Ann. sc. math. Québec, 1982, vol. 6, no 2, pp. 215-220

# BEHAVIOUR OF TRAJECTORIES OF $C_0$-SEMI-GROUPS (II)

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## Résumé

Dans ce travail, on fournit une extension de quelques définitions et propriétés simples dans la théorie des systèmes dynamiques qui s'applique dans une étude des trajectoires complètes des semi-groupes de classe $C_o$ dans un espace de Banach .


## Introduction

In this paper we extend some simple definitions and properties of dynamical systems theory (as exposed in [5]) in order to cover (complete) trajectories of strongly continuous semi-groups of linear operators in a Banach space. For some previous work on the same or on similar, related topics we refer to [1], [6], [3], [4].

1. Let us consider a Banach space $X$ , then a $C_o$-semi-group of linear operators in $X$ , $S(t)$ , $t \in [0,\infty) \to L(X;X)$ (see [2],[7] for definitions and main properties).

A rest-point for $S(t)$ is an element $x \in X$ such that $S(t)x = x$ , $\forall t \geq 0$ . Remark that the set of all rest-point of $S(t)$ is a closed set in $X$ .

[1] This research is supported by the N.S.E.R.C. of Canada

We first state the following

THEOREM 1. *Let* $x \in X$ *with following property: for any* $\delta > 0$ , *there is an* $y \in X$ , *such that* $\|y-x\| < \delta$ *and* $\|S(t)y-x\| < \delta$ , $\forall t > 0$ . *Then* $x$ *is a rest-point for* $S(t)$ . (Compare with Theorem 2.10 in [5]; the present result is "almost" identical.)

COROLLARY. *Let us assume that* $x = \lim_{t\to\infty} S(t)\overline{y}$ , *for* $\overline{y} \in X$ . *Then* $x$ *is a rest-point for* $S(t)$ .

Let us give now the following

DEFINITION. A (complete) trajectory of the semi-group $S(t)$ is a function $x(\sigma)$ , $R \to X$ , verifying the (functional) relation $x(\sigma) = S(\sigma-t)x(t)$ for any $t \in R$ and $\forall\sigma \geq t$ .

Any complete trajectory is a (strongly) continuous function, $\mathbb{R} \to X$ . The following is true:

PROPOSITION 1. *A point* $\overline{x} \in X$ *is a rest-point for* $S(t)$ *if and only if the function* $x(t)$ , $R \to X$ , *given by* $x(t) = \overline{x}$ , $\forall t \in R$ *is a complete trajectory of* $S(t)$ .

PROOF. i) Assume that $S(t)\overline{x} = \overline{x}$ , $\forall t \in R^+$ . Then, for $a \in R$ , $\tau \geq 0$ , $x(a+\tau) = \overline{x}$ and $S(\tau)x(a) = S(\tau)\overline{x} = \overline{x}$ , $\forall\tau \geq 0$ . Therefore $x(a+\tau) = S(\tau)x(a)$ , $a \in R$ , $\tau \geq 0$ . If $a + \tau = \sigma$ , we get $x(\sigma) = S(\sigma-a)x(a)$ , $\forall\sigma \geq a$ , $\forall a \in R$ .

ii) Let us assume that $x(t) = \overline{x}$ , $\forall t \in R$ is a trajectory of $S(t)$ . It results that $x(a+\tau) = S(\tau)x(a)$ , $\forall a \in R$ , $\forall\tau \geq 0$ . Therefore $\overline{x} = S(\tau)\overline{x}$ , $\forall\tau \geq 0$ , as requested.

2. We way that a subset of $X$ , say $A \subset X$ is invariant under $S(t)$ if $S(t)a \in A$ $\forall a \in A$ , $\forall t \in R^+$ . It is immediate that closure of invariant sets is again an invariant set. Remark also

PROPOSITION 2. *Any complete trajectory of the semi-group* $S(t)$ *is an invariant set in* $X$ .

Let $x(t)$ , $R \to X$ be a complete trajectory of $S(t)$ . Let be $y = x(\bar{t})$ , $\bar{t}$ given in $R$ . Then $S(t)y = S(t)x(\bar{t}) = x(t+\bar{t})$ , $\forall t \geq 0$ , as requested.

As usual, a point $y \in X$ is an $\omega$-limit point of a complete trajectory $x(t)$ of a $C_o$-semi-group if $\exists (t_n)_1^\infty$ - a monotone increasing sequence of reals convergent to $+\infty$ , such that $\lim_{n\to\infty} x(t_n) = y$ . Also, $z \in X$ is an $\alpha$-limit point of the complete trajectory $x(t)$ , when $\exists (t_n')_1^\infty$ - a monotone decreasing sequence of reals convergent to $-\infty$ , such that $\lim_{n\to\infty} x(t_n') = z$ . One denotes:

$$\Omega x(\cdot) = \{\text{set of all } \omega\text{-limit points for } x(t)\} ;$$

$$A x(\cdot) = \{\text{set of all } \alpha\text{-limit points for } x(t)\} .$$

We state the following:

THEOREM 2. *For any complete trajectory* $x(t)$ *of a* $C_o$*-semi-group* $S(t)$ , *both sets* $Ax(\cdot)$ *and* $\Omega x(\cdot)$ *are closed and invariant.*

