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A CATEGORICAL ANALOGUE OF Z_n

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Résumé

Si W_o est un objet non nul du N catégoriel (désigné CN), l'on construit une catégorie $C\mathbb{Z}_{W_o}$, qui — si l'on impose certaines restrictions au concept de catégorie quotient (cf. [11]) — est un quotient de CN. Comme CN, $C\mathbb{Z}_{W_o}$ est une catégorie monoïdale, mais possède en plus une structure de groupe catégoriel (cf. [10]).

If W_o is a non-zero object of the categorical N (denoted by CN), a category $C\mathbb{Z}_{W_o}$ is constructed, which, under certain restrictions on the concept of a quotient category (cf. [11]), is a quotient of CN. Like CN, $C\mathbb{Z}_{W_o}$ is a monoidal category, but has, in addition, a categorical group structure (cf. [10]).

0. Introduction

If we agree to call "categorical algebra" the algebra of those structures which are defined on categories rather than on sets, the "binary relations" occurring in "set-theoretical algebra" might be replaced by "(iso)morphisms". Consequently, when quotient categories are formed, instead of simply identifying the "equivalent" objects, one only has to construct canonical isomorphisms between them. Since the morphisms already existing in the original category must be also taken into account, one considers pairs of "equivalent" morphisms and constructs two isomorphisms, one between the domains, the other one between the codomains, of such morphisms, so that the diagram resulting in the quotient structure is commutative. This approach was taken in [11], where we dealt with the general situation.

We note that C. Ehresmann has introduced and studied the concept of *quasi-quotient structures* [1]. It would be interesting to study how the above concept of quotient relates to that of quasi-quotient, and to describe possible situations when the two concepts intersect.

In the present paper, we apply, in a certain sense, the results of [11] to a specific case. Namely, we consider the categorical analogue of the construction of the additive *group* $\mathbb{Z}_n$ of cosets modulo a (positive) integer n from the additive *monoid* N of natural numbers. In the set-theoretical case, natural numbers that are congruent modulo n are "identified". In the categorical case, starting from the categorical analogue of N (cf. [7], VII.1.), which we denote by CN, and which is a monoidal category, and from a non-zero object W_o of CN ("non-zero" meaning "different from the unit object"), we only add isomorphisms and obtain a categorical group — alternatively, a category with group structure — $C\mathbb{Z}_{W_o}$, which is the categorical analogue of $\mathbb{Z}_n$. (For the concept of categorical group, see [10]; an earlier variant was called "groupe dans une catégorie", cf. [9]; for the concept of category with group structure, see [12], I and [5]; all of these are slight variations of the same concept.) More specifically, the objects of $C\mathbb{Z}_{W_o}$ are the objects of CN, i.e., words built up with two symbols, (+) and I, and the morphisms are, besides those between words of the same length, as in CN, those which link words of lengths congruent modulo the length of W_o (thus, the construction depends only on the length of W_o).

In this connection, we mention the construction used by Lambek ([4], 1.4.) to adjoin a new assumption to a conjunction calculus and to include all proofs based on this assumption.

Under certain restrictions — called the *coherence conditions* — on the concept of a quotient category, $C\mathbb{Z}_{W_o}$ turns out to be a quotient monoidal category of CN (the analogue of the fact that $\mathbb{Z}_n$ is a quotient monoid of N). Namely, $C\mathbb{Z}_{W_o}$ has the universal property for the monoidal categories $\mathcal{D}$ and the homomorphisms $CN \to \mathcal{D}$ such that $\mathcal{D}$ contains isomorphisms as above and the coherence conditions are satisfied. This is proved in section 3 of the paper. In section 6, we construct another categorical analogue of $\mathbb{Z}_n$, denoted $\overline{C\mathbb{Z}}_{W_o}$, this time starting from the categorical analogue $C\mathbb{Z}$ of $\mathbb{Z}$ (cf. [10], 2, Ex. (5)). And $\overline{C\mathbb{Z}}_{W_o}$ turns out to be a quotient category of $C\mathbb{Z}$ as a categorical group; that is, the universal property holds with respect to categorical groups and their homomorphisms such that, for the "canonical" isomorphisms, (additional) coherence conditions are satisfied. In section 5, we take a first step in the direction of proving a categorical analogue of the property that a (set-theoretical) monoid of finite "exponent" is a group. Essentially, what is missing is the (di)naturality (cf. [7], IX.4.; also [2]) of the reciprocity isomorphisms. We suggest that this difficulty might be, however, overcome by using other types of quotient structures.

1. In [7], VII.2., there is given the construction of the categorical analogue of N. This analogue, denoted in the sequel by CN, is the free monoidal category on one generator (cf. loc. cit.). Intuitively, instead of identifying the different "objects" obtained by adding together the same number of copies of 1 (the generator), one distinguishes between these objects; only, they are to be isomorphic. Formally, CN is a category having as objects formal expressions (or words) in two symbols, (+) and I (the latter called sometimes the empty word), connected by a symbol $\square$. That is, (+) and I are words; and if W_1 and W_2 are words, so is $(W_1)\square(W_2)$. (Subsequently, we will omit the parentheses whenever possible, as, for example, in the last expression.) The *length* $|W|$ of a word W is defined by induction as follows: $|(+)| = 1$, $|I| = 0$, $|W_1 \square W_2| = |W_1| + |W_2|$. The morphisms of CN are the following: If W and V are words of the same length, there is exactly one morphism $W \to V$; otherwise, none. Since between any two objects there is at most one morphism, composition is clear. We note that all morphisms are isomorphisms. The category CN becomes a monoidal category if we define the multiplication of words by $(W_1, W_2) \mapsto W_1 \square W_2$ and correspondingly for morphisms. The unit

object is I. Since all diagrams commute, naturality and coherence are trivially satisfied.

