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AN UNIQUENESS PROPERTY FOR A FAMILY OF MEASURABLE FUNCTIONS

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INTRODUCTION.

This paper is related to our previous work on the heat equation, [1] precisely to the paragraph II concerning uniqueness of L^2-bounded solutions on the real axis.

The polynomial $P_0(s_1 \dots s_n) = -(s_1^2 + \dots s_n^2) = -|s|^2$ which appears there in connection with equation $\frac{\partial u}{\partial t} = \Delta u$ via L^2-Fourier transform is here replaced by the complex-valued polynomial

$$P(s) = P(s_1 \dots s_n) = -(s_1^2 + \dots s_n^2) + \sum_{p=1}^{n} a_p(is_p) + b \text{ where } a_1,\dots,a_n,b$$

are real constant coefficients and $i = \sqrt{-1}$.

The result that we obtain, even if expressed in a form independent of the theory of partial differential equations, has an obvious application, as in [1], to prove uniqueness of $L^2(\mathbb{R}^n)$-bounded solutions on the real axis for the general heat equation:

$$\frac{\partial u}{\partial t} = \Delta u + \sum_{p=1}^{n} a_p \frac{\partial u}{\partial x_p} + bu.$$

§1. Let us consider a measurable function $f(s_1 \dots s_n) = f(s)$ defined on $\mathbb{R}^n - E$, where E is a given set of (Lebesgue) null measure, having range in the extended complex plane. Then, for any real value of t, the function $e^{tP(s)}f(s)$ is again defined on $\mathbb{R}^n - E$, is measurable, taking values in the extended complex plane.

We have thus defined a map: $t \in \mathbb{R}^1 \Rightarrow e^{tP(s)}f(s) \in M$ where M is the set of complex-valued measurable functions, defined almost-everywhere in $\mathbb{R}^n$.

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Let us assume now, furthermore, that f(s) is square-integrable on $\mathbb{R}^n$, so that

$$\int_{\mathbb{R}^n} |f(s)|^2 ds < \infty \text{ and } f \in L^2(\mathbb{R}^n).$$

We see then that, for any $t \in \mathbb{R}^1$, it is:

$$|e^{tP(s)}f(s)| = e^{t(b-|(s)|^2)}|f(s)|;$$

hence it follows:

$$\int_{\mathbb{R}^n} |e^{tP(s)}f(s)|^2 ds = \int_{\mathbb{R}^n} e^{2t(b-|s|^2)}|f(s)|^2 ds =$$

$$= \int_{\{s;|s|^2\le b\}} e^{2t(b-|s|^2)}|f(s)|^2 ds + \int_{\{s;|s|^2\ge b\}} e^{2t(b-|s|^2)}|f(s)|^2 ds.$$

If b is ≤ 0, the first integral is not considered, and the second is convergent for any $t \ge 0$, because $e^{2t(b-|s|^2)}$ is then ≤ 1.

If b is > 0 we have to consider also the first integral which is estimated by:

$$e^{2tb} \int_{|s|^2\le b} |f(s)|^2 ds$$

when $t \ge 0$.

Hence we proved the following:

PROPOSITION:

If $f(s) \in L^2(\mathbb{R}^n)$ *then, for any* $t \in \mathbb{R}^+_1$, *the function:* $e^{tP(s)}f(s) \in L^2(\mathbb{R}^n)$ *too.*

§2. Let us consider again the map: $t \in \mathbb{R}^1 \Rightarrow e^{tP(s)}f(s) \in M$; for some particular choice of $f(s) \in L^2(\mathbb{R}^n)$ it can happens that: $e^{tP(s)}f(s) \in L^2(\mathbb{R}^n)$ for any real value of t. For example, if f(s) has compact support or if f(s) is very rapidly decreasing, say $f(s) = e^{-|s|^3}$. In the first case:

$$\int_{\mathbb{R}^n} |e^{2tP(s)}||f(s)|^2 ds = \int_K e^{2t(b-|s|^2)}|f(s)|^2 ds \le C(t,b,K) \int_{\mathbb{R}^n} |f(s)|^2 ds$$

for $t \in \mathbb{R}^1$. In the second case,

$$\int_{\mathbb{R}^n} e^{2t(b-|s|^2)}e^{-2|s|^3} ds = e^{2tb} \int_{\mathbb{R}^n} e^{-2|s|^2(t+|s|)} ds.$$

For $|s| > |t|$, $|s| + t > \delta > 0$ so that $e^{-2|s|^2(t+|s|)} < e^{-2\delta|s|^2}$ which is in $L^2(\mathbb{R}^n)$ again.

Hence we can define the non-void class of functions (complex-valued) $C_P = \{f(s) \in L^2(\mathbb{R}^n)$, such that $e^{tP(s)}f(s) \in L^2(\mathbb{R}^n)$ for any real value of $t\}$.

THEOREM:

Let us assume that $f(s) \in C_P$ *and that*

$$L = \sup_{t\in\mathbb{R}^1} \int_{\mathbb{R}^n} |e^{tP(s)}f(s)|^2 ds \text{ is } < \infty.$$

Then $f(s) = 0$ *almost-everywhere on* $\mathbb{R}^n$.

