that $y^* \leq z\}$ for all $x \in D(f)$.

THEOREM 2.6. *For every multifunction* $f: D(f) \subseteq X \to \mathcal{P}(\overline{Z})$ *we have*

(i) *if* $x_0 \in D(f)$ *with* $f(x_0) \neq \{+\infty\}$ *and* $T \in \partial_V f(x_0)$, *then* $x_0 \in \partial_V f^*(T)$:

(ii) *if* $x_0 \in D(f) \cap D(f^{**})$ *and* $f(x_0), f^{**}(x_0) \neq \{+\infty\}$, *then* $\partial_V f(x_0) \subseteq \partial_V f^{**}(x_0)$;

(iii) *if* $x_0 \in D(f) \cap D(f^{**})$, $f^{**}(x_0) \neq \{+\infty\}$, $f^{**}(x) \subseteq f(x)$ *and for every* $y \in f(x)$, *there exists* $y^{**} \in f^{**}(x)$ *such that* $y^{**} \leq y$ *whenever* $x \in D(f)$, *then* $\partial_V f(x_0) = \partial_V f^{**}(x_0)$;

(iv) *if* $x_0 \in D(f)$ *and* $f(x_0) \neq \{+\infty\}$, *then* $\partial_V f(x_0) \neq \Phi$ *implies* $f^{**}(x_0) \cap f(x_0) \neq \Phi$.

PROOF. (i) If $x_0 \in D(f)$ and $T \in \partial_V f(x_0)$, then, by virtue of Theorem 2.2, there exists $y_0 \in f(x_0)$ such that $Tx_0 - y_0 \in f^*(T)$. Hence, from the definition of the multifunction f^* it follows that for every $S \in D(f^*)$ and every $y^* \in f^*(S)$ we have $Tx_0 - Sx_0 \nleq (Tx_0 - y_0) - y^*$, that is, $x_0 \in \partial_V f^*(T)$.

(ii) For every $x_0 \in D(f) \cap D(f^{**})$ and every $T \in \partial_V f(x_0)$ from (i) we obtain: $T \in \partial_V f(x_0) \Rightarrow x_0 \in \partial_V f^*(T) \Rightarrow T \in \partial_V f^{**}(x_0)$.

(iii) is an immediate consequence of (ii) together with the definition of the vectorial subdifferential.

(iv) If $x_0 \in D(f)$ and f is vectorial subdifferentiable at x_0, then taking into account Theorem 2.2 there exist $T \in L(X,Z)$ and $y_0 \in f(x_0)$ such that $Tx_0 - y_0 \in f^*(T)$. We shall prove that $y_0 \in f^{**}(x_0)$. Indeed, for every $S \in L(X,Z)$ and every $z \in Z$ for which there exists $y^* \in f^*(S)$ with $y^* \leq z$ we have $y_0 \nless Sx_0 - z$ since, in the contrary case, $y_0 < Sx_0 - z$ implies $y_0 < Sx_0 - y^*$, that is, $y^* < Sx_0 - y_0$, contradiction. On the other hand, if $\varepsilon < y_0$, then $\varepsilon < Tx_0 - (Tx_0 - y_0)$ with $Tx_0 - y_0 \in f^*(T)$. Hence, by virtue of the relation (2.10) from Theorem 2.5 we have $y_0 \in f^{**}(x_0)$ and (iv) is proved.

REMARK 2.8. The above theorems generalize for multifunctions basic results concerning the conjugate, the biconjugate and the subdifferential from [2], [3], [5, Chapter 6], [7] and [8].