In [10], 2., Ex. (5), the categorical analogue of $\mathbb{Z}$ (denoted $C\mathbb{Z}$) is defined. Since this time we need a "negative generator", we use an additional symbol, $(-)$. Formally, the objects are words built up with three symbols, $(+)$, $(-)$, and I, connected by the symbol $\Box$. Now we allow negative lengths. We define $|(+)| = 1$, $|(-)| = -1$, $|I| = 0$, $|W_1 \Box W_2| = |W_1| + |W_2|$. Again, there is exactly one morphism $W \to V$ if $|W| = |V|$; otherwise, none. But now, for example, there are (iso)morphisms $(+)\Box(-) \xrightarrow{\sim} I$ and $(-)\Box(+) \xrightarrow{\sim} I$. We obtain a monoidal category by defining the multiplication in the same way as for CN, with I the unit object. There is, in addition, a reciprocity functor $(\ )^{\sim}: (C\mathbb{Z})^{op} \to C\mathbb{Z}$, defined by $(+)^{\sim} = (-)$, $(-)^{\sim} = (+)$, $I^{\sim} = I$, $(W_1 \Box W_2)^{\sim} = (W_2)^{\sim} \Box (W_1)^{\sim}$. We have $(W^{\sim})^{\sim} = W$ for any word W; and if we denote by $\lambda_W: W^{\sim} \Box W \to I$, $\rho_W: W \Box W^{\sim} \to I$ the only possible morphisms (with the corresponding domains and codomains), we have $\lambda_W{\sim} = \rho_W$, $\rho_W{\sim} = \lambda_W$. Thus, we obtain a categorical group (cf. [10] and [9]); also a category with group structure (cf. Ulbrich [12], I. and Laplaza [5]).

2. We define the categorical analogue $C\mathbb{Z}_{W_o}$ of $\mathbb{Z}_n$, the "additive" group of congruence classes modulo n as obtained by starting from natural numbers. Instead of identifying natural numbers to obtain congruence classes, we use isomorphisms to link those words from CN which have congruent lengths modulo the length of the given word, W_o, which plays the role of n.

Let $W_o \neq I$ be a fixed word in CN. (If $W_o = I$, there will be no difference between CN and $C\mathbb{Z}_{W_o}$, and the latter will not have a group structure in the manner defined for the case $W_o \neq I$.) We construct the following category, denoted $C\mathbb{Z}_{W_o}$, the *categorical analogue of* $\mathbb{Z}_n$: The objects are those of CN. If W and V are such objects, there is exactly one arrow $W \to V$ if $|W| \equiv |V| \pmod{|W_o|}$; otherwise, none. In particular, all the morphisms of CN are also morphisms of $C\mathbb{Z}_{W_o}$. Again, there is at most one morphism between two given words, and therefore all diagrams commute; and each morphism is an isomorphism. We define a multiplication on the set of words by the same rule as for CN. And if $f_i: W_i \to V_i$, $i = 1,2$, are morphisms, we have $|W_i| \equiv |V_i| \pmod{|W_o|}$, $i = 1,2$, and, consequently, $|W_1 \square W_2| \equiv |V_1 \square V_2| \pmod{|W_o|}$, so that there is a morphism $f_1 \square f_2$: $W_1 \square W_2 \to V_1 \square V_2$. The multiplication is a functor, and if we define the unit object to be I, and the associativity and left and right unit isomorphisms in the only possible way, we obtain a monoidal category.

Now, we can use a selection process to choose, for each object W, a $W^{\sim}$ such that there are isomorphisms $\rho_W: W \square W^{\sim} \rightleftarrows I$ and $\lambda_W: W^{\sim} \square W \rightleftarrows I$. (This means that $\mathbf{CZ}_{W_o}$ is defined up to this selection process.) Indeed, since the set of words has the cardinality of N, let us consider a well-ordering of type ω in this set, with I the first element. We define $I^{\sim} \doteq I$; and if a W is already of the form $V^{\sim}$ for a previous V, we define $W^{\sim} = V$; if this is not the case, we select arbitrarily a $W^{\sim}$ such that $|W| + |W^{\sim}|$ is a multiple of $|W_o|$ and $W^{\sim}$ has not been previously utilized; this is possible since at each step only finitely many objects have been previously selected as $W^{\sim}$s. Now, since $|W| + |W^{\sim}| \equiv 0 \pmod{|W_o|}$, there are unique isomorphisms ρ_W and λ_W as above, which satisfy the dinaturality conditions (cf. [7], IX.4.; also [2]) since in $\mathbf{CZ}_{W_o}$ all diagrams commute. And since by the above selection process we have $(W^{\sim})^{\sim} = W$ for each W, all the axioms for a categorical group are satisfied, including $\lambda_{W^{\sim}} = \rho_W$, $\rho_{W^{\sim}} = \lambda_W$ for any W.

