Ann. Sc. Math. Québec, 1978, vol. II, No 2, pp. 281-288

AN ERGODIC THEOREM FOR VECTOR VALUED POSITIVE CONTRACTIONS

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1. INTRODUCTION

Let (X,m) be a σ-finite measure space, and let $L_p(X) = L_p$, $1 < p < \infty$, be the usual Banach space. A linear operator T , from L_p to L_p , is said to be positive if it maps non-negative functions into non-negative functions. It is a contraction if its norm is not bigger than 1 .

For f in L_p , and n a positive integer we consider the average $A(T,n)f = n^{-1}\sum_0^{n-1}T^k f$. Akcoglu [1], has shown that if T is a positive contraction, then $A(T,n)f(x)$ converges for almost all x .

The purpose of this note is to show that this result can be extended to functions taking values in the Banach space ℓ_r , $1 < r < \infty$.

Our main result is the following theorem

Theorem 1. Let T be a positive contraction in L_p . Let $\overline{f} = (f_i)$ be a function in $L_{p,r}$ (i.e. $|f(x)|_r = (\Sigma|\overline{f}_i(x)|^r)^{1/r}$ is in L_p) . We define $T\overline{f} = (Tf_i)$. Then the averages $A(T,n)\overline{f}$ converge almost everywhere in the ℓ_r norm.

Remarks. Although we use the same letter for the operator in L_p , and the one acting on $L_{p,r}$, there is no risk of confusion.

In view of theorem 1, it seems natural to conjecture that a similar theorem holds for any contraction in $L_{p,r}$ that maps functions with non-negative coordinates into functions with non-negative coordinates.

2. THE DOMINATED ESTIMATE FOR POSITIVE ISOMETRIES

The result in [1], relies on a dominated theorem for positive, invertible isometries, due to A. Ionescu-Tulcea [4]. In this section we use the methods of [5], and [6], to obtain a dominated theorem in our case.

Lemma. Let T be a positive isometry in L_p. Then T as an operator in $L_{p,r}$ is also an isometry.

Proof. It is easy to see that a positive isometry maps functions with disjoint support into functions with disjoint support. Now it is shown in [5], that then T is of the form $Tf(x) = h(x)Sf(x)$, where h is a positive function and S is positive, linear and $|Sf|^q = S|f|^q$ for all q.

Let $\overline{f} = (f_i)$, and let's define $S\overline{f} = (Sf_i)$. Writing $|\overline{f}(x)|_r = (\Sigma|f_i(x)|^r)^{1/r}$, we have

$$|T\overline{f}(x)|_r = h(x)(\Sigma|Sf_i(x)|^r)^{1/r} = h(x)S|\overline{f}(x)|_r = T|\overline{f}(x)|_r ,$$

from which the statement of the lemma follows.

Let N be a positive integer, we define a truncated maximal function $M(T,N)\overline{f}(x) = (M(T,N)f_i(x))$, where $M(T,N)f_i(x) = \sup_{n<N}|A(T,n)f_i(x)|$. The maximal function $M(T)\overline{f}(x)$ is defined by letting N go to infinity, i.e. $M(T)\overline{f}(x)$ is the function with coordinates $M(T)f_i(x) = \lim_{N\to\infty} M(T,N)f_i(x)$.

Theorem (2.1). Let T be a positive isometry in L_p, and let $M(T)\overline{f}$ be defined as above. Then there exists $C = C(p,r)$ such that

$$\| \, |M(T)\overline{f}|_r \|_p < C \| \, |\overline{f}|_r \|_p$$

for all $\overline{f}$ in $L_{p,r}$.

Proof. First we observe that if our measure space is the set of integers with counting measure, and T is defined by $Tg(k) = g(k+1)$, then theorem (2.1) is true. More precisely, if $G(k) = (g_1(k), \dots g_n(k), \dots)$, and $MG(k) = (Mg_i(k))$, where $Mg_i(k)$ is defined as $\lim_{N\to\infty} M(N)g_i(k)$, and $M(N)g_i(k) = \sup_{n<N} |n^{-1}\sum_0^{n-1} g_i(k+m)|$, then there exists $C = C(p,r)$ such that

$$\Sigma(\Sigma|Mg_i(k)|^r)^{p/r} \le C^p\Sigma(\Sigma|g_i(k)|^r)^{p/r} . \tag{2.2}$$

For the proof of (2.2) see [3], where it is proved in the continuous case. It is easy to see that the same proof works for the discrete case.