3. Duality

Let X and Y be two real linear spaces, $f: D(f) \subseteq X \to \mathcal{P}(\overline{Z})$ a multifunction with $D(f) = \{x \in X: f(x) \neq \Phi\}$, $F: D(F) \subseteq X \times Y \to \mathcal{P}(\overline{Z})$ a point-to-set map defined on $D(F) = \{(x,y) \in X \times Y: F(x,y) \neq \Phi\}$ such that $F(x,0) = f(x)$ for every $x \in D(f)$ and the Z-duality $(.,.)$ between $X \times Y$ and $L(X,Z) \times L(Y,Z)$ defined by

$$(3.1) \quad ((x,y),(T,S)) = Tx + Sy, \quad \forall x \in X, \ \forall y \in Y, \ \forall T \in L(X,Z), \ \forall S \in L(Y,Z)$$

where by $L(X,Z)(L(Y,Z))$ we denote the real linear space of all linear operators $T: X \to Z$ $(S: Y \to Z)$ with the remark that, as in the Section 2, we shall consider $L(X,Z)(L(Y,Z))$ containing only the linear and continuous operators from X to Z (from Y to Z) whenever X and Z (Y and Z) are linear topological spaces.

In accordance with the Definition 2.3, the vectorial conjugate of F with respect to the above Z-duality is the multifunction $F^*: D(F^*) \to \mathcal{P}(\overline{Z})$ defined by

$$(3.2) \qquad F^*(T,S) = \mathrm{SUP}_1\{Tx + Sy - z: (x,y) \in D(F) \text{ and } z \in F(x,y)\}$$

where $D(F^*) = \{(T,S) \in L(X,Z) \times L(Y,Z): \mathrm{SUP}_1\{Tx + Sy - z: (x,y) \in D(F)$ and $z \in F(x,y)\}$ is nonempty$\}$.

Therefore,

$$(3.3)\qquad F^* = (0,y^*) = \mathrm{SUP}_1\{y^*y - z:\ (x,y) \in D(F) \text{ and } z \in F(x,y)\}$$

for every $y^* \in L(Y,Z)$ with $(0,y^*) \in D(F^*)$.

We consider the following vectorial problems:

$$(\mathcal{P})\qquad \mathrm{INF}_1[\bigcup_{x\in D(f)} f(x)] \cap [\bigcup_{x\in D(f)} f(x)] = \mathrm{MIN}(f)$$

and

$$(\mathcal{P}^*)\qquad \mathrm{SUP}_1[\bigcup_{(p,y^*)\in D(F^*)} (-F^*(0,y^*))] \cap [\bigcup_{(0,y^*)\in D(F^*)} (-F^*(0,y^*))] = \mathrm{MAX}[-F^*(0,\cdot)]$$

The following theorem shows the immediate connections between the feasible solutions of $(\mathcal{P})$ and $(\mathcal{P}^*)$.

THEOREM 3.1.

(i) $\mathrm{INF}_1[\bigcup_{x\in D(f)} f(x)] \subseteq \mathrm{SUP}[\bigcup_{(p,y^*)\in D(F^*)} (-F^*(0,y^*))]$;

(ii) $\mathrm{SUP}_1[\bigcup_{(p,y^*)\in D(F^*)} (-F^*(0,y^*))] \subseteq \mathrm{INF}[\bigcup_{x\in D(f)} f(x)]$;

(iii) *if every two elements in* Z *are* "$\le$" *comparable and for every* $x_o \in D(f)$ *there exists* $(0,y_o^*) \in D(F^*)$ *such that* $y_o^*(y) \le z - z_o$, $\forall (x,y) \in D(F)$, $\forall z \in F(x,y)$, $\forall z_o \in F(x_o,0)$, *then*

$$\mathrm{SUP}[\bigcup_{(0,y^*)\in D(F^*)} (-F^*(0,y^*))] \subseteq \mathrm{SUP}[\bigcup_{x\in D(f)} f(x)].$$

PROOF. (i) Let $p \in \mathrm{INF}_1[\bigcup_{x\in D(f)} f(x)]$ and assume that $p \notin \mathrm{SUP}[\bigcup_{(0,y^*)\in D(F^*)} [(-F^*(0,y^*))]$. Then there exist $y^* \in L(Y,Z)$ and $-f^* \in -F^*(0,y^*)$ such that $p < -f^*$. Since $p \in \mathrm{INF}_1[\bigcup_{x\in D(f)} f(x)]$, it follows that there exist $x \in D(f)$ and $y \in f(x)$ such that $f^* < y^*(0) - y$, in contradiction with the definition of $F^*(0,y^*)$.