3. We want to prove that $\mathbf{CZ}_{W_o}$ is a quotient monoidal category of CN with respect to some set S of pairs of morphisms, in a sense slightly modified from that of [11], 3., Th. 3. In that paper, a (small) monoidal category C was given together with a set S of pairs of morphisms of C. We constructed a monoidal category C/S and a strict homomorphism $T: C \to C/S$ such that for every $(u,v) \in S$, $u: A \to A'$, $v: B \to B'$, there are isomorphisms $f_{u,v}: TA \rightleftarrows TB$, $f'_{u,v}: TA' \rightleftarrows TB'$ rendering commutative the diagram

$$(1)\qquad \begin{array}{ccc} TA & \xrightarrow{f_{u,v}} & TB \\ {\scriptstyle Tu}\downarrow & & \downarrow{\scriptstyle Tv} \\ TA' & \xrightarrow{f'_{u,v}} & TB' \end{array} .$$

Besides, $(C/S,T)$ has the universal property that for each (small) monoidal category $\mathcal{D}$ and (not necessarily strict) homomorphism $(F,\Phi,\mu): C \to \mathcal{D}$ (cf. Definition 1, below) such that, for $(u,v) \in S$ as above, there are given isomorphisms $g_{u,v}$: $FA \cong FB$, $g'_{u,v}$: $FA' \cong FB'$ rendering commutative the diagram

$$(2)\qquad \begin{array}{ccc} FA & \xrightarrow{g_{u,v}} & FB \\ {\scriptstyle Fu}\downarrow & & \downarrow{\scriptstyle Fv} \\ FA' & \xrightarrow{g'_{u,v}} & FB' \end{array}$$

there exists a unique homomorphism $(G,\Psi,\nu): C/S \to \mathcal{D}$ such that $(G,\Psi,\nu) \circ (T,id,id) = (F,\Phi,\mu)$ and $G(f_{u,v}) = g_{u,v}$, $G(f'_{u,v}) = g'_{u,v}$ for any $(u,v) \in S$. Here we require an additional condition, namely, the one given by the following definition:

DEFINITION 1. Let C, $\mathcal{D}$ be monoidal categories, with the multiplication functor denoted by $\otimes$ in both, and with the unit objects I, I', respectively. Let S be a set of ordered pairs of morphisms of C. Let $(F,\Phi,\mu): C \to \mathcal{D}$ be a homomorphism of monoidal categories, where $F: C \to \mathcal{D}$ is a functor between the underlying categories, Φ is a natural isomorphism such that $\Phi_{X,Y}$: $F(X\otimes Y) \cong FX\otimes FY$ for objects X, Y of C, and μ is an isomorphism $FI \cong I'$. Let us assume that for each $(u,v) \in S$, $u: A \to A'$, $v: B \to B'$, there are given isomorphisms $g_{u,v}$: $FA \cong FB$, $g'_{u,v}$: $FA' \cong FB'$ of $\mathcal{D}$ such that diagram (2) commutes. Then the following conditions on the $g_{u,v}$s, $g'_{u,v}$s, and (F,Φ,μ) are called the *coherence conditions*: (A) If (u,v), (v,w) and (u,w) are pairs from S, then $g_{v,w} \circ g_{u,v} = g_{u,w}$ and $g'_{v,w} \circ g'_{u,v} = g'_{u,w}$; (B) If $(u,v) \in S$ and $\mathrm{domain}(u) = \mathrm{domain}(v)$, then $g_{u,v}$ is the identity; and if $\mathrm{codomain}(u) = \mathrm{codomain}(v)$, then $g'_{u,v}$ is the identity; (C) If (u_1,v_1), (u_2,v_2) and $(u_1\otimes u_2, v_1\otimes v_2)$ are pairs from S, with u_i: $A_i \to A'_i$, v_i: $B_i \to B'_i$, $i = 1,2$, then the following diagrams commute

$$(3)\qquad \begin{array}{ccc} F(A_1\otimes A_2) & \xrightarrow{g_{u_1\otimes u_2,v_1\otimes v_2}} & F(B_1\otimes B_2) \\ {\scriptstyle \Phi_{A_1,A_2}}\downarrow & & \downarrow{\scriptstyle \Phi_{B_1,B_2}} \\ FA_1\otimes FA_2 & \xrightarrow{g_{u_1,v_1}\otimes g_{u_2,v_2}} & FB_1\otimes FB_2 \end{array} \qquad \begin{array}{ccc} F(A_1'\otimes A_2') & \xrightarrow{g'_{u_1\otimes u_2,v_1\otimes v_2}} & F(B_1'\otimes B_2') \\ {\scriptstyle \Phi_{A_1',A_2'}}\downarrow & & \downarrow{\scriptstyle \Phi_{B_1',B_2'}} \\ FA_1'\otimes FA_2' & \xrightarrow{g'_{u_1,v_1}\otimes g'_{u_2,v_2}} & FB_1'\otimes FB_2' \end{array}$$

(Actually, the commutativity of diagrams (3) is a relaxation of the conditions $g_{u_1\otimes u_2,v_1\otimes v_2} = g_{u_1,v_1}\otimes g_{u_2,v_2}$, etc., which correspond to the case when Φ is the identity.) □

The approach in [11] was a general one, and, besides, we were not concerned about the relationships between, for example, $g_{u,v}$, $g_{v,w}$, and $g_{u,w}$. It will turn out (see Proposition 1, below) that if we impose these more restrictive conditions, $\mathcal{CZ}_{W_o}$ is, up to an isomorphism, "the" quotient category CN/S for some convenient S; if we did not require these conditions, the quotient category would probably have to be "richer", i.e., to contain more morphisms. Thus, the next result is a characterization of $\mathcal{CZ}_{W_o}$ as a quotient monoidal category of CN under the assumption that the coherence conditions hold. (And Proposition 2 of section 6 has a similar interpretation, only, in terms of categorical groups instead of monoidal categories.) A general treatment of the quotient categories, as described in Part I of [11], but such that the coherence conditions are satisfied, should be done in another paper.

