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GENERAL SOLUTION METHODS FOR FREDHOLM INTEGRAL EQUATIONS WITH KERNEL $K(z,z_0) = g(zz_0)$ ON THE INTERVAL $(0, \infty)$

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Résumé

Les équations intégrales de Fredholm de première espèce de la forme

$$\varphi(z) = \int_0^\infty K(z,z_0)\chi(z_0)\, dz_0 ,$$

où $K(z,z_0) = g(zz_0)$ peuvent être résolues avec une transformée de Laplace par rapport à la variable z, donnant $f(z) \longleftrightarrow F(s)$, suivie par une transformée de Mellin par rapport à s de même que z_0. La méthode est illustrée pour le noyau $K(z,z_0) = (4/\pi)\sin^4(zz_0/2)$. Les équations de Fredholm de deuxième ou de première espèce peuvent souvent être résolues par la substitution $z_0 = 1/u$, et en employant des transformées de Mellin partielles; cependant cette méthode est souvent plus fastidieuse que la méthode mentionnée ci-haut.

Abstract

Fredholm integral equations of the first kind, of the type

$$\varphi(z) = \int_0^\infty K(z,z_0)\chi(z_0)\, dz_0 ,$$

where $K(z,z_0) = g(zz_0)$, can generally be solved by applying first a Laplace transform with respect to the variable z, mapping $f(z) \longleftrightarrow F(s)$, and subsequently a Mellin transform with respect to both s and z_0. The method is illustrated for the kernel $K(z,z_0) = (4/\pi)\sin^4(zz_0/2)$. Fredholm equations of the first or second

kind can often be solved by the substitution $z_0 = 1/u$, using further partial Mellin transforms; this method is usually more cumbersome than the aforementioned method.

1. Introduction

Integral equations of the Fredholm and Volterra types, can often be solved using the convolution theorem of an appropriate integral transform pair. Thus, Fredholm equations, both of the first and second kind, with kernel $v(z \pm z_0)$ on the interval $(-\infty,\infty)$ are solvable by Fourier transform of the entire equation (i.e. with respect to both z and z_0), with an extension of the method on the interval $(0,\infty)$ by the Wiener-Hopf procedure; Volterra equations with kernels $v(x-x_0)$ on the interval $(0,x)$ or (x,∞) are solvable by Laplace transform of the entire equation; Fredholm equations of the first or second kind with kernel $v(z/z_0)z_0^{-1}$ on the interval $(0,\infty)$ are solvable by Mellin transform of the entire equation. These methods are well documented in the literature, see e.g. Morse and Feshbach [1], Green [2].

Recently in the theory of spectral densities we came upon a Fredholm integral equation of the first kind with kernel $\sin^4(zz_0/2)$ and we formulated its general solution [3]. It then appeared to us that all Fredholm integral equations of the first kind, with kernel $g(zz_0)$ are solvable by the same procedure. Now, of course, these equations can formally be carried over in those with a kernel $v(z/u_0)u_0^{-1}$, by the substitution $z_0 = 1/u_0$; however, the function $u_0^{-1}\chi(1/u_0)$ on which the kernel now works often renders the method cumbersome, see section 4. For that reason, we also present another technique, set forth in section 2. In section 3 we give an example.

2. The method

We consider the equation

$$\varphi(z) = \int_0^\infty g(zz_0)\chi(z_0)\, dz_0 \ . \tag{2.1}$$

Let $\Phi(s)$ be the Laplace transform of $\varphi(z)$, and let $G(s)$ be the Laplace transform of $g(w)$; then,

$$\int_0^\infty g(zz_o)e^{-sz}\,dz = \int_0^\infty g(w)e^{-(s/z_o)w}\,dw/z_o = G(s/z_o)z_o^{-1} \,. \tag{2.2}$$

Consequently, assuming that the Laplace integral and the integral in (2.1) are interchangeable, we obtain

$$\Phi(s) = \int_0^\infty G(s/z_o)\chi(z_o)\,\frac{dz_o}{z_o} \tag{2.3}$$

which is exactly of the form required for the applicability of the Mellin transform. Thus, taking the Mellin transform of both members in (2.3), we recall the convolution theorem

$$M\int_0^\infty G(\frac{s}{z_o})\chi(z_o)\,\frac{dz_o}{z_o} \equiv \int_0^\infty ds\; s^{p-1}\int_0^\infty G(\frac{s}{z_o})\chi(z_o)\,\frac{dz_o}{z_o} = V(p)\mathcal{X}(p) \;, \tag{2.4}$$

where

$$V(p) = \int_0^\infty ds\; s^{p-1}G(s) < \infty \quad \text{for} \quad \alpha < \text{Re } p < \gamma \tag{2.5}$$

and

$$\mathcal{X}(p) = \int_0^\infty ds\; s^{p-1}\chi(s) < \infty \quad \text{for} \quad \bar{\sigma}_o < \text{Re } p < \bar{\tau}_o \,. \tag{2.6}$$

Let also $F(p)$ be the Mellin transform of $\Phi(s)$ [1]

$$F(p) = \int_0^\infty ds\; s^{p-1}\Phi(s) < \infty \quad \text{for} \quad \sigma_o < \text{Re } p < \tau_o \tag{2.7}$$

then eq. (2.3) is transformed to

$$F(p) = V(p)\mathcal{X}(p) \quad , \quad \text{Re } p \in \text{common domain of analyticity.} \tag{2.8}$$

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[1] Usually one assumes for the existence that the Lebesgue integral $\int_0^\infty ds\; |\Phi(s)|^2 s^{2\beta-1} < \infty$ for $\sigma_o < \beta < \tau_o$; the domain of analyticity of the transform is then $\sigma_o < \text{Re } p < \tau_o$. This condition is sufficient, but not necessary, however, as is exemplified by $M \sin z = \Gamma(p)\sin p\pi/2$, $\text{Re } p > -1$.