Proof of Theorem:

Consider in fact an arbitrary function $\varphi(s) \in L^2(\mathbb{R}^n)$ (complex-valued and then consider the integral corresponding to the $L^2(\mathbb{R}^n)$-inner product:

$$\int_{\mathbb{R}^n} f(s)\,\overline{\varphi(s)}\,ds$$

which exists by Schwartz inequality. Then write, for a $\delta > 0$, the relation:

$$\int_{R^n} f(s)\bar{\varphi}(s)ds = \int_{b<|s|^2<b+\delta} f(s)\bar{\varphi}(s)ds + \int_{|s|^2>b+\delta} f(s)\bar{\varphi}(s)ds$$

$$+ \int_{b-\delta<|s|^2<b} f(s)\bar{\varphi}(s)ds + \int_{|s|^2<b-\delta} f(s)\bar{\varphi}(s)ds = I_1+I_2+I_3+I_4.$$

Now, if $b < 0$, choose δ such that $\delta+b \leq 0$; then only I_2 has to be considered so that

$$\int_{R^n} f(s)\bar{\varphi}(s)ds = I_2.$$

Then:

$$I_2 = \int_{|s|^2>b+\delta} e^{P(s)(-n)}f(s)e^{nP(s)}\bar{\varphi}(s)ds$$

$$\leq \Big(\int_{|s|^2>b+\delta} |e^{-nP(s)}f(s)|^2ds\Big)^{\frac{1}{2}}\Big(\int_{|s|^2>b+\delta} |e^{nP(s)}\bar{\varphi}(s)|^2ds\Big)^{\frac{1}{2}}$$

$$\leq L\Big(\int_{|s|^2>b+\delta} e^{2n(b-|s|^2)}|\varphi(s)|^2ds\Big)^{\frac{1}{2}},$$

because

$$\sup_{t\in\mathbb{R}^1} \|e^{tP(s)}f(s)\|_{L^2} = L < \infty.$$

Furthermore, for $|s|^2-b > \delta$, $e^{2n(b-|s|^2)} < e^{-2n\delta}$, so that:

$$I_2 \le L\, e^{-n\delta}\Big(\int_{\mathbf{R}^n} |\varphi(s)|^2 ds\Big)^{\frac{1}{2}};$$

as $n \to \infty$, the right-hand side tends to zero, so that $I_2 = I = 0$.

Take now the case of $b \ge 0$. Then I_1 writes as:

$$I_1 = \int_{b<|s|^2<b+\delta} f(s)\bar{\varphi}(s)ds,$$

then:

$$I_1 \le \Big(\int_{b<|s|^2<b+\delta} |\varphi(s)|^2 ds\Big)^{\frac{1}{2}}\|f\|_{L^2}.$$

Given any $\varepsilon > 0$, $\exists\ \delta(\varepsilon)$ such that $I_1 < \varepsilon$. Now I_2 estimates through:

$$I_2 \le L\Big(\int_{|s|^2>b+\delta} e^{2n(b-|s|^2)}|\varphi(s)|^2 ds\Big)^{\frac{1}{2}} \le L\, e^{-\delta n}\Big(\int_{\mathbf{R}^n} |\varphi(s)|^2 ds\Big)^{\frac{1}{2}}.$$

Hence, for $n>$, $N(\varepsilon)$, I_2 is $< \varepsilon$; here δ is already chosen in order that I_1 be $< \varepsilon$ too.

Now, when $b=0$, we do not have to consider I_3 and I_4; then

$$I = \int_{R^n} f(s)\bar{\varphi}(s)ds = I_1+I_2 < 2\varepsilon \text{ for } \delta = \delta(\varepsilon) \text{ and } n \ge N(\varepsilon).$$

But I is independent of δ and n so that $I=0$ necessarily. When b is strictly > 0 we have also to consider

$$I_3 \le \|f\|_{L^2}\Big(\int_{b-\delta<|s|^2<b} |\varphi(s)|^2 ds\Big)^{\frac{1}{2}} \text{ which is } < \varepsilon \text{ for } \delta = \delta(\varepsilon)$$

$$I_4 = \int_{0\le|s|^2<b-\delta} e^{nP(s)}f(s)e^{-nP(s)}\bar{\varphi}(s)ds \le L\Big(\int_{|s|^2<b-\delta} e^{-2n(b-|s|^2)}|\varphi(s)|^2 ds\Big)^{\frac{1}{2}}$$

$$\le e^{-2n\delta}L\|\varphi\|_{L^2} \to 0 \text{ as } n \to \infty.$$

Hence, for $b > 0$, $I = I_1+I_2+I_3+I_4 < 4\varepsilon$ for $\delta(\varepsilon)$ well-chosen and n big enough. As I is independent of δ and n, I=0. This proves the theorem.

REFERENCE

1. ZAIDMAN, S., Soluzioni limitate e quasi-periodiche dell equazione del calore non-omogenea, I, Rend. Accad. Naz. Lincei, Dicembre 1961.

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