Let S be the set of all ordered pairs (u,v) of morphisms of CN such that, if $u\colon W \to W'$, $v\colon V \to V'$, then $|W| \equiv |V| \pmod{|W_o|}$ (and hence, necessarily, $|W'| \equiv |V'| \pmod{|W_o|}$).

We have the following result:

PROPOSITION 1. *There exists a strict homomorphism of monoidal categories* $T\colon CN \to \mathcal{CZ}_{W_o}$ *such that, for each* $(u,v) \in S$ *as above, there are isomorphisms* $f_{u,v}\colon TW \simeq TV$, $f'_{u,v}\colon TW' \simeq TV'$ *in* $\mathcal{CZ}_{W_o}$ *with the property that* $f'_{u,v} \circ Tu = TV \circ f_{u,v}$ *and that the* $f_{u,v}$*s,* $f'_{u,v}$*s, and* T *satisfy the coherence conditions. And for any monoidal category* $\mathcal{D} = (\mathcal{D},\otimes,I',a',\ell',r')$ *and homomorphism of monoidal* *categories* $(F,\Phi,\mu)\colon CN \to \mathcal{D}$ *such that for each* $(u,v) \in S$ *as above there are given isomorphisms* $g_{u,v}\colon FW \not\simeq FV$, $g'_{u,v}\colon FW' \not\simeq FV'$ *in* $\mathcal{D}$ *with the property that* $g'_{u,v} \circ Fu = Fv \circ g_{u,v}$ *and that the* $g_{u,v}$*s,* $g'_{u,v}$*s, and* (F,Φ,μ) *satisfy the coherence conditions, there is a unique homomorphism of monoidal categories* $(G,\Psi,\nu)\colon CZ_{W_o} \to \mathcal{D}$ *such that* $(G,\Psi,\nu) \circ (T,id,id) = (F,\Phi,\mu)$ *and* $G(f_{u,v}) = g_{u,v}$, $G(f'_{u,v}) = g'_{u,v}$ *for each* $(u,v) \in S$.

PROOF. Let T be the functor which is the identity on objects and maps each morphism to itself. If $(u,v) \in S$ as above, then, by the definition of CZ_{W_o}, there is exactly one isomorphism $f_{u,v}\colon TW = W \not\simeq TV = V$, and one $f'_{u,v}\colon TW' = W' \not\simeq TV' = V'$. By its very definition, T is a strict homomorphism of monoidal categories. The equations $f'_{u,v} \circ Tu = Tv \circ f_{u,v}$ and the coherence conditions hold since in CZ_{W_o} all diagrams commute.

Now, for the universal property, let us define $G(W) = F(W)$ for any word W. If $f\colon W \to V$ is a morphism of CZ_{W_o}, then $|W| \equiv |V| \pmod{|W_o|}$. Consequently, the pair $(1_W, 1_V)$ is in S, and there are isomorphism $g_{1_W,1_V}$, $g'_{1_W,1_V}$ from FW to FV in $\mathcal{D}$ such that the corresponding diagram (2) commutes, that is, $g'_{1_W,1_V} = g_{1_W,1_V}$. We define $G(f) = g_{1_W,1_V}$. (In the sequel, we will write $g_{W,V}$ instead of $g_{1_W,1_V}$.) Then G is a functor, according to the coherence conditions (A) and (B).

We define Ψ by $\Psi_{W_1,W_2} = \Phi_{W_1,W_2}$; and ν by $\nu = \mu$. Then Ψ is natural. Indeed, if $f_i\colon W_i \to V_i$, $i = 1,2$, are morphisms of CZ_{W_o}, then $G(f_1 \Box f_2) = g_{W_1 \Box W_2, V_1 \Box V_2}$, and the naturality of Ψ follows from the coherence condition (C).

Let us show that $G \circ T = F$. The two functors have the same effect on words. Let $f\colon W \to V$ be a morphism of CN. Then $|W| = |V|$, and hence the pairs $(1_W, f)$ and $(f, 1_V)$ are in S. Using conditions (A) and (B), we obtain the (commutative) diagram

(4)
$$\begin{array}{ccccc} FW & \xrightarrow{g_{1_W,f}=id} & FW & \xrightarrow{g_{f,1_V}} & FV \\ {\scriptstyle F1_W}\downarrow & & \downarrow{\scriptstyle Ff} & & \downarrow{\scriptstyle F1_V} \\ FW & \xrightarrow[g'_{1_W,f}]{} & FV & \xrightarrow[g'_{f,1_V}=id]{} & FV \end{array}$$

and $G(f) = g_{W,V} = g_{f,1_V} \circ g_{1_W,f} = g_{f,1_V} = F(f)$.

To prove that (G,Ψ,ν) is a homomorphism of monoidal categories, we must show that

$$a'_{GW_1,GW_2,GW_3} \circ (\Psi_{W_1,W_2} \otimes 1_{GW_3}) \circ \Psi_{W_1 \square W_2,W_3} = (1_{GW_1} \otimes \Psi_{W_2,W_3}) \circ \Psi_{W_1,W_2 \square W_3} \circ G(a_{W_1,W_2,W_3}),$$

$$\ell'_{GW} \circ (\nu \otimes 1_{GW}) \circ \Psi_{I,W} = G(\ell_W), \qquad r'_{GW} \circ (1_{GW} \otimes \nu) \circ \Psi_{W,I} = G(r_W)$$

for any words W_1, W_2, W_3, W. This is easy, since we may replace everywhere G by F, Ψ by Φ, and ν by μ, and the resulting identities hold because (F,Φ,μ) is a homomorphism. By the definitions of Ψ and ν it follows that $(G,\Psi,\nu) \circ (T,id,id) = (F,\Phi,\mu)$.