In order for (2.8) to be valid there must be an overlap of the domains specified in (2.5), (2.6) and (2.7), see Fig. 1. The validity of (2.8) in the rest of the complex plane, except at poles or branch points, is obtained by analytic continuation of the functions involved. Let β' be a real number from the common domain of analyticity; the inversion of (2.8) yields

$$\chi(z) = \frac{1}{2\pi i}\int_{\beta'-i\infty}^{\beta'+i\infty} \frac{dp}{z^p}\frac{F(p)}{V(p)} . \tag{2.9}$$

This integral is usually obtained by closing the line $\mathrm{Re}\ p = \beta'$ with a suitable contour, see Fig. 1.

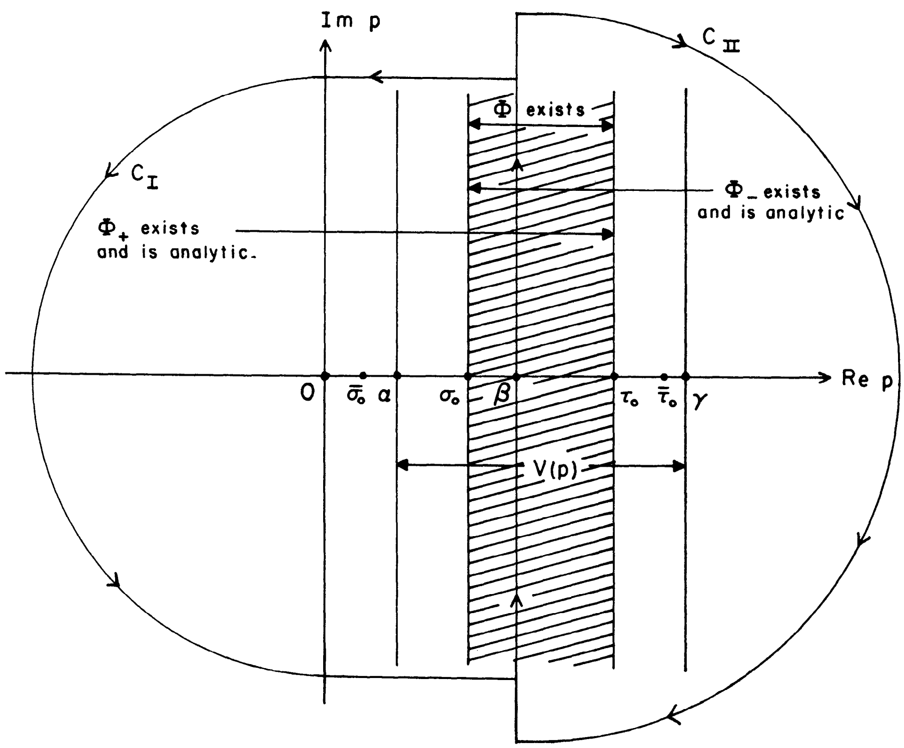

Fig. 1

The case $\sigma_o < \tau_o$ for which the domain of analyticity of the full Mellin transform overlaps with the domain of $V(p)$. The shaded area gives the analyticity of the transformed equation. The inverse is along $\mathrm{Re}\ p = \beta'$ with $\sigma_o < \beta' < \tau_o$.

We note that (2.9) has poles where $F(p)$ has poles (that are not poles of $V(p)$) or where $V(p)$ has zeros (which are not also zeros of $F(p)$). The zeros of $V(p)$ are particularly noteworthy when they provide poles that fall within the common region of analyticity of $V(p)$ and $\Phi(p)$, for they then determine the limits $\bar{\sigma}_o$ and $\bar{\tau}_o$ which were initially unknown, and β' must be chosen from

$$\max(\sigma_o, \bar{\sigma}_o, \alpha) < \beta' < \min(\tau_o, \bar{\tau}_o, \gamma) \quad . \tag{2.10}$$

There may be some ambiguity in associating the limits $\bar{\sigma}_o$ or $\bar{\tau}_o$ with one of the above mentioned poles; generally, if the domain due to F and V is limited only on one side, say $\min(\tau_o, \gamma)$, β' should be taken between this bound and the first pole prior to this bound, which refers to $\bar{\sigma}_o$.