(ii) Let $p \in \mathrm{SUP}_1[\bigcup_{(0,y^*)\in D(F^*)} (-F^*(0,y^*))]$ and suppose that

$p \notin INF[\bigcup_{x\in D(f)} f(x)]$. Then there exist $x \in D(f)$ and $y \in f(x)$ such that $y < p$. Since $p \in SUP_1[\bigcup_{(0,y^*)\in D(F^*)} (-F^*(0,y^*))]$ it follows that there exist $y^* \in L(Y,Z)$ and $-f^* \in -F^*(0,y^*)$ with $y < -f^*$. The result follows as in the proof of (i).

(iii) Let $p \in SUP[\bigcup_{(0,y^*)\in D(F^*)} (-F^*(0,y^*))]$ and let us assume that $p \notin SUP[\bigcup_{x\in D(f)} f(x)]$. Then there exist $x_o \in D(f)$ and $y_o \in f(x_o)$, such that $p < y_o$. By virtue of the hypotheses, there exists $y_o^* \in L(Y,Z)$ such that $y_o^*(y) - z \le -y_o < -p$ for every $(x,y) \in D(F)$ and every $z \in F(x,y)$, which implies $z_o^* \le -z_o < -p$, $\forall z_o \in F(x_o,0)$, $\forall z_o^* \in F^*(0,y_o^*)$, that is, $p < -z_o^*$, in contradiction with our assumption.

COROLLARY. $MIN(f) \neq \Phi$, $MAX[-F^*(0,\cdot)] \neq \Phi$ *and* $MIN(f) \cap MAX[-F^*(0,\cdot)] \neq \Phi$ *if and only if there exist* $x_o \in D(f)$, $y_o \in f(x_o)$, $y^* \in L(Y,Z)$ *and* $f^* \in F^*(0,y^*)$ *such that* $y_o + f^* = 0$.

The proof follows the same line as the proof of the above theorem and for this reason we have omitted it.

Now we introduce the concepts of vectorial Lagrangian and saddle point for the problem (P) in order to show that these notions are closely connected with the solutions of problems (P) and (P^*) especially for the case when the objective multifunction f takes values in a quasi-complete locally convex space ordered by a closed and nuclear cone.

DEFINITION 3.1. We shall call the vectorial Lagrangian map of problem (P) the multifunction $L: D(f) \times L(Y,Z) \to P(\overline{Z})$ defined by

$$(3.4) \quad L(x,T) = INF_1\{z - Ty: (x,y) \in D(F) \text{ and } z \in F(x,y)\} \cap \{z - Ty: (x,y) \in D(F) \text{ and } z \in F(x,y)\}.$$

DEFINITION 3.2. A point $(\hat{x},\hat{T}) \in D(f) \times L(Y,Z)$ will be called a *saddle point of* L if

$$(3.5) \quad L(\hat{x},\hat{T}) \cap SUP_1[\bigcup_{T\in L(Y,Z)} L(\hat{x},T)] \cap INF_1[\bigcup_{x\in D(f)} L(x,\hat{T})] \neq \Phi.$$

REMARK 3.1. As in the case of vectorial subdifferential and vectorial conjugate it is easy to see that the notions of vectorial Lagrangian and saddle point generalize the classical similar concepts.

Before stating the theorem which gives the connection between the solutions of problems $(\mathcal{P})$, $(\mathcal{P}^*)$ and the saddle points of $L(.,.)$, we recall a few definitions and we establish some lemmas which will be used in the proof.

DEFINITION 3.3. A linear topological space is said to be *quasi-complete* if every non-void subset closed and bounded is complete.

Let E be a Hausdorff locally convex space with the topology induced by a family $\mathcal{Q} = \{q_\alpha\}_{\alpha\in I}$ of semi-norms and the dual space E'.