Let us prove that if $(u,v) \in S$, with $u: W \to W'$, $v: V \to V'$, then $G(f_{u,v}) = g_{u,v}$, $G(f'_{u,v}) = g'_{u,v}$. Since $(1_{W'},u)$ and $(v,1_{V'})$ are in S, we obtain, using conditions (A) and (B), the (commutative) diagram

$$\begin{array}{ccccccc} FW' & \xrightarrow{g_{1_{W'},u}} & FW & \xrightarrow{g_{u,v}} & FV & \xrightarrow{g_{v,1_{V'}}} & FV' \\ {\scriptstyle F1_{W'}}\downarrow & & \downarrow{\scriptstyle Fu} & & {\scriptstyle Fv}\downarrow & & \downarrow{\scriptstyle F1_{V'}} \\ FW' & \xrightarrow[g'_{1_W,u}=id]{} & FW' & \xrightarrow[g'_{u,v}]{} & FV' & \xrightarrow[g'_{v,1_{V'}}=id]{} & FV' \end{array}$$

and $G(f'_{u,v}) = g_{W',V'} = g'_{W',V'} = g'_{v,1_{V'}} \circ g'_{u,v} \circ g'_{1_{W'},u} = g'_{u,v}$. The other equation is proved in a similar fashion.

To prove that (G,Ψ,ν) is unique with the above properties, we only show that if (G',Ψ',ν') is another such homomorphism, then $G'(f) = G(f)$ for any morphism $f: W \to V$ of $\mathcal{CZ}_{W_o}$. Indeed, since $(1_W,1_V) \in S$, there must be an $f_{1_W,1_V}$: $W \to V$ in $\mathcal{CZ}_{W_o}$. Then $f = f_{1_W,1_V}$, and $G'(f) = G'(f_{1_W,1_V}) = g_{1_W,1_V} = G(f)$. The other points of the uniqueness proof are clear. $\square$

One can take a step further, and deal with natural transformations. Let, in addition to (F,Φ,μ), the $g_{u,v}$s and $g'_{u,v}$s as in Proposition 1, a homomorphism $(\overline{F},\overline{\Phi},\overline{\mu})$ and isomorphisms $\overline{g}_{u,v}$ and $\overline{g}'_{u,v}$ be given, satisfying the same conditions, and let $\sigma\colon (F,\Psi,\mu) \to (\overline{F},\overline{\Phi},\overline{\mu})$ be a monoidal natural transformation (cf. [3], II.1); that is, σ is a natural transformation $F \to \overline{F}$ and, in addition,

$$(\sigma_{W_1} \otimes \sigma_{W_2}) \circ \Phi_{W_1,W_2} = \overline{\Phi}_{W_1,W_2} \circ \sigma_{W_1 \Box W_2}, \qquad \overline{\mu} \circ \sigma_I = \mu$$

for any words W_1, W_2. Let us assume that, in addition to the coherence conditions (A) - (C), the following condition is satisfied: (T) For any $(u,v) \in S$, $u\colon W \to W'$, $v\colon V \to V'$, the diagrams

$$\begin{array}{ccc} FW & \xrightarrow{g_{u,v}} & FV \\ {\scriptstyle \sigma_W}\downarrow & & \downarrow{\scriptstyle \sigma_V} \\ \overline{F}W & \xrightarrow{\overline{g}_{u,v}} & \overline{F}V \end{array} \qquad \begin{array}{ccc} FW' & \xrightarrow{g'_{u,v}} & FV' \\ {\scriptstyle \sigma_{W'}}\downarrow & & \downarrow{\scriptstyle \sigma_{V'}} \\ \overline{F}W' & \xrightarrow{\overline{g}'_{u,v}} & \overline{F}V' \end{array}$$

are commutative. Intuitively, these conditions mean that if F is to "change" into $\overline{F}$, the change must be consistent with the change in the $g_{u,v}$s and the $g'_{u,v}$s. Then, denoting by (G,Ψ,ν) the homomorphism $\mathbf{CZ}_{W_o} \to \mathcal{D}$ given by Proposition 1, and by $(\overline{G},\overline{\Psi},\overline{\nu})$ the corresponding homomorphism for $(\overline{F},\overline{\Phi},\overline{\mu})$, there is a unique monoidal natural transformation $\tau\colon (G,\Psi,\nu) \to (\overline{G},\overline{\Psi},\overline{\nu})$ such that $\tau \circ T = \sigma$, where $\circ$ represents the horizontal composition in the 2-category of monoidal categories, monoidal functors, and monoidal natural transformations. Indeed, one defines $\tau_W = \sigma_W$ for any object W of CN (or of $\mathbf{CZ}_{W_o}$).

4. The categories CN, $\mathbf{CZ}$, and $\mathbf{CZ}_{W_o}$ are in fact symmetric monoidal categories (cf. [7], VII.7). Moreover, the group structures on $\mathbf{CZ}$ and $\mathbf{CZ}_{W_o}$ are abelian (cf. [12] and [5]), or symmetric categorical groups (cf. [11], 6). This means that there is a symmetry natural isomorphism $c_{W_1,W_2}\colon W_1 \Box W_2 \rightleftarrows W_2 \Box W_1$ which is jointly coherent with the associativity and unit isomorphisms. We can extend the properties given in section 3, so that they would refer to symmetric monoidal categories. Indeed, $T\colon \text{CN} \to \mathbf{CZ}_{W_o}$ is a strict homomorphism of symmetric monoidal

categories. And if $\mathcal{D}$, as defined in the statement of Proposition 1 of section 3, is, in addition, a symmetric monoidal category, with the symmetry natural isomorphism c', and (F,Φ,μ) is a homomorphism for this structure (meaning that, in addition to being a homomorphism of monoidal categories, the equations $c'_{FW_1,FW_2} \circ \Phi_{W_1,W_2} = \Phi_{W_2,W_1} \circ F(c_{W_1,W_2})$ are satisfied for any words W_1, W_2), then (G,Ψ,ν) of section 3 is also a homomorphism of symmetric monoidal categories; and, of course, it is unique.