In many cases the full Mellin transform does not exist for any p, since the integral is improper at both ends. We must then split $\Phi(s)$ into two parts and likewise for χ such that

$$\Phi(s) = \Phi_-(s)U(1-s) + \Phi_+(s)U(s-1) \quad , \tag{2.11}$$

where $U(x) = 0$ for $x < 1$ and $U(x) = 1$ for $x > 1$. We then introduce the Mellin transforms for the two parts, a procedure well described in Morse and Feshbach [2]:

$$F_-(p) = \int_0^\infty ds\; s^{p-1}\Phi(s)U(1-s) = \int_0^1 ds\; s^{p-1}\Phi_-(s) \quad , \tag{2.12a}$$

$$F_+(p) = \int_0^\infty ds\; s^{p-1}\Phi_+(s)U(s-1) = \int_1^\infty ds\; s^{p-1}\Phi_+(s) \quad ; \tag{2.12b}$$

clearly

$$\mathcal{M}\Phi(s) = F_-(p) + F_+(p) \quad . \tag{2.13}$$

Similarly we introduce

$$\mathcal{X}_-(p) = \int_0^1 ds\; s^{p-1}\chi_-(s) \tag{2.14a}$$

$$\mathcal{X}_+(p) = \int_1^\infty ds\; s^{p-1}\chi_+(s) \quad . \tag{2.14b}$$

Let now

$$\int_0^1 ds \ |\Phi_-(s)|^2 s^{2\sigma-1} < \infty \quad \text{for} \quad \sigma > \sigma_o \ , \tag{2.15a}$$

and

$$\int_1^\infty ds \ |\Phi_+(s)|^2 s^{2\tau-1} < \infty \quad \text{for} \quad \tau < \tau_o \ , \tag{2.15b}$$

$$\int_0^1 ds \ |\chi_-(s)|^2 s^{2\bar{\sigma}-1} < \infty \quad \text{for} \quad \bar{\sigma} > \bar{\sigma}_o \ , \tag{2.16a}$$

$$\int_1^\infty ds \ |\chi_+(s)|^2 s^{2\bar{\tau}-1} < \infty \quad \text{for} \quad \bar{\tau} < \bar{\tau}_o \ . \tag{2.16b}$$

Since the full Mellin transform does not exist, this implies that $\tau_o \le \sigma_o$, and $\bar{\tau}_o \le \bar{\sigma}_o$ in contrast to the inequalities figuring in (2.6) and (2.7). Equation (2.8) is now meaningful if the lower bound α , set by the existence of V(p) , is less than τ_o and $\bar{\tau}_o$ while the higher bound γ , set by the existence of V(p) , is greater than σ_o , and $\bar{\sigma}_o$. We can then choose a σ' and τ' from the region common for the existence of V(p) and Φ_- , χ_- or Φ_+ , χ_+ , respectively, see Fig. 2. Thus the region of analyticity consists of two strips near α and γ . In the rest of the plane, except at poles or branchpoints the transformed integral equation is validated by analytic continuation. Equation (2.8) now reads explicitly,

$$F_-(p) + F_+(p) = V(p)[\chi_-(p) + \chi_+(p)] \ . \tag{2.17}$$

Using a theorem of Morse and Feshbach (op cit p. 463) one can conclude that

$$F_-(p) - V(p)\chi_-(p) = -F_+(p) + V(p)\chi_+(p) = S(p) \ , \tag{2.18}$$

where S(p) is analytic in the entire region $\alpha < \text{Re } p < \beta$. The function S(p) is the Mellin transform of the solution of the homogeneous equation,

$$\int_0^\infty g(zz_o) \ \Sigma \ (z_o) \ dz_o = 0 \ . \tag{2.19}$$

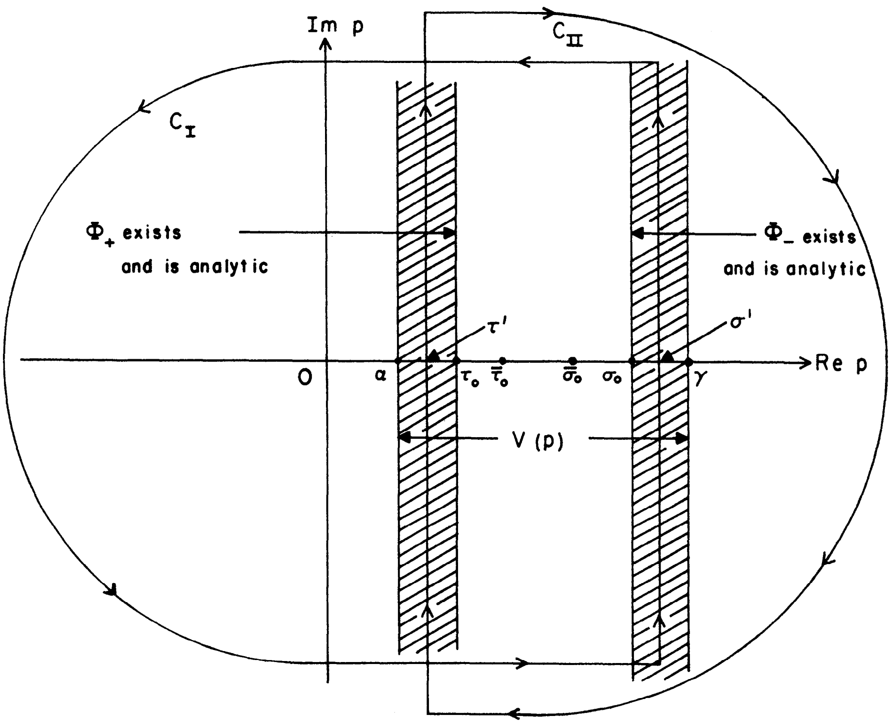

Fig. 2

Shaded areas are domains of analyticity for the transformed equation. We pictured the case $\sigma_o > \tau_o$; outside the shaded areas the equation can be validated by analytical continuation except at poles and branch points. For $\sigma_o = \tau_o$ the same situation prevails since $\tau' < \tau_o = \sigma_o < \sigma'$ gives still different contours for the inverse.