Now, since T is positive, we may assure $\overline{f} \ge 0$ (i.e. $f_i \ge 0$ for all i). Using that T^k separates supports and that it is positive, we have

$$T^k M(T,N) f_i(x) = \Sigma T^k 1_{E_j} A(T,j) f_i(x) \le M(T,N) T^k f_i(x) .$$

Here 1_{E_j} is the indicator function of the set of x's where $M(T,N)f_i(x) = A(T,j)f_i(x)$. It follows that $|T^k M(T,N)\overline{f}(x)|_r \le |M(T,N)T^k\overline{f}(x)|_r$. This, together with the lemma, implies

$$\int|M(T,N)\overline{f}(x)|_r^p = \int|T^k M(T,N)\overline{f}(x)|_r^p \le \int|M(T,N)T^k\overline{f}(x)|_r^p .$$

Now, if g is any function defined on the integers, L is a positive integer, and k is less than L, then it is clear that $M(N)g(k) = M(N)(gI)(k)$, where I is the indicator function of the integers from 0 to $L + N - 1$. This means that we can obtain a local version of (2.2),

$$\begin{aligned}\sum_{K=0}^{L-1} (\sum_i|M(N)g_i(k)|^r)^{p/r} &= \sum_{k=0}^{L-1} (\sum_i|M(N)(g_iI)(k)|^r)^{p/r} \\ &\le C^p\sum_k (\sum_i|(g_iI)(k)|^r)^{p/r} = C^p\sum_{k=0}^{L+N-1} (\sum_i|g_i(k)|^r)^{p/r} .\end{aligned} \tag{2.4}$$

If, for x fixed, we take $T^k f_i(x)$ as $g_i(k)$, and observe that $M(T,N)T^k f_i(x) = M(N)g_i(k)$, we have from (2.3) and (2.4) that

$$\int |M(T,N)\overline{f}(x)|_r^p \le L^{-1} \sum_0^{L-1} \int |M(T,N)T^k\overline{f}(x)|_r^p$$

$$\le C^p L^{-1} \int \sum_0^{L+N-1} |T^k\overline{f}(x)|_r^p = C^p (L+N)/L \int |\overline{f}(x)|_r^p .$$

Letting L go to infinity, we obtain the theorem.

3. THE DOMINATED ESTIMATE FOR POSITIVE CONTRACTIONS

Since we already have an estimate for isometries, we can use the dilation theorem of [1], to reduce the problem to the isometry case.

Lemma (3.1). Let T be a strictly positive contraction on $L_p(X)$, where X is a space with a finite number of points. Then

$$\int |M(T)\overline{f}|_r^p \le C^p \int |\overline{f}|_r^p$$

for all $\overline{f}$ in $L_{p,r}(X)$, with the same C as in (2.1).

Proof. We may assume $\|T\| = 1$. Let $x_1, x_2, \ldots x_N$, be the points of X. By theorem 2.13 of [1], there exists a measure space (Z,μ) such that $Z = \cup_1^N E_i$, with $\mu(E_i) = m(x_i)$, and a positive invertible isometry Q on $L_p(Z)$ such that, if f is in $L_p(X)$, then

$$\zeta T^n f = EQ^n \zeta f \quad \text{for all} \quad n \ge 0 . \tag{3.2}$$

Here ζ is the isometry of $L_p(X)$ into $L_p(Z)$, defined by $\zeta f(z) = \Sigma f(x_i) 1_{E_i}$, and E is the conditional expectation induced in Z by the partition $E_1, \ldots E_N$. From the definition of ζ it is clear that if $\overline{f} = (f_1, \ldots f_n, \ldots)$ is in $L_{p,r}(X)$ and we consider $\zeta\overline{f} = (\zeta f_i)$, then

$$\int |\zeta\overline{f}|_r^p = \int \zeta |\overline{f}|_r^p .$$

From (3.2) it follows, using that E is a positive operator, that

$$\zeta M(T) f \le EM(Q)\zeta f \quad \text{for all} \quad f \quad \text{in} \quad L_p .$$

We are now in position to use theorem (2.1) to obtain

$$\int_X |M(T)\overline{f}|_r^p = \int_Z |\zeta|M(T)\overline{f}|_r)^p = \int_Z |\zeta M(T)\overline{f}|_r^p \le \int_Z |EM(Q)\zeta\overline{f}|_r^p \le \int_Z |M(Q)\zeta\overline{f}|_r^p$$

$$\le C^p \int_Z |\zeta\overline{f}|_r^p = C^p \int_Z |\overline{f}|_r^p \ .$$

The following lemma will allow us to remove the restriction of strict positivity.