5. This section may be considered as a first step in the direction of obtaining a categorical analogue of the property that a (set-theoretical) monoid of finite exponent (that is, such that there is a natural $n \neq 0$ with the property that $x^n = 1$ for any x) is a group. Of course, instead of assuming that $x^n = 1$, we assume that there is a natural isomorphism $\pi_A\colon A^{W_o} \cong I'$ for a fixed $W_o \neq I$ from CN, with the property that π is jointly coherent with the associativity, etc. It is likely that such a structure can be constructed using methods described in [11]. Here we only recover part of the group structure. Essentially, what is missing is the (di)naturality of the reciprocity isomorphisms.

Let $\mathcal{D} = (\mathcal{D},\otimes,I',a',\ell',r')$ be a monoidal category. If W is an object of CN, then a functor $(\ )^W\colon \mathcal{D} \to \mathcal{D}$ will be defined in the following, inductive, manner: If $W = I$, then $(\ )^W = K_{I'}$, the constant functor I'; if $W = (+)$, then $(\ )^W = id_{\mathcal{D}}$; if $W = (W_1)\square(W_2)$, then $A^W = A^{W_1} \otimes A^{W_2}$ for any object A, and similarly for morphisms.

Now, let $W_o \neq I$ be a fixed object of CN, and let us assume that there is given a natural isomorphism

$$\pi\colon (\ )^{W_o} \cong K_{I'} .$$

Let us assume, in addition, that a', ℓ', r', and π are jointly coherent, which here means that any diagram in $\mathcal{D}$ whose arrows are built up via $\otimes$ and composition with instances of a', ℓ', r', π, their inverses, and identities is commutative. Let us call such arrows *canonical*.

Then, for any object A of $\mathcal{D}$, there exists a strict homomorphism F_A: $CN \to \mathcal{D}$ such that, for the S defined in 3, there are $g_{u,v}$s and $g'_{u,v}$s in $\mathcal{D}$, $(u,v) \in S$, making the analogues of diagram (2) commute and such that the coherence conditions (cf. Definition 1 of 3) are satisfied. Indeed, we define $F_A(W)$ to be A^W and if $f: W \to V$ is a morphism of CN, then $F_A(f)$ is the unique morphism $A^W \to A^V$ which is built up with instances of a', ℓ', r', their inverses, and identities (the πs are not included). Then F_A is a strict homomorphism. (We note that this is the homomorphism described as $w \mapsto w_b$ for a fixed b by Mac Lane in [7], VII.2.)

If W, V are objects of CN such that $|W| \equiv |V| \pmod{|W_o|}$, then there is a (unique) canonical arrow $F_A W \to F_A V$. Indeed, let $|W| = n$, $|W_o| = n_o$, and let $n = kn_o + r$ with $0 \le r < n_o$. Since for words S, T having the same length there is a canonical morphism $A^S \to A^T$, it suffices to show that if U is a word of length r and if we set $U_o = U$, $U_{t+1} = W_o \square U_t$, then there is a canonical morphism b_t: $A^{U_t} \to A^U$ for any natural t. We define b_o: $A^U \to A^U$ to be the identity, and b_1 the composite of

$$A^{U_1} = A^{W_o} \otimes A^U \xrightarrow{\pi_A \otimes id} I' \otimes A^U \xrightarrow{\ell'} A^U;$$

and if b_t has been defined, b_{t+1} is the composite of

$$A^{U_{t+1}} = A^{W_o} \otimes A^{U_t} \xrightarrow{id \otimes b_t} A^{W_o} \otimes A^U \xrightarrow{b_1} A^U.$$

Consequently, if $(u,v) \in S$, with $u: W \to W'$, $v: V \to V'$, then $|W| \equiv |V| \pmod{|W_o|}$ and $|W'| \equiv |V'| \pmod{|W_o|}$, so that there are canonical morphisms $A^W \to A^V$ and $A^{W'} \to A^{V'}$; and we define $g_{u,v}$ and $g'_{u,v}$ to be, respectively, these morphisms. Since $F_A u$, $F_A v$, $g_{u,v}$, $g'_{u,v}$ are canonical, we must have $g'_{u,v} \circ Fu = Fv \circ g_{u,v}$, as a, ℓ, r, π are jointly coherent; and, for the same reason, the coherence conditions (A) - (C) of section 3 must be satisfied.

According to Proposition 1 of 3, there is a unique (necessarily strict) homomorphism G_A: $C\mathbb{Z}_{W_o} \to \mathcal{D}$ such that $G_A \circ T = F_A$ and $G_A(f_{u,v}) = g_{u,v}$,

$G_A(f'_{u,v}) = g'_{u,v}$ for each $(u,v) \in S$.