Generally, the kernel will not have an eigenvalue zero, in which case $\Sigma \equiv 0$, $S \equiv 0$. Thus from (2.18),

$$\chi(z) = \frac{1}{2\pi i}\int_{\sigma'-i\infty}^{\sigma'+i\infty} \frac{dp}{z^p}\frac{F_-(p)}{V(p)} + \int_{\tau'-i\infty}^{\tau'+i\infty}\frac{dp}{z_o^p}\frac{F_+(p)}{V(p)} , \tag{2.20}$$

where σ' and τ' are respectively, from the strips

$$\max(\bar{\sigma}_o,\sigma_o) < \sigma' < \gamma \quad \text{and} \quad \min\alpha < \tau' < \min(\bar{\tau}_o,\tau_o) \ . \tag{2.21}$$

Since $\chi_\pm(p)$ and $F_\pm(p)$ are analytic outside the interval (α,β) there must be no poles in these regions caused by zeros of $V(p)$. Under this conditions, (2.20) provides the complete solution. An example is given in the next section.

3. The kernel $g(zz_o) = (4/\pi)\sin^4(zz_o/2)$

In Ref. 3 we considered the solutions of

$$\varphi(z) = \frac{4}{\pi}\int_0^\infty \sin^4(zz_o/2)\chi(z_o)\,dz_o \ , \tag{3.1}$$

for various functions $\chi(z)$. The Laplace transformed kernel (with respect to z) is

$$\int_0^\infty dz\, e^{-sz}\sin^4(zz_o/2) = \frac{3}{2}\,\frac{z_o^4}{(s^2+4z_o^2)(s^2+z_o^2)s} \tag{3.2}$$

so that (3.1) is carried over into

$$\Phi(s) = \frac{6}{\pi}\int_0^\infty \frac{dz_o}{z_o}\,\frac{1}{\frac{s}{z_o}(\frac{s^2}{z_o^2}+4)(\frac{s^2}{z_o^2}+1)}\,\chi(z_o) \ , \tag{3.3}$$

which is of the form required for the Mellin convolution theorem. For the Mellin transform of the new kernel we have

$$V(p) = \frac{6}{\pi}\int_0^\infty ds\,\frac{s^{p-2}}{(s^2+4)(s^2+1)} = -\,\frac{1-2^{p-3}}{\cos p\pi/2} \ , \quad 1 < \mathrm{Re}\ p < 5 \ . \tag{3.4}$$

Thus, in Fig. 1, now $\alpha = 1$, $\gamma = 5$. Equation (3.4) is to be substituted into (2.9) or (2.20).

We consider first the case that

$$\varphi(z) = \eta^{-2}(4e^{-\eta z}-e^{-2\eta z}+2\eta z-3) \quad , \quad \eta > 0 \ . \tag{3.5}$$

$$\Phi(s) = 4\left[\frac{2\eta-s}{s^2(s^2-4\eta^2)} + \frac{s-\eta}{s^2(s^2-\eta^2)}\right] ; \tag{3.6}$$

the full Mellin transform of $\Phi(s)$ exists [4]:

$$F(p) = \frac{2\pi\eta^{p-3}(1-2^{p-3})}{\sin p\frac{\pi}{2} \cos p\frac{\pi}{2}} \quad , \quad 2 < \text{Re } p < 4 \ . \tag{3.7}$$

(Thus, for this case $2 = \sigma_o < \tau_o = 4$.) There is an overlap of the domains indicated in (3.7) and (3.4); $V(p)$ has no zeros, so we take β' such that $2 < \beta' < 4$. The poles of $F(p)/V(p)$ occur at the zeros of $\sin(p\pi/2)$, i.e. at $p = 2m$, $m = 0,\pm 1,\pm 2,\ldots$ Explicitly, from (3.4), (3.7) and (2.9) we have

$$\chi(z) = \frac{i}{\eta z^2} \int_{\beta'-i\infty}^{\beta'+i\infty} dp \ (\frac{\eta}{2})^{p-2} \frac{1}{\sin \frac{1}{2}p\pi} \ . \tag{3.8}$$

For $z < \eta$ we close the contour with a semicircle in the left half plane, and for $z > \eta$ with a semicircle in the right half plane; the poles to be included are $m = 0,1,-1,-2,\ldots$ or $m = 2,3$, etc. respectively. In both cases the residue series is summable, and we obtain a Lorentzian:

$$z^2\chi(z) = \frac{4\eta}{\eta^2 + z^2} \ . \tag{3.9}$$

Secondly, let

$$\varphi(z) = Kz^{\mu} \quad , \quad 0 < \mu < 4 \ . \tag{3.10}$$

$$\Phi(s) = K\Gamma(\mu+1)s^{-\mu-1} \quad , \quad \text{Re } s > 0 \ . \tag{3.11}$$