Lemma (3.3). Let T be a positive contraction on $L_p(X)$, where X is a σ-finite measure space. Let's assume that there exists C such that for any $\overline{f}$ with a finite number of non zero coordinates, we have

$$\int |M(T)\overline{f}|_r^p \le C^p \int |\overline{f}|_r^p \ .$$

Then, the same estimate holds for any $\overline{f}$ in $L_{p,r}$.

Proof. Let $\overline{f}$ be in $L_{p,r}$. For any positive integer L, $\overline{f}_L$ will denote a function that has its first L coordinates equal to those of $\overline{f}$, and the rest are zero. By monotone convergence, $\overline{f}_L$ converges to $\overline{f}$ in $L_{p,r}$. Our lemma will be proved if we show that $M(T)\overline{f}_L$ converges to $M(T)\overline{f}$ in $L_{p,r}$. Clearly, $M(T)\overline{f}_{L+K} - M(T)\overline{f}_L = M(T)(\overline{f}_{L+K} - \overline{f}_L)$, and therefore

$$\int |M(T)\overline{f}_{L+K} - M(T)\overline{f}_L|_r^p \le C^p \int |\overline{f}_{L+K} - \overline{f}_L|_r^p \ .$$

This means that $M(T)\overline{f}_L$ is a Cauchy sequence in $L_{p,r}$, and therefore it converges in the $L_{p,r}$ norm. It is clear that the limit has to be $M(T)\overline{f}$.

The lemma implies that if the dominated estimate fails, then it fails for a function with a finite number of non zero coordinates.

Corollary. The word "strictly" can be eliminated from lemma (3.1).

Proof. If not, there exists T, positive contraction, $\overline{f}$ with a finite number of non zero coordinates, and an integer N such that

$$\int |M(T,N)\overline{f}|_r^p > C^p \int |\overline{f}|_r^p \ .$$

Then the same inequality is satisfied replacing T by sT, for some $s < 1$. But then there exists $u > 0$ so that the operator $T' = sT + u$, is a strictly positive contraction on L_p for which the inequality above still holds, but this contradicts (3.1).

Theorem (3.4). Let T be a positive contraction on $L_p(X)$, where X is a σ-finite measure space. Then the dominated estimate holds for $L_{p,r}(X)$, with the same constant as in (2.1).

Proof. If not, then there exists L and a function $\overline{f}$ with only L non zero coordinates, such that

$$\int |M(T,L)\overline{f}|_r^p > C^p \int |\overline{f}|_r^p .$$

By lemmas (3.1) and (3.2) of [1], for any $e > 0$, there exists a conditional expectation E, such that

$$\|Ef_i - f_i\|_p < e \quad \text{for} \quad 0 < i \le L$$

and

$$\|M(T,L)f_i - M(ET,L)Ef_i\|_p < e \quad \text{for} \quad 0 < i \le L .$$

But this implies

$$\int |M(T,L)\overline{f} - M(ET,L)E\overline{f}|_r^p \le L^{p/r} \int \Sigma |M(T,L)f_i - M(ET,L)Ef_i|^p \le L^{1+p/r} e \quad .$$

In the same way

$$\int |\overline{f} - E\overline{f}|_r^p \le L^{1+p/r} e .$$

Choosing e small enough, we have

$$\int |M(ET,L)E\overline{f}|_r^p > C^p \int |E\overline{f}|_r^p \quad .$$

Let $(E_1, E_2, \ldots E_n)$ be the partition of X corresponding to the conditional expectation E. Then the functions of the form Ef can be thought of as functions on L_p of a space with a finite number of points, and ET defines a positive contraction on this space. But this contradicts the corollary to lemma (3.3).

Theorem 1 now follows in a standard way from the dominated estimate, plus the fact that since $L_{p,r}$ is reflexive, the functions of the form

$$\overline{h} + (\overline{g} - T\overline{g})$$

are dense in $L_{p,r}$. See [2] for details.

Remark. I want to express my thanks to the referee, for his helpful indications. In particular he observed that, from our first lemma in section 2, and the theorem of Ionescu-Tulcea one can obtain, in the case of a positive isometry

$$\int(\sup_n |A_n\overline{f}|_r)^p \le \int(\sup_n A_n|\overline{f}|_r)^p \le C_p \int|\overline{f}|_r^p .$$

This is enough to prove the a.e. convergence in this case. Unfortunately it does not seem to be possible to get the general case from this one. The reason is that, in the reduction of the problem to a space with a finite number of points, it is essential to use that, if the dominated estimate fails, then it already does fail for a function with a finite number of non zero coordinates. This is easy to see for $M(T)\overline{f}$. But I don't see how to do it for the maximal operator defined on this remark.

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Manuscrit reçu le 15 juillet 1977.
Revisé le 20 décembre 1977.