We give a proof for $Ax(\cdot)$ only. Take therefore an element $y \in Ax(\cdot)$ : there exists a decreasing sequence of reals, convergent to $-\infty$ , $(t_n')_1^\infty$ , such that $y = \lim_{n\to\infty} x(t_n')$ . For any $\bar{t} \in R^+$ we have $S(\bar{t})y = \lim_{n\to\infty} S(\bar{t})x(t_n') = \lim_{n\to\infty} x(\bar{t}+t_n')$ . But $(\bar{t}+t_n')_1^\infty$ is also a decreasing sequence convergent to $-\infty$ , thus $S(\bar{t})y \in Ax(\cdot)$ as requested. Let now $y \in \overline{Ax(\cdot)}$ ; there is a sequence $(y_n)_1^\infty \subset Ax(\cdot)$ , $y_n \to y$ as $n \to \infty$ . Also, $\forall n = 1,2,\dots$ , there exists a decreasing sequence convergent to $-\infty$ , $(t_n^k)_{k=1}^\infty$ , such that $\lim_{k\to\infty} x(t_n^k) = y_n$ , $n = 1,2,\dots$ Choose now $t_1^{k_1} < -1$ such that $\|x(t_1^{k_1})-y_1\| < \frac{1}{2}$ . Then take $t_2^{k_2} < \min(t_1^{k_1},-2)$ , such that $\|x(t_2^{k_2})-y_2\| < \frac{1}{2^2}$ . Also $t_3^{k_3} < \min(t_2^{k_2},-3)$ , such that $\|x(t_3^{k_3})-y_3\| < \frac{1}{2^3}$ , and so on.

The sequence $(\tau_n)_1^\infty = (t_n^{k_n})_{n=1}^\infty$ is obviously a decreasing sequence convergent to $-\infty$ . Furthermore,

$$\|x(\tau_n)-y\| \leq \|x(\tau_n)-y_n\| + \|y_n-y\| < \frac{1}{2^n} + \|y_n-y\| \to 0$$

as $n \to \infty$ . This proves Theorem 2.

Remark that $\Omega x(\cdot) \subset \overline{\{x(t)\}_{t\in R^+}}$ , $Ax(\cdot) \subset$ closure $\{x(t)\}_{t\in R^-}$ . Also, if the set $\{x(t)\}_{t\in R^+}$ is relatively compact, $\Omega x(\cdot)$ is a non-void compact set in $X$ , and if $\{x(t)\}_{t\in R^-}$ is relatively compact, $Ax(\cdot)$ is also a non-void compact set in $X$ . We see that if $x(t) = \bar{x}$ , $\forall t \in R$ , $\Omega x(\cdot) = Ax(\cdot) = \{\bar{x}\}$ ; if $x(t) \to \bar{x}$ as $t \to -\infty$ , then $Ax(\cdot) = \{\bar{x}\}$ and $\bar{x}$ is a rest-point for $S(t)$ . This last assertion because, $\forall t \geq 0$ , $S(t)\bar{x} = \lim_{n\to\infty} S(t)x(t_n) = \lim_{n\to\infty} x(t+t_n) = \bar{x}$ , $\{t_n\}_1^\infty$ being any sequence decreasing to $-\infty$ .

We have also

PROPOSITION 3. *Let* $x(t)$ , $R \to X$ *be a periodic complete trajectory of* $S(t)$ — *i.e.* $\exists\omega > 0$ *such that* $x(t+\omega) = x(t)$ , $\forall t \in R$ — . *Then* $Ax(\cdot) = \Omega x(\cdot) = \{x(t)\}_{t\in R}$ . (Cf. [5], Th. 3.04 .)

PROOF. i) Let $y = x(t_o)$ . Then $y = x(t_o \pm n\omega)$ , $n = 1,2,\dots$ ; thus $y \in \Omega x(\cdot) \cap Ax(\cdot)$ .

ii) Let, say, $y \in Ax(\cdot)$ ; then $y = \lim_{n\to\infty} x(t_n)$ , where $t_n \downarrow -\infty$ . We write $t_n = k_n\omega + t'_n$ where $k_n$ is a convenient integer and $0 \leq t'_n < \omega$ : we may assume that $t'_n \to t_o \in [0,\omega]$ . Then $x(t_n) = x(t'_n) \to x(t_o)$ . Therefore $y = x(t_o)$.

Let us give now the following

DEFINITION. A $C_o$-semi-group $S(t)$ has the backward uniqueness property if any two complete trajectories having the same value for $t = t_o \in R$ coincide for any other $t$ . (Compare [8], page 248.)

Our last result is

PROPOSITION 4. *Let us assume that* $S(t)$ *has the backward uniqueness property and that* $x(t)$ *is a trajectory such that* $x(t_1) = x(t_2)$ *for a couple* $t_1 \neq t_2$ . *Then* $x(t)$ *is a periodic trajectory with period* $t_2 - t_1$ .

First observe that for any fixed $a \in R$, $y_a(t) = x(t+a)$ is again a complete trajectory of $S(t)$; (we have $x(t+\tau) = S(\tau)x(t)$, $\forall\tau \geq 0$, $t \in R$; therefore $y_a(t+\tau) = x(t+\tau+a) = S(\tau)x(t+a) = S(\tau)y_a(t)$, $\forall\tau \geq 0$, $t \in R$). In particular, for $a = t_2 - t_1$, $x(t+t_2-t_1)$ is complete trajectory of $S(t)$, as well as $x(t)$. For $t = t_1$, the first equals $x(t_2)$, the second equals $x(t_1) = x(t_2)$. Therefore $x(t+t_2-t_1) = x(t)$, $\forall t \in R$.

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*Manuscrit reçu le 14 juillet 1981.*