The partial Mellin transforms are

$$F_-(p) = \frac{K}{p - \mu - 1}\Gamma(\mu+1) \quad , \quad \text{Re } p > \sigma_o = 1 + \mu \ , \tag{3.12}$$

$$F_+(p) = -\frac{K}{p - \mu - 1}\Gamma(\mu+1) \quad , \quad \text{Re } p < \tau_o = 1 + \mu \ . \tag{3.13}$$

With $0 < \mu < 4$, the σ_o and τ_o are located as in Fig. 2. For (2.19) we now obtain

$$\chi(z) = -\frac{1}{2\pi i}\int_{\sigma'-i\infty}^{\sigma'+i\infty} \frac{dp}{z^p} \frac{K\Gamma(\mu+1)\ \cos(p\pi/2)}{(p-\mu-1)(1-z^{p-3})}$$

$$+ \int_{\tau'-i\infty}^{\tau'+i\infty} \frac{dp}{z^p} \frac{K\Gamma(\mu+1)\ \cos(p\pi/2)}{(p-\mu-1)(1-z^{p-3})} \ , \tag{3.14}$$

where σ' is such that $1 + \mu < \sigma' < 5$ and τ' is such that $1 < \tau' < 1 + \mu$; the situation is as sketched in Fig. 2. Note that $p = 3$ does not provide a pole since the numerator is also zero, and the ratio is finite. Thus there is only one pole at $p = \mu + 1$, occurring as expected outside the domains of analyticity.

For $z < 1$, the contours are closed with a semicircle in the left half plane. The integral over $F_+(p)/V(p)$ now vanishes since the integrand is analytic. From the other integral one obtains

$$\chi(z) = -\frac{K}{z^{\mu+1}} \lim_{p\to 1+\mu} \frac{\cos\frac{1}{2}p\pi}{1 - z^{p-3}} \Gamma(\mu+1) . \tag{3.15}$$

Hence for $\mu \neq 2$:

$$\chi(z) = \frac{K}{z^{\mu+1}} \frac{\sin\frac{1}{2}\mu\pi}{1 - 2^{\mu-2}} \Gamma(\mu+1) \quad , \quad \mu \neq 2 . \tag{3.16}$$

For $\mu = 2$ one finds from de l'Hôpital's rule

$$\chi(z) = \pi/z^3 \log 2 . \tag{3.17}$$

For $z > 1$, the same result is obtained by closing the contours with a semicircle in the right half plane. We note that the functions $\chi(z)$ obtained are highly singular at the origin, yet the transform method works quite well.

4. Remarks on Fredholm equations of the second kind

The equation

$$\chi(z) = \varphi(z) + \lambda \int_0^\infty g(zz_0)\chi(z_0)\, dz_0 \tag{4.1}$$

cannot be solved by the techniques of section 2. The only method here is substitution of $u = 1/z_0$. Then (4.1) becomes

$$\chi(z) = \varphi(z) + \lambda \int_0^\infty g(\frac{z}{u})\chi(\frac{1}{u}) \frac{1}{u}\frac{du}{u} . \tag{4.2}$$

With $\chi(1/u)u^{-1} \equiv \psi(u)$ this equation is now in a form that the convolution theorem of the Mellin transform can be employed.

First we consider again the case that the full Mellin transform of $\varphi(z)$ exists. Thus let

$$\Phi(p) = \int_0^\infty dz\ z^{p-1}\varphi(z) \equiv M\varphi(z) < \infty \quad \text{for} \quad \sigma_o < \text{Re } p < \tau_o \ . \tag{4.3}$$

Let further the Mellin transform for $\chi(z)$ be defined by

$$\mathcal{X}(p) = M\chi(z) < \infty \quad \text{for} \quad \bar{\sigma}_o < \text{Re } p < \bar{\tau}_o \ . \tag{4.4}$$

From this we also find

$$\mathcal{X}(1-p) = \int_0^\infty dz\ z^{-p}\chi(z) = \int_0^\infty du\ u^{p-1}\chi(1/u)u^{-1} = M\chi(1/z)z^{-1} \ . \tag{4.5}$$

Further for the kernel

$$W(p) = Mg(z) < \infty \quad , \quad \alpha < \text{Re } p < \gamma \ . \tag{4.6}$$

We require that there is a domain of overlap for analyticity of (4.3), (4.4) and (4.6). The analyticity of $\chi(1-p)$ for the Mellin transform (4.5) is then established by the transformed equation. Thus, we consider the domain of analyticity

$$\max(\bar{\sigma}_o,\sigma_o,\alpha) < \text{Re } p < \min(\bar{\tau}_o,\tau_o,\gamma) \ . \tag{4.7}$$

It is noted that the bounds $\bar{\sigma}_o$ and $\bar{\tau}_o$ are not known. We may therefore want to initially ignore the $\bar{\sigma}_o$, $\bar{\tau}_o$ requirement in fixing a tentative domain, which then later on may have to be reduced at one or both sides. If $\bar{\sigma}_o,\sigma_o > \alpha$ and $\bar{\tau}_o,\tau_o < \gamma$, we have the situation as pictured in Fig. 1. Applying now the Mellin transform to both sides of (4.2), we find with the convolution theorem (2.4):

$$\mathcal{X}(p) = \Phi(p) + \lambda W(p)\mathcal{X}(1-p) \ ; \tag{4.8}$$

the domain of initial validity of this equation is as in (4.7), see the shaded area in Fig. 1; outside of this area the result (4.8) is validated by analytical continuation except at poles or branch points.