DEFINITION 3.4 [4]. A convex cone $K \subseteq E$ is said to be *supernormal* (*nuclear*) if for every $q_\alpha \in \mathcal{Q}$, there exists $f_\alpha \in E'$ such that

$$q_\alpha(x) \leq f_\alpha(x), \quad \forall x \in K. \tag{3.6}$$

REMARK 3.2. The importance of the nuclear cones for the existence of the solution for vectorial problems in locally convex spaces was emphasized for the first time in [4].

Let K be a convex cone and $A \subseteq E$ a nonempty set. Following [8], we say that A is *K-bounded* if there exists $a_0 \in E$ such that $A \subseteq a_0 + K$ and A is said to be *K-closed* if $A + K$ is closed. Moreover, A is called *K-compact* if it is K-bounded and K-closed.

The following theorem and its corollary generalizes in locally convex spaces the result established by Tanino and Sawaragi in Lemma 2.1 and Lemma 2.2 of [8] concerning the existence of the efficient points for nonempty sets in finite dimensional Euclidian spaces and it is fundamental for our purpose.

THEOREM 3.2 [4]. *For every nonempty and* K-*compact set* A *of a quasi-complete locally convex space ordered by a closed and nuclear cone* K *we have* $INF_1(A) \cap A \neq \Phi$.

COROLLARY. *If* A *is a set which satisfies the above conditions then* $A \subseteq INF_1(A) \cap A + K$.

In all our further considerations we suppose that Z is a quasi-complete locally convex space ordered by a closed and nuclear cone Z_+ .

LEMMA 3.1. *If* $F_1: D(F_1) \subseteq X \to P(\overline{Z})$, $F_2: D(F_2) \subseteq X \to P(\overline{Z})$ *are multi-functions with* $D(F_1) \cap D(F_2) \neq \Phi$, *there exists* $F_1(x) + F_2(x)$ *for all* x *from* $D(F_1) \cap D(F_2)$ *and such that the set* $F_2(x)$ *is* Z_+*-compact for every* $x \in D(F_1) \cap D(F_2)$, *then*

$$(3.7)\quad INF_1[\bigcup_{x\in D(F_1)\cap D(F_2)} [F_1(x)+F_2(x)] = INF_1 \bigcup_{x\in D(F_1)\cap D(F_2)} [F_1(x)+F_2(x)\ INF_1\ F_2(x)].$$

PROOF. By virtue of Remark 1.1 together with the above Corollary, we have:

$$\begin{aligned} INF_1 \bigcup_{x\in D(F_1)\cap D(F_2)} [F_1(x)+F_2(x)] &= INF_1 \bigcup_{x\in D(F_1)\cap D(F_2)} [F_1(x)+F_2(x)+Z_+] \\ &= INF_1 \bigcup_{x\in D(F_1)\cap D(F_2)} [F_1(x)+(F_2(x)+Z_+)] \\ &= INF_1 \bigcup_{x\in D(F_1)\cap D(F_2)} [F_1(x)+(F_2(x)\cap INF_1\ F_2(x)+Z_+)] \\ &= INF_1 \bigcup_{x\in D(F_1)\cap D(F_2)} [F_1(x)+F_2(x)\cap INF_1\ F_2(x)+Z_+] \\ &= INF_1 \bigcup_{x\in D(F_1)\cap D(F_2)} [F_1(x)+F_2(x)\cap INF_1\ F_2(x)] . \end{aligned}$$

LEMMA 3.2. *If* $S \in L(Y,Z)$ *and the set* $\{z - Sy: (x,y) \in D(F)$ *and* $z \in F(x,y)\}$ *is* Z_+*-compact for every* $x \in D(f)$ *for which there exists* $y \in Y$ *such that* $(x,y) \in D(F)$, *then*

$$(3.8)\qquad -F^*(0,S) = INF_1[\bigcup_{x\in D(f)} L(x,S)].$$

PROOF. It follows from the relations (3.3), (3.4) and Lemma 3.1 applied for $F_1(x) = \{0\}$ and $F_2(x) = \{z - Sy: (x,y) \in D(F)$ and $z \in F(x,y)\}$, $\forall x \in D(f)$.