Now, we can find for the given A an $A^{\sim}$ and canonical isomorphisms $\lambda'_A\colon A^{\sim}\otimes A \gtrsim I'$, $\rho'_A\colon A\otimes A^{\sim} \gtrsim I'$. Indeed, we set $A^{\sim} = G_A(-) = G_A((+)^{\sim})$. Since there are isomorphisms $(-)\square(+) \gtrsim I$ and $(+)\square(-) \gtrsim I$ in $C\mathbb{Z}_{W_o}$, their images under G_A are canonical, and these are λ'_A and ρ'_A, respectively. Since in diagram (1) of [10] (with s, d, σ, δ changed to ℓ, r, λ, ρ, respectively) all arrows are canonical, it is commutative. Of course, we can do that for any object A of $\mathcal{D}$.

All this does not mean, however, that $\mathcal{D}$ with the described structure is a categorical group, as we did not prove the λ's and ρ's to be (di)natural (cf. [7], IX.4; also [2]), neither $(A^{\sim})^{\sim}$ to be A, etc. We only showed that, under the above hypotheses, we recover part of the group structure.

6. The construction of $C\mathbb{Z}_{W_o}$, as provided in section 2, is, of course, analogous to that of $\mathbb{Z}_n$, n natural, starting from the monoid N of natural numbers and forming congruence classes modulo n, of natural numbers. The structure becomes a group just by adding new relations, and this was actually shown to happen, for this special case, also in the categorical setting, provided we replace the "relations" by "(iso)morphisms". We can start, however, from the categorical group structure carried by $C\mathbb{Z}$ (cf. section 1) and construct another categorical analogue of $\mathbb{Z}_n$.

If we do that, this "new" analogue, which we denote by $\bar{C}\mathbb{Z}_{W_o}$, is to be defined in a somewhat different way. First, we start with $C\mathbb{Z}$ instead of CN (cf. section 2). Next, let W_o be an object of $C\mathbb{Z}$ such that $|W_o| > 0$. (If $|W_o| = 0$, we do not obtain additional morphisms, and $\bar{C}\mathbb{Z}_{W_o}$ is the same as $C\mathbb{Z}$, as in the set-theoretical case; if $|W_o| < 0$, we can replace W_o by $W_o^{\sim}$, which has positive length, and obtain the same $\bar{C}\mathbb{Z}_{W_o}$.) The objects of $\bar{C}\mathbb{Z}_{W_o}$ are the objects of $C\mathbb{Z}$. There is exactly one morphism $W \to V$ if $|W| \equiv |V| \pmod{|W_o|}$; otherwise, none. The multiplication and reciprocity functors are defined on objects in the same manner as those of $C\mathbb{Z}$; and then, on morphisms, in the only possible manner.

We note that, in contrast to $\mathcal{CZ}_{W_o}$, there is no longer a selection process involved. The functor $\overline{T}: \mathcal{CZ} \to \overline{\mathcal{CZ}}_{W_o}$ which maps each object and morphism to itself is now a strict homomorphism of categorical groups (cf. [11], 6, Déf. 6; also [12], II, 1), that is, $\overline{T}$ is a strict homomorphism of monoidal categories and, besides, $\overline{T}(W^{\sim}) = (\overline{T}W)^{\sim}$, $\overline{T}(f^{\sim}) = (\overline{T}f)^{\sim}$.

Let S be the set of those pairs (u,v) of morphisms of $\mathcal{CZ}$, $u: W \to W'$, $v: V \to V'$, such that $|W| \equiv |V| \pmod{|W_o|}$. Then, for $(u,v) \in S$ as above, there are isomorphisms $f_{u,v}: \overline{T}W \cong \overline{T}V$, $f'_{u,v}: \overline{T}W' \cong \overline{T}V'$ such that, necessarily, $f'_{u,v} \circ \overline{T}u = \overline{T}v \circ f_{u,v}$.

Let $\mathcal{D} = (\mathcal{D},\otimes,I',(\ )^{\sim},a',\ell',r',\lambda',\rho')$ be a categorical group, with $(\ )^{\sim}$ the reciprocity functor and λ' and ρ' the reciprocity isomorphisms, and let $(F,\Phi,\mu,\xi): \mathcal{CZ} \to \mathcal{D}$ be a homomorphism of categorical groups (cf. loc. cit.; this means that (F,Φ,μ) is a homomorphism of monoidal categories and ξ is a natural isomorphism, where $\xi_W: F(W^{\sim}) \cong (FW)^{\sim}$, such that $\lambda'_{FW} \circ (\xi_W \otimes 1_{FW}) \circ \Phi_{W^{\sim},W} = \mu \circ F(\lambda_W)$ and $\rho'_{FW} \circ (1_{FW} \otimes \xi_W) \circ \Phi_{W,W^{\sim}} = \mu \circ F(\rho_W)$ for each W); we assume, in addition, that, for each $(u,v) \in S$ as above, there are given isomorphisms $g_{u,v}: FW \cong FV$, $g'_{u,v}: FW' \cong FV'$ with the property that $g'_{u,v} \circ Fu = Fv \circ g_{u,v}$ and satisfying, besides the coherence conditions (A) - (C) of section 3, the following one: (D) For $(u,v) \in S$ as above, the following diagrams commute

(5)
$$
\begin{array}{ccc}
F(V'^{\sim}) & \xrightarrow{g_{v^{\sim},u^{\sim}}} & F(W'^{\sim}) \\
\downarrow{\scriptstyle \xi_{V'}} & & \downarrow{\scriptstyle \xi_{W'}} \\
(FV')^{\sim} & \xrightarrow{g'^{\sim}_{u,v}} & (FW')^{\sim}
\end{array}
\qquad\qquad
\begin{array}{ccc}
F(V^{\sim}) & \xrightarrow{g'_{v^{\sim},u^{\sim}}} & F(W^{\sim}) \\
\downarrow{\scriptstyle \xi_{V}} & & \downarrow{\scriptstyle \xi_{W}} \\
(FV)^{\sim} & \xrightarrow{g^{\sim}_{u,v}} & (FW)^{\sim}
\end{array}
$$