In order to solve (4.8) we also write down the result for $1 - p$:

$$\chi(1-p) = \Phi(1-p) + \lambda W(1-p)\chi(p) \; ; \tag{4.9}$$

the analyticity of the functions in this equation must cover the same domain, i.e. $W(1-p)$ and $\Phi(1-p)$ must be analytic in the domain where both $W(p)$ and $\Phi(p)$ are analytic (if this is not the case, the method fails). Solving now from (4.8) and (4.9) for $\chi(p)$ we obtain

$$\chi(p) = \frac{\Phi(p) + \lambda\Phi(1-p)W(p)}{1 - \lambda^2 W(p)W(1-p)} \; . \tag{4.10}$$

Then by inversion, if β' is a value of $\mathrm{Re}\, p$ in the interval (4.7)

$$\chi(z) = \frac{1}{2\pi i}\int_{\beta'-i\infty}^{\beta'+i\infty} \frac{dp}{z^p}\,\frac{\Phi(p) + \lambda\Phi(1-p)W(p)}{1 - \lambda^2 W(p)W(1-p)} \; . \tag{4.11}$$

We note that if the condition for (4.9) is satisfied, the two terms in the numerator of the r.h.s. are both regular in the tentative domain of analyticity. We now consider the poles of the r.h.s., given by

$$1 - \lambda^2 W(p_r)W(1-p_r) = 0 \; . \tag{4.12}$$

If there are no poles in the tentative domain, then $\chi(z)$ is regular and the tentative domain is the definite domain satisfying (4.7). This means that $\bar{\sigma}_o$ is lying to the left or coincides with $\max(\sigma_o,\alpha)$ and $\bar{\tau}_o$ is lying to the right or coincides with $\min(\tau_o,\gamma)$. If there are poles within the tentative domain, then the definite domain is construed by one of the bands between adjacent poles which fall entirely within the tentative domain (again there is some ambiguity in the choice, as for the case of Eq. (2.9)), or by a partial band composed of $\max(\sigma_o,\alpha)$ and the first pole to the right or composed of $\min(\tau_o,\gamma)$ and the first pole to the left. The line $\mathrm{Re}\, p = \beta'$ must be within this band or partial band. We thus note that, since the poles are a function of λ , it is the choice of λ which determines *a posteriori* the bounds $\bar{\sigma}_o$ and $\bar{\tau}_o$. To the solution (4.11) we must add the solutions of the homogeneous equation, see below. If the number of poles is infinite there are an infinite number of solutions of the homogeneous equation.

We illustrate the result for the kernel $g(zz_0) = (4/\pi)\sin^4(zz_0/2)$ of section 3. Its Mellin transform exists; in fact, from the second example in section 3 one easily obtains,

$$W(p) = \frac{4}{\pi}\int_0^\infty dz\, z^{p-1}\,\sin^4(z/2) = -\,\frac{1 - 2^{-p-2}}{\sin p\frac{\pi}{2}\Gamma(1-p)}$$

$$= -\,\frac{1 - 2^{-p-2}}{\pi/2}\cos p\frac{\pi}{2}\Gamma(p) \quad , \quad -4 < \text{Re } p < 0 \; , \tag{4.13}$$

Where we used the reflection property

$$\Gamma(p)\Gamma(1-p) = \pi/\sin p\pi \; . \tag{4.14}$$

Taking $\varphi(z)$ as in (3.5) we find a Mellin transform[1]

$$\Phi(p) = 4\Gamma(p)\eta^{-p-2}(1-2^{-p-2}) \quad , \quad \sigma_0 = -3 \quad , \quad \tau_0 = -1 \; . \tag{4.15}$$

Using (4.13) and (4.14) we find for the poles

$$1 - \lambda^2 W(p)W(1-p) = 1 - \frac{2\lambda^2}{\pi}(1-2^{-p-2})(1-2^{p-3}) = A(p-p_1)(p-p_2) = 0 \; . \tag{4.16}$$

Let $2^p = x$ and let $2\lambda^2/\pi = a$. Then x_r is a solution of $4ax^2 + (32-33a)x + 8a = 0$. Suppose $0{,}722 < a < 1{,}476$. Then x_r is complex with modulus $|x| = \sqrt{x_1 x_2} = \sqrt{2}$. Hence for both roots, $\text{Re } p_r = \log|x|/\log 2 = \frac{1}{2}$. This is well outside the limits set by (4.15), which comprise therefore the definite domain. For the second term in the numerator of (4.11) we have

$$\lambda\Phi(1-p)W(p) = \frac{4\lambda\eta^{p-3}(1-2^{p-3})(1-2^{-p-2})}{\sin p\pi/2} \; ; \tag{4.17}$$

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[1] It is easily seen that the first three terms of a Taylor series for $\varphi(z)$ vanish, so $\varphi(z) = O(z^3)$ for $z \to 0$. For the individual terms of $\varphi(z)$ the Mellin transforms Φ_- and Φ_+ are found for stricter conditions, viz. $\text{Re } p > 0$ and $\text{Re } p < -1$, respectively. For the total expression Φ_- can now be analytically continued to $\text{Re } p > -3$, after which it overlaps with Φ_+ ; we can then find Φ by adding Φ_+ and Φ_- .

this is indeed analytic on $-3 < \text{Re}\ p < -1$, as it should. Hence (4.11) is a valid solution, β' being anywhere in $-3 < \beta' < -1$. The total collection of poles of the integrand in (4.11) is $\ldots,6,4,2,\frac{1}{2}\pm bi,0,-1,-3,-4,\ldots$ The line $\text{Re}\ p = \beta'$ in (4.11) is closed with a suitable contour as in Fig. 1.