REMARK 3.3. In the conditions of Lemma 3.2 we have

$$(3.9)\quad SUP_1[\bigcup_{(0,y^*)\in D(F^*)}(-F^*(0,y^*))] = SUP_1[\bigcup_{T^*\in L(Y,Z)} INF_1[\bigcup_{x\in D(f)} L(x,T)]].$$

LEMMA 3.3. *If* $f_1\colon D(f_1) \subseteq X \to \overline{Z}$ *is a point-to-point map and* $F_1\colon D(F_1) \subseteq X\times Y \to P(\overline{Z})$ *is a multifunction such that*

(a) *the set* $\{t\colon f_1(x) \le t\} + F_1(x,y)$ *is nonempty for every* $x \in D(f_1)$ *and every* $(x,y) \in D(F_1)$;

(b) $F_1(x,y)$ *is* Z_+*-compact for every* $(x,y) \in D(F_1)$, *then*

$$(3.10)\quad INF_1 \bigcup_{\substack{y\in Y\\(x,y)\in D(F_1)}} [\{t: f_1(x)\le t\}+F_1(x,y)]] = \{f_1(x)\}+F_1(x,y)\cap INF_1[\bigcup_{\substack{y\in Y\\(x,y)\in D(F_1)}} F_1(x,y)]$$

for every $x \in D(f_1)$.

PROOF. It follows from Definition 1.2 and Lemma 3.1.

REMARK 3.4. By the definition of vectorial Lagrangian and Lemma 3.3 we conclude that if the perturbation multifunction F is of the form $F(x,y) = \{t: f_1(x) \le t\} + F_1(x,y)$ with f_1 and F_1 satisfying the conditions (a) and (b), then

$$(3.11)\qquad L(x,T) = \{f_1(x)\} - F^*_{1x}(T)$$

where $F_{1x}\colon D(F_{1x}) \subseteq Y \to P(\overline{Z})$ is the multifunction defined by $F_{1x}(y) = F_1(x,y)$ for every $(x,y) \in D(F_1)$.

REMARK 3.5. If, in addition, F_{1x} coincides with its vectorial biconjugate and $\{f_1(x)\} + F_1(x,0) = f(x)$ for every $x \in D(f)$, then taking into account Lemma 3.1, Lemma 3.3 and the relation (3.11) we have

$$\begin{aligned} SUP_1[\bigcup_{T\in L(Y,Z)} L(x,T)] &= SUP_1[\bigcup_{T\in D(F^*_{1x})} [\{f_1(x)\} - F^*_{1x}(T)]] \\ &= \{f_1(x)\} + SUP_1[\bigcup_{T\in D(F^*_{1x})} (-F^*_{1x}(T))] \\ &= \{f_1(x)\} + F^{**}_{1x}(0) = \{f_1(x)\} + F_1(x,0) = f(x), \end{aligned}$$

$\forall x \in D(f)$.

Therefore

$$\text{INF}_1[\bigcup_{x\in D(f)} f(x)] = \text{INF}_1[\bigcup_{x\in D(f)} \text{SUP}_1[\bigcup_{T\in L(Y,Z)} L(x,T)]]. \tag{3.12}$$

The next result is an immediate consequence of the above discussions and generalizes the Theorem 5.1 of [8].

THEOREM 3.3. *Under the assumption which assure the validity of the above lemmas and remarks, the following conditions are equivalent:*

(i) $(\hat{x},\hat{T})$ *is a saddle point for the vectorial Lagrangian;*

(ii) $\hat{x}$ *is a solution of* (P), $\hat{T}$ *is a solution of* (P*) *and* $-F(\hat{x},0) \cap F^*(0,\hat{T}) \neq \Phi$.

REMARK 3.6. The Theorem 3.2 and its corollary (therefore, all our results of Section 3) remain valid if we consider that a nonempty set A is K-bounded when $A \subseteq A_o + K$ with A_o bounded. I observed this immediately after the acceptance of the paper and I communicated it to Professor G. Isac to which I thank for helpful discussions and comments on this and related subjects.

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