(We note that for $\overline{\mathcal{CZ}}_{W_o}$, $\overline{T}$, etc., these conditions are trivially satisfied.) Then we construct a (unique) homomorphism of categorical groups $(G,\Psi,\nu,\eta): \overline{\mathcal{CZ}}_{W_o} \to \mathcal{D}$ such that $(G,\Psi,\nu,\eta) \circ (\overline{T},id,id,id) = (F,\Phi,\mu,\xi)$ and $G(f_{u,v}) = g_{u,v}$, $G(f'_{u,v}) = g'_{u,v}$ for each $(u,v) \in S$. Indeed, G, Ψ, and ν are defined as in the previous case (cf. section 3), whereas η is defined by $\eta_W = \xi_W$. We note that the naturality of η follows from the commutativity of diagrams (5).

Thus, we have the following result:

PROPOSITION 2. *There exists a strict homomorphism of categorical groups* $\overline{T}$: $CZ \to \overline{CZ}_{W_o}$ *such that, for each* $(u,v) \in S$, $u: W \to W'$, $v: V \to V'$, *there are isomorphisms* $f_{u,v}: \overline{T}W \simeq \overline{T}V$, $f'_{u,v}: \overline{T}W' \simeq \overline{T}V'$ *with the property that* $f'_{u,v} \circ \overline{T}u = \overline{T}v \circ f_{u,v}$ *and that the* $f_{u,v}$s, $f'_{u,v}$s, *and* $\overline{T}$ *satisfy the coherence conditions* (A) - (D). *And for any categorical group* $\mathcal{D}$ *as above, and homomorphism of categorical groups* $(F,\Phi,\mu,\xi): CZ \to \mathcal{D}$ *such that, for* $(u,v) \in S$ *as above, there are given isomorphisms* $g_{u,v}: FW \simeq FV$, $g'_{u,v}: FW' \simeq FV'$ *with the property that* $g'_{u,v} \circ Fu = Fv \circ g_{u,v}$ *and that the* $g_{u,v}$s, $g'_{u,v}$s, *and* (F,Φ,μ,ξ) *satisfy the coherence conditions* (A) - (D), *there is a unique homomorphism of categorical groups* $(G,\Psi,\nu,\eta): \overline{CZ}_{W_o} \to \mathcal{D}$ *such that* $(G,\Psi,\nu,\eta) \circ (\overline{T},id,id,id) = (F,\Phi,\mu,\xi)$ *and* $G(f_{u,v}) = g_{u,v}$, $G(f'_{u,v}) = g'_{u,v}$ *for each* $(u,v) \in S$. □

The meaning of the above proposition is that, under the more restrictive coherence conditions (A) - (D), $\overline{CZ}_{W_o}$ is a quotient category of CZ as categorical groups.

Similarly to the treatment for CZ_{W_o} (cf. section 3), one can also consider natural transformations between homomorphisms (F,Φ,μ,ξ) and $(\overline{F},\overline{\Phi},\overline{\mu},\overline{\xi})$ from CZ to $\mathcal{D}$. Only, we consider group-like natural transformations. By a *group-like natural transformation* $\sigma: (F,\Phi,\mu,\xi) \to (\overline{F},\overline{\Phi},\overline{\mu},\overline{\xi})$ we mean a monoidal natural transformation, $\sigma: (F,\Phi,\mu) \to (\overline{F},\overline{\Phi},\overline{\mu})$ such that, in addition, the following diagram is commutative for any object W

$$\begin{array}{ccc} F(W^{\sim}) & \xrightarrow{\xi_W} & (FW)^{\sim} \\ {\scriptstyle \sigma_{W^{\sim}}}\downarrow & & \uparrow{\scriptstyle \tilde{\sigma}_W} \\ \overline{F}(W^{\sim}) & \xrightarrow{\overline{\xi}_W} & (\overline{F}W)^{\sim} \end{array}$$

(cf. also [12], II, 2, diagram (D.11)). Of course, the definition is valid for arbitrary categorical groups.

Thus, let, in addition to (F,Φ,μ,ξ), a homomorphism $(\overline{F},\overline{\Phi},\overline{\mu},\overline{\xi})$ of categorical groups be given, and isomorphisms $\overline{g}_{u,v}$ and $\overline{g}'_{u,v}$, similar to the $g_{u,v}$s and $g'_{u,v}$s, such that conditions (A) - (D) are satisfied, and let $\sigma\colon (F,\Phi,\mu,\xi) \to (\overline{F},\overline{\Phi},\overline{\mu},\overline{\xi})$ be a group-like natural transformation satisfying condition (T) of section 3. If $(\overline{G},\overline{\Psi},\overline{\nu},\overline{\eta})$ is the homomorphism $C\mathbb{Z}_{W_0} \to \mathcal{D}$ corresponding by Proposition 2 to $(\overline{F},\overline{\Phi},\overline{\mu},\overline{\xi})$, then there is a unique group-like natural transformation $\tau\colon (G,\Psi,\nu,\eta) \to (\overline{G},\overline{\Psi},\overline{\nu},\overline{\eta})$ such that $\tau\circ\overline{T} = \sigma$. Indeed, τ is defined by $\tau_W = \sigma_W$ for any word W.

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