Next, we consider the case that $\varphi(z)$ has only partial Mellin transforms. Thus, let analogous to (2.11)

$$\varphi(z) = \varphi_-(z)U(1-z) + \varphi_+(z)U(z-1)\ , \tag{4.18}$$

$$\chi(z) = \chi_-(z)U(1-z) + \chi_+(z)U(z-1)\ , \tag{4.19}$$

$$\chi(1/z) = \chi_+(1/z)U(1-z) + \chi_-(1/z)U(z-1)\ , \tag{4.20}$$

where we noted

$$U(1-1/z) = U(z-1)\ . \tag{4.21}$$

The relevant Mellin transforms are

$$\Phi_-(p) = \int_0^\infty dz\ z^{p-1}\varphi_-(z)U(1-z) = \int_0^1 dz\ z^{p-1}\varphi_-(z)\quad , \quad \text{Re}\ p > \sigma_o\ , \tag{4.22}$$

$$\Phi_+(p) = \int_0^\infty dz\ z^{p-1}\varphi_+(z)U(z-1) = \int_1^\infty dz\ z^{p-1}\varphi_+(z)\quad , \quad \text{Re}\ p < \tau_o\ , \tag{4.23}$$

$$\mathcal{X}_-(p) = M_-\chi_-(z)\quad , \quad \text{Re}\ p > \overline{\sigma}_o\ , \tag{4.24}$$

$$\mathcal{X}_+(p) = M_+\chi_+(z)\quad , \quad \text{Re}\ p < \overline{\tau}_o\ . \tag{4.25}$$

In addition we have,

$$\mathcal{X}_-(1-p) = \int_0^1 dz\ z^{-p}\chi_-(z) = \int_1^\infty du\ u^{p-1}\chi_-(1/u)u^{-1} = M_+\chi_-(1/z)z^{-1}\ , \tag{4.26}$$

$$\mathcal{X}_+(1-p) = \int_1^\infty dz\ z^{-p}\chi_+(z) = \int_0^1 du\ u^{p-1}\chi_+(1/u)u^{-1} = M_-\chi_+(1/z)z^{-1}\ . \tag{4.27}$$

The transformed equation now reads

$$\mathcal{X}_-(p) + \mathcal{X}_+(p) = \Phi_-(p) + \Phi_+(p) + \lambda W(p)[\mathcal{X}_+(1-p) + \mathcal{X}_-(1-p)] \tag{4.28}$$

or

$$\{\chi_-(p) - \Phi_-(p) - \lambda W(p)\chi_+(1-p)\} = \{-\chi_+(p) + \Phi_+(p) + \lambda W(p)\chi_-(1-p)\} \ . \tag{4.29}$$

The first { } are all M_- transforms of the relevant functions and the second { } are M_+ transforms of the relevant functions. We assume that there are strips of analyticity for the terms within the first accolades

$$\max(\sigma_o, \overline{\sigma}_o) < \text{Re } p < \gamma \tag{4.30a}$$

and for the terms within the second accolades,

$$\alpha < \text{Re } p < \min(\tau_o, \overline{\tau}_o) \ . \tag{4.30b}$$

The situation is as in Fig. 2.

For the inverse transforms we must choose lines $\text{Re } p = \sigma'$ and $\text{Re } p = \tau'$, with σ' and τ' within the strips of Eqs. (4.30a) and (4.30b), respectively. Equation (4.29) now leads to (see [1], loc. cit.)

$$\chi_-(p) - \lambda W(p)\chi_+(1-p) = \Phi_-(p) + S(p) \ , \tag{4.31}$$

$$\chi_+(p) - \lambda W(p)\chi_-(1-p) = \Phi_+(p) - S(p) \ , \tag{4.32}$$

where $S(p)$ is an analytic function in the entire domain $\alpha < \text{Re } p < \beta$. In Eq. (4.32) we change p into $(1-p)$. Again, this is only useful if the new functions are analytic in the strips, i.e. $\Phi_+(1-p)$ and $W(1-p)$ must be analytic in the strip (4.30a), where $\Phi_-(p)$ and $W(p)$ are analytic. We can then solve from (4.31) and the switched result (4.32) for $\chi_-(p)$ and $\chi_+(1-p)$; the latter also yields $\chi_+(p)$. So we finally arrive at

$$\chi_-(p) = \frac{\Phi_-(p) + \lambda\Phi_+(1-p)W(p)}{1 - \lambda^2 W(p)W(1-p)} \ , \tag{4.33}$$

$$\chi_+(p) = \frac{\Phi_+(p) + \lambda\Phi_-(1-p)W(p)}{1 - \lambda^2 W(p)W(1-p)} \ . \tag{4.34}$$

This yields for $\chi(z)$

$$\chi(z) = \frac{1}{2\pi i}\int_{\sigma'-i\infty}^{\sigma'+i\infty} \frac{dp}{z^p}\frac{\Phi_-(p) + \lambda\Phi_+(1-p)W(p)}{1-\lambda^2 W(p)W(1-p)}$$

$$+ \frac{1}{2\pi i}\int_{\tau'-i\infty}^{\tau'+i\infty} \frac{dp}{z^p}\frac{\Phi_+(p) + \lambda\Phi_-(1-p)W(p)}{1-\lambda^2 W(p)W(1-p)} + \chi_o(z) \ , \qquad (4.35)$$

where $\chi_o(z)$ is the solution of the homogeneous equation,

$$\chi_o(z) = \frac{1}{2\pi i}\int_C \frac{dp}{z^p}\frac{S(p) - \lambda S(1-p)W(p)}{1-\lambda^2 W(p)W(1-p)} \qquad (4.36)$$

where C is a counterclockwise closed contour consisting of the lines $\mathrm{Re}\ p = \tau'$ and $\mathrm{Re}\ p = \sigma'$. The solution is quite similar to that for kernels of the type $v(z+z_o)$, see [1, p. 970] .

As to the homogeneous solution, let the poles within the area of the contour be p_r (see below), and let

$$\frac{1}{1-\lambda^2 W(p)W(1-p)} = \frac{b_s}{(p-p_r)^s} + \frac{b_{s-1}}{(p-p_r)^{s-1}} + \dots \qquad (4.37)$$

The solutions are then of the form

$$\chi_o(z) = \sum_{r,s} A_{rs}(\log z)^{s-1} z^{-p_r} \ , \qquad (4.38)$$

where A_{rs} are constants to be determined.

Returning to (4.33) and (4.34) the r.h.s. has the same domain of analyticity as the l.h.s.; in particular, the numerators are analytic to the right of the bounds $\max(\bar{\sigma}_o,\sigma_o)$ and $\min(\bar{\tau}_o,\tau_o)$, respectively. Hence the poles given by (4.12) cannot occur in these regions, so we find

$$\min(\bar{\tau}_o,\tau_o) < \mathrm{Re}\ p_r < \max(\bar{\sigma}_o,\sigma_o) \ . \qquad (4.39)$$

It is thus again *a posteriori* possible to fix $\bar{\sigma}_o$ and $\bar{\tau}_o$. Also, (4.39) puts a restriction on the possible values of λ ; the largest leeway on λ occurs when $\bar{\tau}_o$ is pushed to the bound α and $\bar{\sigma}_o$ is pushed to the bound γ . Thus the roots must satisfy

$$\alpha + \varepsilon \le \mathrm{Re}\; p_r(\lambda) \le \gamma - \varepsilon \tag{4.40}$$

for ε arbitrarily small. The shaded areas of Fig. 2 thus become very thin. The lines σ' and τ' will likewise be pushed very close to α and γ ; this fixes the lines of integration in (4.35). Again, these lines are closed by suitable contours.

Lastly, we remark that the method can also be used for Fredholm equations of the first kind. For the full Mellin transform case one finds,

$$\chi(z) = \frac{1}{2\pi i}\int_{\beta'-i\infty}^{\beta'+i\infty} \frac{dp}{z^p}\frac{\Phi(1-p)}{W(1-p)} , \tag{4.41}$$

while for the partial Mellin transform one has

$$\chi(z) = \frac{1}{2\pi i}\int_{\sigma'-i\infty}^{\sigma'+i\infty} \frac{dp}{z^p}\frac{\Phi_+(1-p)}{W(1-p)} + \frac{1}{2\pi i}\int_{\tau'-i\infty}^{\tau'+i\infty} \frac{dp}{z^p}\frac{\Phi_-(1-p)}{W(1-p)} . \tag{4.42}$$

We now need the domains where $\Phi(1-p)$, $\Phi_\pm(1-p)$, and $W(1-p)$ are analytic. With the bounds on $\Phi(p)$, $\Phi_\pm(p)$ and $W(p)$ as before, one finds for $W(1-p)$: $\alpha < \mathrm{Re}(1-p) < \gamma$, or

$$1 - \gamma < \mathrm{Re}\; p < 1 - \alpha . \tag{4.43}$$

In the case of existence of a full Mellin transform, the domain of analyticity for eq. (4.41) is

$$1 - \min(\tau_o,\overline{\tau}_o,\gamma) < \mathrm{Re}\; p < 1 - \max(\sigma_o,\overline{\sigma}_o,\alpha) . \tag{4.44}$$

The line $\mathrm{Re}\; p = \beta'$ must be in this domain. For the case of (3.5), (4.13) and (4.15) one finds for (4.43) the domain $1 < \mathrm{Re}\; p < 5$ and for (4.44) the domain $2 < \mathrm{Re}\; p < 4$; it is now easily found that (4.41) confirms (3.8) and (3.9).

For the case of the partial Mellin transforms the domains of analyticity for $\Phi_\pm$ are:

$$\Phi_-(1-p) : \mathrm{Re}\; p < 1 - \sigma_o , \tag{4.45a}$$

$$\Phi_+(1-p) \ : \ \mathrm{Re}\ p > 1 - \tau_o \ . \qquad (4.45b)$$

The strips of analyticity for eq. (4.42) become as in Fig. 2, but with Φ_- at the left and Φ_+ at the right; σ' and τ' must be within these strips. Because of the change in bounds and different transformed kernels, this method is usually less straightforward than that of section 2, though it gives identical results.

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